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A GUIDE TO FUNCTIONAL ANALYSIS

Steven G. Krantz

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A GUIDE TO FUNCTIONAL ANALYSIS

Steven G. Krantz
Washington University in St. Louis

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To the memory of Stefan Banach.

PREFACE

Functional analysis is a child of the twentieth century. Rapid developments in the theory of differential equations and especially in harmonic analysis (the theory of Fourier series) made it desirable to study entire spaces of functions. These were usually infinite dimensional spaces, which revealed new worlds of harmony and truth. Functional analysis gave analysis a new set of techniques and an entirely new way of looking at things. It created the idea of "soft" analysis (as opposed to "hard" analysis). It often was able to prove in a few lines results that were hard work to verify by classical means.

Functional analysis is abstract mathematics at its best. It requires a good deal of the reader, and particularly of the end user. It is a demanding discipline, but one which yields many fruits.

Most graduate students are required to learn some functional analysis as part of the qualifying exam system. Working analysts have to have some functional analysis under their belts. It is part of our toolkit, just as Galois theory is for the algebraist.

The creators of functional analysis are also legend. Stefan Banach and Stanislaw Ulam, to name just two, were part of the Scottish Cafe team in pre-war Poland, and they helped to set a standard for how mathematics is practiced today. A bit later, John von Neumann played a critical role in establishing the importance of Hilbert space theory, both in mathematics and in physics. Some of the most important and powerful mathematicians today are functional analysts.

The purpose of this book is to introduce the reader with minimal background to the basic scripture of functional analysis. Readers should know some real analysis and some linear algebra. Measure theory rears its ugly head in some of the examples and also in the treatment of spectral theory. The latter is unavoidable and the former allows us to present a rich variety of examples. The nervous reader may safely skip any of the measure theory and still derive a lot from the rest of the book. Apart from this caveat, the book is almost completely self-contained; in a few instances we mention easily accessible references.

A feature that sets this book apart from most other functional analysis texts is that it has a *lot* of examples and a *lot* of applications. This helps to make the material more concrete, and relates it to ideas that the reader has already seen. It also makes the book more accessible to a broader audience.

I thank Don Albers for being a worldly and gentle editor. I thank Jerry Folland for helpful comments at various junctures. And I thank the staff at the MAA for another delightful publishing experience.

St. Louis, Missouri Steven G. Krantz

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CHAPTER 1

FUNDAMENTALS

1.1 WHAT IS FUNCTIONAL ANALYSIS?

The mathematical analysts of the nineteenth century (Cauchy, Riemann, Weierstrass, and others) contented themselves with studying one function at a time. As a sterling instance, the Weierstrass nowhere differentiable function is a world-changing example of the real function theory of "one function at a time." Some of Riemann's examples in Fourier analysis give other instances. This was the world view 150 years ago. To be sure, Cauchy and others considered sequences and series of functions, but the end goal was to consider the single limit function.

A major paradigm shift took place, however, in the early twentieth century. For then people began to consider *spaces* of functions. By this we mean a linear space, equipped with a norm. The process began slowly. At first people considered very specific spaces, such as the square-integrable real functions on the unit circle. Much later, people branched out to more general classes of spaces. An important feature of the spaces under study was that they must be complete. For we want to pass to limits, and completeness guarantees that this process is reliable.

Thus was born the concepts of Hilbert space and Banach space. People like to joke that, in the early 1940s, Hilbert went to one of his colleagues in Göttingen and asked, "What is a Hilbert space?" Perhaps he did. For it was a new idea at the time, and not well established. Banach spaces took even longer to catch on. But indeed they did. Later came topological vector spaces. These all proved to be powerful and flexible tools that provide new insights and new power to the study of classical analysis. They also afford a completely different point of view in the subjects of real and complex analysis. Functional analysis is a lovely instance of how mathematical abstraction enables one to see new things, and see them very clearly.

The purpose of this book is to introduce the reader to the wealth of ideas that is functional analysis. This will not be a thorough grounding, but rather a taste of what the subject is like. We shall make a special effort to provide examples and concrete applications of the abstract ideas, just so that the neophyte can get a concrete grip on the techniques. Certainly we provide references to more advanced and more comprehensive texts.

As readers work through the book, they may find it useful to refer to some of the great classic texts, such as [DUS], [RES], [RUD2], and [YOS].

1.2 Normed Linear Spaces

Let X be a collection of objects equipped with a binary operation $+$ of addition and also with a notion of scalar multiplication. Thus, if $x, y \in X$, then $x + y \in X$. Also, if $x \in X$ and $c \in \mathbb{C}$ then $cx \in X$. (The scalar field can be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. For us it will usually be $\mathbb{C}$, but there will be exceptions. When we want to refer to the scalar field generically, we use the letter k.) We equip X with a norm; thus, if $x \in X$, then $\|x\| \in \mathbb{R}^+ \equiv \{t \in \mathbb{R} : t \geq 0\}$. We demand that

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ if and only if $x = 0$,
3. If $x \in X$ and $c \in \mathbb{C}$ then $\|cx\| = |c| \cdot \|x\|$,
4. If $x, y \in X$, then $\|x + y\| \leq \|x\| + \|y\|$.

We call X a *normed linear space* (or NLS).

