# Introduction to Complex Analysis

## Second Edition

Rolf Nevanlinna
Veikko Paatero

AMS Chelsea Publishing
American Mathematical Society • Providence, Rhode Island

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# FOREWORD

The present textbook is based upon lectures given by the authors at Helsinki University and at the University of Zürich, and is a translation of the German edition, *Einführung in die Funktionentheorie*, published by Birkhäuser Verlag, Basel, in 1964.

It is assumed that the reader is acquainted with analytic geometry and the calculus, so that this introduction to the theory of functions may be begun in the third or fourth year of undergraduate study in college.

As the Table of Contents indicates, the present volume is limited to the presentation of the elements of the theory of functions, and the authors have attempted to make the material both comprehensible and precise. Among the sections in which this volume deviates more or less from other presentations we must mention the following: the introduction of the complex numbers, the concept of homotopy and its application, the integral theorems, the theory and application of harmonic functions, in particular harmonic measure, and the correspondence of boundaries under conformal mapping.

Exercises have been placed at the end of each chapter, and all 320 of these exercises should be solved by the student for better insight into the subject matter, whether he learns the subject through lectures or by self-study.

In introducing the elementary functions (Chapters 2–7) we have followed in many places the presentation given by Ernst Lindelöf in his Finnish textbook, *Johdatus funktioteoriaan* (introduction to the theory of functions). This is particularly true for a considerable number of the exercises of these chapters.

We have received assistance in our work from various sources. First we owe thanks to Dr. G. S. Goodman and Dr. T. Kövari for the effort and interest which they have put into the translation of the book. We also express our appreciation to Addison-Wesley Publishing Company and, in particular, to Professor A. J. Lohwater for the valuable advice and generous assistance which he has given in the editing of this edition.

Helsinki, September, 1968

Rolf Nevanlinna
V. Paatero

# CONTENTS

CHAPTER 1

# THE CONCEPT OF AN ANALYTIC FUNCTION

The theory of functions is concerned with complex-valued functions of a complex variable. Our study is confined to those functions which are *differentiable* in a sense which will be made precise later on; such functions are known as *analytic functions*. In order to create a basis for the theory, we begin by introducing the complex numbers in a manner which will lead us naturally to their interpretation as vectors in the plane.

## §1. THE COMPLEX NUMBERS

### 1.1. Two-dimensional Vector Spaces

We begin by stating the axioms for a two-dimensional vector space over the real numbers.

Let there be given a set $R$, whose elements $a, b, \ldots, x, y, \ldots$ shall be called *points* or *vectors*, satisfying the following conditions.

I. To every two elements $a, b \in R$ there corresponds an element $c \in R$, known as their sum and written $c = a + b$, obeying the following rules:

I.1. $a + b = b + a$ (the commutative law).

I.2. $a + (b + c) = (a + b) + c$ (the associative law).

I.3. There is a zero in $R$, denoted $x = 0$, with the property that $a + 0 = a$ for every $a \in R$.

I.4. The equation $a + x = b$ has one and only one solution, $x = b - a \in R$.

II. To every vector $a$ and every real number $\lambda$ there corresponds a vector $b = \lambda a \in R$, known as their product, and obeying the following rules.

II.1. $\lambda(\mu a) = (\lambda\mu)a$ ($\lambda$, $\mu$ real numbers).

II.2. $(\lambda + \mu)a = \lambda a + \mu a$, $\lambda(a + b) = \lambda a + \lambda b$ (the distributive law).

II.3. $1 \cdot a = a$.

II.4. The product $\lambda a$ vanishes if and only if $\lambda = 0$, or $a = 0$, or both $\lambda = 0$ and $a = 0$.†

II.5. *The axiom of dimension:* there exist two vectors $a$ and $b$ in $R$ which are linearly independent, that is, for which the equation

† We shall use the symbol 0 for the number zero as well as for the vector zero without, we trust, provoking any confusion.

$\lambda a + \mu b = 0$ has only the solution $\lambda = \mu = 0$, but every three vectors $a$, $b$, $c$ in $R$ are linearly dependent, that is, the equation $\lambda a + \mu b + \nu c = 0$ always has a solution such that at least one of the three numbers $\lambda$, $\mu$, $\nu$ does not vanish.

This axiom asserts that the dimension of the vector space $R$ is equal to *two*. In the resulting "affine plane" every vector $x$ admits a representation in terms of its coordinates in a two-dimensional reference system. Such a system is given by a *basis* for $R$, that is, by two linearly independent vectors $e_1$, $e_2 \in R$. From II.5 it follows that every vector $x \in R$ has two numbers $\xi_1$ and $\xi_2$ associated with it (its coordinates in this reference system) such that

$$x = \xi_1 e_1 + \xi_2 e_2.$$

### 1.2. Plane Euclidean Geometry

Axioms I and II define a two-dimensional vector space whose geometry is the geometry of the affine plane. It becomes a Euclidean geometry once we introduce a (Euclidean) measure of length and angle. We can arrive at such a measure by defining, for any two vectors $x$ and $y$ in $R$, a scalar product $(x, y)$ with the following properties.

III.1. $(x, y)$ is a real, symmetric function of its arguments $x$ and $y$: $(x, y) = (y, x)$.

III.2. $(x, y)$ is linear in each argument.†

III.3. $(x, y)$ is positive definite, that is, $(x, x) \geqq 0$, and equality holds only for $x = 0$.

The *length*, *norm*, or *modulus* $|x|$ of a vector $x$ is defined by

$$|x| = +\sqrt{(x, x)}.$$

It is easily proved (Exercises 1 and 2)‡ that the following inequalities hold:

1) *Schwarz's inequality* $(x, y)^2 \leqq |x|^2\,|y|^2$;

2) *The triangle inequality* $|x + y| \leqq |x| + |y|$.

The angle $[x, y]$ between two vectors $x, y\,(\neq 0)$ is defined by

$$\cos\,[x, y] = \frac{(x, y)}{|x|\,|y|}.$$

Two vectors are therefore orthogonal if $(x, y) = 0$.

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† A function $f(x)$ is said to be linear, if $f(\lambda x) = \lambda f(x)$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$. The linearity of the scalar product $(x, y)$ asserts, therefore, that this product obeys the distributive law.

‡ Unless there are indications to the contrary, the numbers will always refer to the exercises at the end of the chapter.

If $e_1$, $e_2$ is a basis for $R$ and if the vectors $x$, $y$ have the representations

$$x = \xi_1 e_1 + \xi_2 e_2, \qquad y = \eta_1 e_1 + \eta_2 e_2,$$

in terms of this basis, then

$$(x, y) = (\xi_1 e_1 + \xi_2 e_2, \eta_1 e_1 + \eta_2 e_2) = \sum_{i,k=1}^{2} g_{ik}\xi_i \eta_k,$$

where the $g_{ik}$ denote the real constants

$$g_{ik} = (e_i, e_k) \qquad (g_{12} = g_{21}).$$

The square of the norm of $x$ is the quadratic form

$$|x|^2 = (x, x) = \sum_{i,k} g_{ik}\xi_i\xi_k = g_{11}\xi_1^2 + 2g_{12}\xi_1\xi_2 + g_{22}\xi_2^2.$$

It reduces to the Pythagorean form

$$|x|^2 = \xi_1^2 + \xi_2^2$$

if and only if the coordinate system is orthonormal; that is,

$$(e_1, e_2) = 0, \qquad |e_1| = |e_2| = 1$$

(the Cartesian coordinate system).

### 1.3. Extension of the Set $R$ to a Vector Algebra

In what follows, we shall not introduce a metric into the plane $R$ for the time being, so that we shall be dealing with an affine geometry on $R$ defined by the postulates in groups I and II. The problem before us is to see whether it is possible to extend I and II so as to give $R$ the structure of a field (or algebra), and, if this is possible, to discover in how many different ways it can be done.

The vector space $R$ becomes an algebra once we are able to define, for any two elements $x, y \in R$, a "product"

$$z = xy \in R$$

which satisfies the following axioms.

- IV.1. The product is commutative: $xy = yx$.
- IV.2. The product is bilinear, that is, linear in each factor.
- IV.3. The product is associative: $x(yz) = (xy)z$.
- IV.4. The product $xy$ vanishes, $xy = 0$, if and only if at least one factor vanishes.

### 1.4.

Our task, then, is *to find all bilinear forms $xy \in R$ which satisfy these axioms IV*. In order to arrive at the general solution to this problem, we shall assume, at first, that we already have a product $xy$ defined on $R$ in accordance with axioms IV and see what this tells us.

If we fix the vector $y \neq 0$ in the product $z = xy$, we obtain a linear transformation in $x$ which maps the plane $R$ into itself. This mapping is one-to-one, for if $z_1 = x_1 y$, $z_2 = x_2 y$, then

$$z_1 - z_2 = (x_1 - x_2)y.$$

Since $y$ was assumed to be different from 0, $z_1 - z_2$ will vanish if and only if $x_1 - x_2 = 0$. Different vectors $x$ therefore have (for each fixed $y \neq 0$) different image vectors $z = xy$.