Notice that X as described above is naturally equipped with balls. If $x \in X$ and $r > 0$ then

$$B(x, r) = \{t \in X : \|x - t\| < r\}$$

is the (open) *ball with center x and radius r*. We may think of the collection of balls as the subbasis for a topology on X. Concomitantly, we say that a sequence $\{x_j\} \subseteq X$ *converges* to $x \in X$ if $\|x_j - x\| \to 0$ as $j \to \infty$. The sequence $\{x_j\}$ is *Cauchy* if, for any $\epsilon > 0$, there is a J so large that $j, k > J$ implies $\|x_j - x_k\| < \epsilon$.

We use the notation $\overline{B}(x, r) \equiv \{t \in X : \|x - t\| \leq r\}$ to denote the *closed ball* with center x and radius r. It is worth commenting that this closed ball is not necessarily the closure of the open ball (exercise).

DEFINITION. Let X be a normed linear space. We say that X is a *Banach space* if X is complete in the topology induced by the norm. That is to say, if $\{x_j\}$ is a Cauchy sequence in X, then there is a limit element $x \in X$ such that $x_j \to x$ as $j \to \infty$.

EXAMPLE. Let $X = \mathbb{R}^N$ equipped with the usual norm: If $x = (x_1, x_2, \ldots, x_N)$ is a point of $\mathbb{R}^N$, then

$$\|x\| = \left(\sum_{j=1}^{N} x_j^2\right)^{1/2} .$$

We certainly know that this norm satisfies the axioms for a norm. It is a standard fact that $\mathbb{R}^N$, equipped with the topology coming from this norm, is complete. So $\mathbb{R}^N$ is a Banach space.

EXAMPLE. Let

$$X = \{f : f \text{ is a continuous function on the unit interval } [0, 1] \text{ with values in } \mathbb{R}\} .$$

We equip X with the norm, for $f \in X$, given by

$$\|f\| = \max_{t\in[0,1]} |f(t)| .$$

It is straightforward to verify that this norm satisfies the four axioms.

Furthermore, if $\{f_j\}$ is a Cauchy sequence in this norm, then in fact $\{f_j\}$ is *uniformly Cauchy*. It is a standard result from real analysis (see [KRA1]) that such a sequence has a continuous limit function f. Hence our space is complete. And X is therefore a Banach space. We usually denote this space by $C([0, 1])$.

EXAMPLE. Let us consider the space $X = \ell^1$ of sequences $\alpha = \{a_j\}$ of real numbers with the property that $\sum_j |a_j| < \infty$. The norm on this space is

$$\|\alpha\| \equiv \sum_j |a_j| .$$

It is easy to check the four axioms of a norm. Addition is defined componentwise, as is scalar multiplication.

If $\alpha^j = \{a_\ell^j\}_{\ell=1}^\infty$ is a Cauchy sequence of elements of X then let $\epsilon > 0$. Choose $K > 0$ such that, if $j, k > K$ then $\|\alpha^j - \alpha^k\| < \epsilon/5$. It follows for such j, k and any index ℓ that

$$|a_\ell^j - a_\ell^k| \le \|\alpha^j - \alpha^k\| < \frac{\epsilon}{5} .$$

By the completeness of the real numbers, we find for each ℓ that the sequence $\{a_\ell^j\}_{j=1}^\infty$ converges to a real limit a'_ℓ. We claim that the sequence $A \equiv \{a'_\ell\}$ lies in ℓ^1 and is the limit in norm of the original Cauchy sequence $\{\alpha^j\}$.

Choose K as above. Select L so large that $\sum_{m=L}^\infty |\alpha_m^K| < \epsilon/5$. If $n > K$ then

$$\sum_{m=L}^\infty |a_m^n| \le \sum_{m=L}^\infty |a_m^n - a_m^K| + \sum_{m=L}^\infty |a_m^K| < \frac{2\epsilon}{5}. \tag{1}$$

As a result,

$$\sum_{m=1}^\infty |a_m^n - a'_m| \le \sum_{m=1}^{L-1} |a_m^n - a'_m| + \sum_{m=L}^\infty |a_m^n| + \sum_{m=L}^\infty |a'_m| < \frac{\epsilon}{5} + \frac{2\epsilon}{5} + \frac{2\epsilon}{5} = \epsilon .$$

Here we use the fact that $a_m^n \to a'_m$, each m, so the first sum is less than $\epsilon/5$ if n is large enough. That the last sum does not exceed $2\epsilon/5$ follows from (1) by letting $n \to \infty$. Therefore the α^j converge to A as desired.

We see that X is complete, so it is a Banach space. We usually denote this space by ℓ^1.

EXAMPLE. It is a fact, and we shall not provide all the details here, that if $1 \le p < \infty$, then the collection of sequences $\alpha = \{a_j\}$ such that $\sum_j |a_j|^p < \infty$ forms a Banach space. The appropriate norm is

$$\|\alpha\| \equiv \left(\sum_j |a_j|^p \right)^{1/p} .$$

We usually denote this space by ℓ^p.

For $p = \infty$, the appropriate space is that of all *bounded* sequences $\alpha = \{a_j\}$ of real numbers. The right norm is

$$\|\alpha\| = \sup_j |a_j| .$$

We denote this space by ℓ^∞. See [RUD2] for a thorough treatment of these spaces.

As previously indicated, the balls $B(x, r)$ in a normed linear space X may be taken to be a subbasis for the topology on X.

PROPOSITION 1.1. *The topology on a normed linear space is Hausdorff.*

Proof. Let $x, y \in X$ be distinct elements. Let $\|x - y\| = \delta > 0$. Then, by the triangle inequality, the balls $B(x, \delta/3)$ and $B(y, \delta/3)$ are disjoint. Hence the space is Hausdorff. □