On the other hand, the range of the mapping $z = xy$ is the whole plane $R$. For, if $x_1$ and $x_2$ are two vectors in $R$, and $\lambda_1$ and $\lambda_2$ are two arbitrary real numbers, then

$$(\lambda_1 x_1 + \lambda_2 x_2)y = \lambda_1 x_1 y + \lambda_2 x_2 y = \lambda_1 z_1 + \lambda_2 z_2,$$

where $z_1 = x_1 y$, $z_2 = x_2 y$. From this we see that the image vectors $z_1$, $z_2$ are linearly independent if and only if the vectors $x_1$, $x_2$ are linearly independent. Hence, if $x_1$, $x_2$ is a basis for $R$, then $z_1$, $z_2$ will also be a basis. If the vector $x$ has the coordinates $\lambda_1$, $\lambda_2$ in the system $(x_1, x_2)$, then its image vector has the same coordinates in the system $(z_1, z_2)$, for $z = xy = \lambda_1 z_1 + \lambda_2 z_2$. Hence, the set of image vectors $z = xy$ covers the plane $R$ exactly once if $x$ runs through all values in $R$ (for $y$ fixed).

Thus, for any given vector $y \neq 0$, there is precisely one vector $x$ which makes the product $xy$ take a prescribed value $z$; that element is the "quotient" $x = z/y$.

### 1.5. Definition of the Unit Vector *e*

If, in particular, we take $z = y (\neq 0)$, then there is a definite vector $e = e_y \in R$ having the property that $e_y y = y$. We shall show that $e_y$ is *independent* of the choice of $y$. Let $y_1$ and $y_2$ be two non-zero vectors. If $e_1 y_1 = y_1$, $e_2 y_2 = y_2$, then

$$e_2 y_2 = y_2 = y_1 \frac{y_2}{y_1} = (e_1 y_1) \frac{y_2}{y_1},$$

and this last expression is, by axiom IV.3 (the associative law), equal to $e_1(y_1 y_2/y_1) = e_1 y_2$. Hence, $e_2 y_2 = e_1 y_2$, or $(e_1 - e_2)y_2 = 0$, from which it follows that $e_1 = e_2$, since $y_2 \neq 0$.

The element $e\ (\neq 0)$ defined uniquely by the equation

$$ey = ye = y \tag{1.1}$$

is called the *unit vector*, or *unit*, in $R$.

### 1.6. Definition of the Vector *i*

Let $a$ be an arbitrary vector in $R$ and consider the equation

$$x^2 = a.$$

If this equation has a solution $x = x_1$, $x_1^2 = a$, then, for every vector $x \in R$, we have

$$x^2 - a = x^2 - x_1^2 = (x - x_1)(x + x_1),$$

so that the equation $x^2 - a = 0$ has, in addition to $x = x_1$, one further solution $x = -x_1$.

Let us choose $a = -e$ and solve the equation

$$x^2 + e = 0. \tag{1.2}$$

The existence of a solution will be shown in an exercise (Exercise 3). We denote the solutions by $x = \pm i$ ($\neq 0$). The vector $i$ is linearly independent of the vector $e$, for, if $i = \lambda e$ ($\lambda$ real), then we would have $-e = i^2 = (\lambda e)^2 = \lambda^2 e^2 = \lambda^2 e$, or $(1 + \lambda^2)e = 0$, which is impossible, since both $1 + \lambda^2 \neq 0$ and $e \neq 0$.

The vectors $x = e$ and $x = i$ span the entire plane $R$. An arbitrary vector $x \in R$ has the coordinate representation

$$x = \xi e + \eta i.$$

This representation has been found under the assumption that there is a product, defined for pairs of vectors $x_1, x_2 \in R$, which satisfies the axioms IV. If $x_1$ and $x_2$ are written in terms of coordinates,

$$x_1 = \xi_1 e + \eta_1 i, \qquad x_2 = \xi_2 e + \eta_2 i,$$

it follows from IV and the definition of the basis vectors $e$ and $i$ via (1.1) and (1.2) that the product $x_1 x_2$ must have the form

$$\begin{aligned} x_1 x_2 &= (\xi_1 e + \eta_1 i)(\xi_2 e + \eta_2 i) \\ &= (\xi_1 \xi_2 - \eta_1 \eta_2)e + (\xi_1 \eta_2 + \eta_1 \xi_2)i. \end{aligned} \tag{1.3}$$

The quotient $x_1/x_2$ ($x_2 \neq 0$) is defined to be that vector $x = \xi e + \eta i$ which, when multiplied by the vector $x_2 = \xi_2 e + \eta_2 i$, yields the vector $x_1 = \xi_1 e + \eta_1 i$. With the aid of (1.3), we can obtain the coordinates $\xi, \eta$ of $x$ from the equations

$$\xi_2 \xi - \eta_2 \eta = \xi_1, \qquad \eta_2 \xi + \xi_2 \eta = \eta_1.$$

Therefore the quotient $x_1/x_2$ is given by the expression

$$\frac{x_1}{x_2} = \frac{\xi_1 e + \eta_1 i}{\xi_2 e + \eta_2 i} = \frac{\xi_1 \xi_2 + \eta_1 \eta_2}{\xi_2^2 + \eta_2^2} e + \frac{\eta_1 \xi_2 - \xi_1 \eta_2}{\xi_2^2 + \eta_2^2} i. \tag{1.4}$$

### 1.7. The Solution of the Extension Problem

We now turn all this around and choose any two linearly independent vectors in $R$, label them $e$ and $i$, and *define* the product $x_1 x_2$ of two vectors $x_j = \xi_j e + \eta_j i$ ($j = 1, 2$) by means of Eq. (1.3). We shall then have $ex = xe = x$ and $i^2 + e = 0$, and all the axioms IV will be satisfied. The verification of axioms IV.1–3 we leave to the reader. To prove IV.4, we observe that the equation $x_1x_2 = 0$ is equivalent, by (1.3), to the coordinate equations

$$\xi_1\xi_2 - \eta_1\eta_2 = 0, \qquad \xi_1\eta_2 + \eta_1\xi_2 = 0.$$

Squaring and then adding, we obtain

$$(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = 0.$$

Consequently, $\xi_1 = \eta_1 = 0$ or $\xi_2 = \eta_2 = 0$, that is, $x_1 = 0$ or $x_2 = 0$ (or $x_1 = x_2 = 0$), as required by axiom IV.4.

We have, therefore, completely solved the problem before us:

*If the vectors e and i are any two arbitrarily chosen linearly independent vectors, then* (1.3) *furnishes a definition for the product of two vectors in R which makes R into a field* (or algebra, that is, a vector space which satisfies the axioms IV), *and this definition of the product is the only one that is compatible with all the axioms.*

### 1.8. Notation for Complex Numbers. Absolute Value and Argument

Having made $R$ into a field in which every vector, or *complex number*, can be written as $\xi_1 e + \xi_2 i$, we want to say something about notation. Vectors $\xi e$ ($\xi$ real) along the $e$-axis we shall denote, for brevity, by $\xi$ alone, by dropping the $e$. In view of the property $xe = ex = x$ which defines the unit, this can hardly lead to confusion. Furthermore, in keeping with a long-standing custom we shall denote the coordinates of a complex number $z = \xi_1 e + \xi_2 i = \xi_1 + \xi_2 i$ by $\xi_1 = x$, $\xi_2 = y$, and write

$$z = x + iy.$$

The real number $x$ is called the *real part* of $z$, and the real number $y$ is called the *imaginary part* of $z$. These terms can be abbreviated to

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

We now introduce a Euclidean metric into the "complex plane" $R$ by defining the scalar product $(z_1, z_2)$ of two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ as

$$(z_1, z_2) = x_1x_2 + y_1y_2.$$

This means that the basis vectors $e$ and $i$ are orthogonal to one another, and that their lengths are one: $|e| = |i| = 1$.

The *modulus* or *absolute value* of a complex number $z = x + iy$ is then given by

$$|z| = +\sqrt{(z, z)} = +\sqrt{x^2 + y^2}.$$

If we go over to polar coordinates, we get

$$z = r(\cos\phi + i\sin\phi),$$

where $r = |z|$, $\phi = \arctan y/x$. The quantity $\phi$ is called the *argument* of $z$:

$$\arg z = \phi = \arctan y/x.$$

As long as $z \neq 0$, $\phi$ is defined up to a multiple of $2\pi$ (we say "modulo $2\pi$," and write "mod $2\pi$").

In this notation, the product of two complex numbers

$$z_k = r_k(\cos \phi_k + i \sin \phi_k) \ (k = 1, 2)$$

is

$$z_1 z_2 = r_1 r_2\{\cos (\phi_1 + \phi_2) + i \sin (\phi_1 + \phi_2)\}.$$

From this it follows that

*The absolute value of the product of two complex numbers is equal to the product of their absolute values, while the argument of the product is equal to the sum* (mod $2\pi$) *of the arguments of the factors:*

$$|z_1 z_2| = |z_1|\,|z_2|, \qquad \arg (z_1 z_2) = \arg z_1 + \arg z_2 \ (\text{mod } 2\pi).$$

The latter rule presupposes that the factors are different from zero, since the argument of the number $z = 0$ is indeterminate.

From the product rule it follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \ (\text{mod } 2\pi).$$

If all $n$ factors, $z = r(\cos \phi + i \sin \phi)$, of a product are equal we obtain

$$[r(\cos \phi + i \sin \phi)]^n = r^n(\cos n\phi + i \sin n\phi).$$

This yields as a special case, for $r = 1$, *de Moivre's formula*

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi. \tag{1.5}$$

The numbers $x + iy$ and $x - iy$ are said to be *complex conjugates*. The complex conjugate of the complex number $z$ is denoted by $\bar{z}$; obviously,

$$z\bar{z} = |z|^2; \qquad \text{Re } z = \frac{1}{2}(z + \bar{z}), \qquad \text{Im } z = \frac{1}{2i}(z - \bar{z}).$$

Geometrically speaking, the addition of complex numbers corresponds to vector addition (according to the parallelogram rule). The difference $z_1 - z_2$ corresponds to a vector whose initial point is at $z_2$ and whose end-point is at $z_1$. The modulus of the difference $|z_1 - z_2|$ gives the distance between the points $z_1$ and $z_2$.

Since the complex numbers form an algebra (axioms I–IV), the rational operations of arithmetic (addition, subtraction, multiplication, and division) obey the same rules as in the real case. Over and beyond this, the defining equation $i^2 = -1$ must be taken into account.

## §2. POINT SETS IN THE COMPLEX PLANE

### 1.9. Convergent Sequences

*A sequence of complex numbers*

$$z_1, z_2, \ldots, z_n, \ldots \tag{1.6}$$

*tends to a limit,*

$$\lim_{n\to\infty} z_n = z, \tag{1.7}$$

*if, to any arbitrarily prescribed number* $\epsilon > 0$, *a number* $n_\epsilon > 0$ *can be found such that*

$$|z_n - z| < \epsilon \qquad \textit{for} \qquad n \geqq n_\epsilon. \tag{1.8}$$

The condition (1.8) says, geometrically, that all the points $z_n$ ($n \geqq n_\epsilon$) lie in a circle about $z$ with radius $\epsilon$.

Let $z = x + iy$, $z_n = x_n + iy_n$.

*Then the conditions*

$$\lim_{n\to\infty} x_n = x \qquad \textit{and} \qquad \lim_{n\to\infty} y_n = y \tag{1.9}$$

*are necessary and sufficient for* (1.7) *to hold.*

The necessity of the condition (1.9) follows immediately from the inequalities

$$|x_n - x| \leqq |z_n - z|, \qquad |y_n - y| \leqq |z_n - z|.$$

Conversely, if (1.9) is fulfilled, then there exists a number $N$ with the property that

$$|x_n - x| < \frac{\epsilon}{2} \qquad \text{and} \qquad |y_n - y| < \frac{\epsilon}{2} \qquad \text{for} \qquad n \geqq N.$$

Consequently, for all $n \geqq N$ we have

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \frac{\epsilon}{\sqrt{2}} < \epsilon,$$

which shows that the condition (1.9) is also sufficient. The following theorem is also easy to prove (Exercise 16):

*If* $z \neq 0$, *the conditions*

$$\lim_{n\to\infty} |z_n| = |z| \qquad \text{and} \qquad \lim_{n\to\infty} \arg z_n = \arg z \ (\text{mod } 2\pi)$$

*are necessary and sufficient for the validity of* (1.7).

When the sequence $z_n$ ($n = 1, 2, \ldots$) is such that

$$\lim_{n\to\infty} |z_n| = \infty,$$

we say that the sequence tends to $\infty$ and write simply

$$\lim_{n\to\infty} z_n = \infty.$$

This limit, $\infty$, is taken as a point, the *point at infinity*, of the complex plane. The plane, completed by the single point at infinity, is called the *extended*, or *closed*, plane.† In many questions, the point $z = \infty$ has an equal status with the finite points of the plane (cf. Section 3.13).

## 1.10. The Topology of the Complex Plane

The set of points $z$ belonging to the interior of a disk of radius $r$ with center at the point $z_0 = x_0 + iy_0 \neq \infty$:

$$K_r: \qquad |z - z_0| < r$$

is called a *circular neighborhood* of $z_0$. A circular neighborhood of the point $z = \infty$ will be taken to mean the set of points which lie outside some circle of radius $r$ about the origin: $|z| > r$.

A set of points $\{z\}$ in the extended plane $|z| \leqq \infty$ is said to be *open* if each of its points is the center of some circular neighborhood which belongs entirely to the set.

An open set of points $\{z\}$ in the extended plane $|z| \leqq \infty$ forms a *domain* if it is possible to join any two points in $\{z\}$ by a polygonal path which lies entirely in $\{z\}$. (This condition makes the open set *connected.*)

Any domain containing the point $z_0$ is called a *neighborhood* of $z_0$.

A point $a$ is called a *cluster point* (or sometimes, a *limit point* or *accumulation point*) of a set of complex numbers $\{z\}$ if every circular neighborhood of $a$ contains at least one point $z \neq a$ of $\{z\}$. From this it follows that every neighborhood of a cluster point $a$ must contain infinitely many points of the set.

If a set contains all of its cluster points, the set is said to be *closed.*

The set of points $|z| < \infty$ is open. The extended plane $|z| \leqq \infty$ is both open and closed (Exercise 18). Open sets and closed sets are important particular classes of sets, but an arbitrary set of points is, in general, neither open nor closed.

A closed set which cannot be split into two disjoint closed subsets is called a *continuum.*

A set of points $\{z\}$ is said to be *compact* if any infinite subset of it has a cluster point belonging to $\{z\}$. (A compact set is therefore closed.) The closed plane $|z| \leqq \infty$ is compact.

The set of points in the plane which do not belong to a given set $\{z\}$ forms what is called the *complement* of $\{z\}$. The complement of an open set is closed, and the complement of a closed set is open (Exercise 20).

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† The finite plane can also be extended in other ways. For example, in projective geometry, there is the so-called line at infinity with its infinitely many, infinitely distant points.

Let $G$ be a domain. If $G$ does not contain every point of $|z| \leqq \infty$, then the points of the complement of $G$ fall into two classes:

a) *Boundary points of* $G$. These do not belong to $G$, but are cluster points of $G$. The set of boundary points forms the *boundary* of $G$.

b) *Exterior points of* $G$. These are points which belong neither to $G$ nor to the boundary of $G$. This set can be empty.

If $\Gamma$ is the boundary of the domain $G$, then the union $G \cup \Gamma$ (that is, the set of all points which belong either to $G$ or $\Gamma$) is a closed set (Exercise 21). It is called the *closure* of the domain $G$.

The union of a domain and its boundary is also called a *closed domain*.

## §3. FUNCTIONS OF A COMPLEX VARIABLE

### 1.11. Definition of a Function. Continuity

Functions of a complex variable are defined in the same manner as functions of a real variable:

*If to every value $z$ in a domain $G$ there corresponds a definite complex value $w$, then the mapping $f: z \to w$ is said to be a function defined in the domain $G$.*

The number $w = f(z)$ is called the value of the function at the point $z$.

In what follows we shall consider first only those functions which assume *finite* values in a *finite* domain, that is, a domain belonging to the finite plane $|z| < \infty$.

The real and imaginary parts, $u$ and $v$, of the function $f(z)$ are real functions of the real variables $x$ and $y$ $(z = x + iy)$:

$$u = u(x, y), \qquad v = v(x, y).$$

Conversely, any two such functions always define one complex function $f(z) = u + iv$ of $z = x + iy$.

Continuity is defined in the same way as in the real case:

*A function $w = f(z)$ is continuous at the point $z$ if, to every positive number $\epsilon$, there corresponds a positive number $\rho_\epsilon$, such that*

$$|f(z + \Delta z) - f(z)| < \epsilon \qquad \text{whenever} \qquad |\Delta z| < \rho_\epsilon.$$

Geometrically speaking, the continuity of a function $w = f(z)$ at $z = z_0$ means this: to an arbitrarily small disk $K_w$ centered at $w_0 = f(z_0)$ there corresponds a disk $K_z$ about $z_0$ with the property that $w = f(z)$ lies in $K_w$ whenever $z$ lies in $K_z$.

The limit of a function is defined in the same way as the limit of a sequence in Section 1.9. Everything that was said there about the limit of a sequence of complex numbers applies here as well. Combining the definitions of