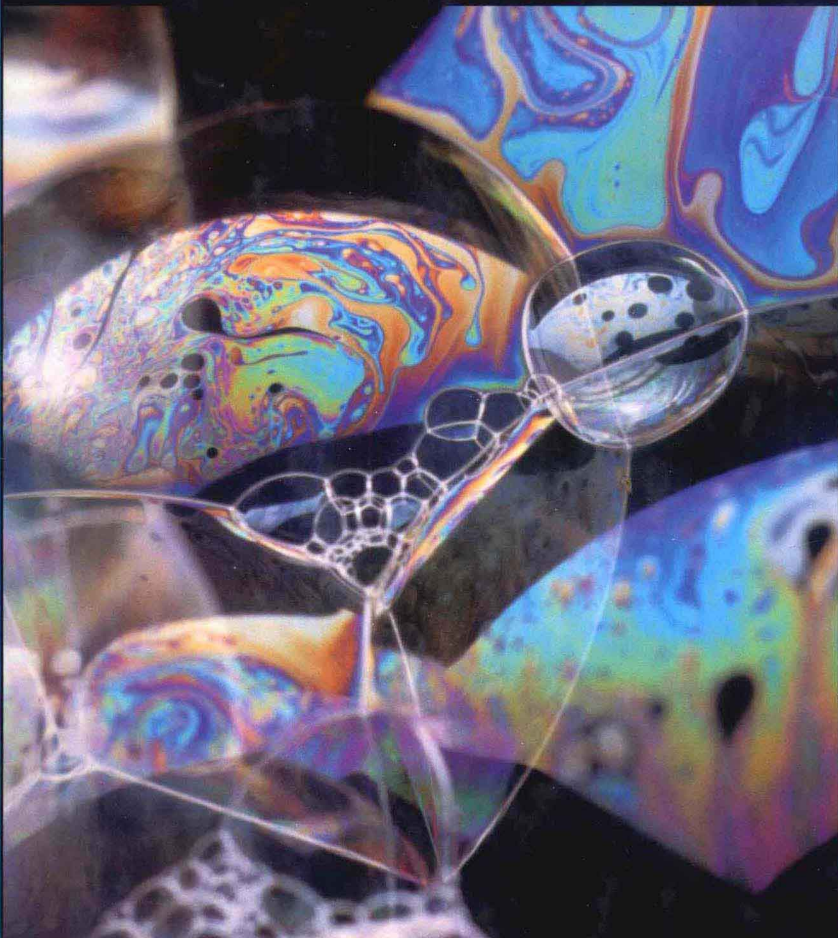
A First Course in Differential Equations

with Modeling Applications

SIXTH EDITION



DENNIS G. ZILL

A FIRST COURSE IN DIFFERENTIAL EQUATIONS
WITH MODELING APPLICATIONS

Sixth Edition

Dennis G. Zill
Loyola Marymount University

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PREFACE

For the Sixth Edition of *A First Course in Differential Equations with Modeling Applications,* the revisions reflect a dual purpose: to ensure that the information is current and relevant to students and yet maintain the rudimental foundations on which the previous editions were constructed. Written with the student in mind, this new book retains the basic level, pedagogical aids, and straight-forward style of presentation of previous editions.

In differential equations, as in many other mathematics courses, instructors are beginning to question certain aspects of traditional teaching methods. This healthy introspection is important in making the subject matter not only more interesting for the students but also more relevant to the world in which they live. Changes in both the content and the style of *A First Course in Differential Equations with Modeling Applications, Sixth Edition*, including the new subtitle, reflect changes the author has observed in the overall approach to teaching differential equations.

Summary of Major Changes

■ *Greater emphasis on differential equations as mathematical models.* The notion of a mathematical model is now interwoven throughout the text, and the construction and pitfalls of such models are discussed.

■ *Five new Modeling Applications.* These applications, illustrated in full color, have been contributed by selected experts in each field. Presented at appropriate points throughout the text, the applications cover timely areas of study, ranging from AZT and survivability with AIDS to the effects on the ecological balance of the reintroduction of the gray wolf to Yellowstone National Park.

■ *Greater emphasis on nonlinear differential equations and on linear and nonlinear systems of differential equations.* Three chapters contain new sections—Sections 3.3, 4.9, 5.2, and 5.3.

■ *Greater emphasis on boundary-value problems for ordinary differential equations.* New to this edition, eigenvalues and eigenfunctions are introduced in Chapter 5.

■ *Greater utilization of technology throughout the text.* Graphics calculators, graphing software, computer algebra systems, and ODE solvers are utilized wherever appropriate in applications and examples, as well as in the exercise sets.

■ *Increase in the number of conceptual problems in exercises.* In many sections, new Discussion Problems have been added. Instead of being asked to solve a differential equation, the student is asked to think about what a differential equation says. In order to encourage students to think, to draw conclusions, and to explore possibilities, the answers to these questions have been intentionally omitted from the end of the book. Some of these problems can serve as individual or group assignments, at the option of the instructor.

Changes in the Sixth Edition by Chapter

Chapter 1 has been expanded to include an introduction to the notions of an initial-value problem and ODE solvers in Section 1.2. The discussion of differential equations as mathematical models, in Section 1.3, has been rewritten for ease of student comprehension.

Chapter 2 now combines discussion of homogeneous first-order equations with discussion of the Bernoulli equation in Section 2.4, *Solutions by Substitution*. Material on the Ricatti and Clairaut equations is now in exercise sets.

Chapter 3 includes a new Section 3.3, *Systems of Linear and Nonlinear Equations*, which shows systems of first-order differential equations as mathematical models. Orthogonal trajectories are now covered in exercises.

Chapter 4 introduces the concept of a linear differential operator in Section 4.1 for the purpose of expediting the proofs of some important theorems. The slightly different manner in which the two equations defining "variable parameters" are presented in Section 4.6 is due to a student, J. Thoo.* The Cauchy-Euler equation is now covered in Section 4.7. Solving systems of linear differential equations with constant coefficients has been moved forward to Section 4.8. A new Section 4.9, *Nonlinear Equations*, begins with a qualitative discussion of the differences between linear and nonlinear equations.

Chapter 5 includes two sections new to this edition. Section 5.2, *Linear Equations: Boundary-Value Problems*, introduces the concepts of eigenvalues and eigenfunctions. Section 5.3, *Nonlinear Equations*, describes modeling with nonlinear higher-order differential equations.

Chapter 6 is now devoted solely to series solutions of linear differential equations.

Chapter 7 now contains, in Section 7.7, the application of the Laplace transform to systems of linear differential equations with constant coefficients. An alternative form of the second translation theorem has been added to Section 7.3.

Chapter 8 covers only the theory and solution of systems of linear first-order differential equations, as material on matrices has been moved to Appendix II. This placement allows the instructor to decide whether to designate the material for a reading assignment or insert it into classroom discussion.

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* J. Thoo, "Timing Is Everything," *The College Mathematical Journal*, Vol. 23, No. 4, September 1992.

Chapter 9 has been completely rewritten. Error analysis for the various numerical techniques is covered in the appropriate section devoted to each method.

Supplements Available

For Students

Student Solutions Manual (Warren S. Wright) provides a solution to every third problem in each exercise set, with the exception of the Discussion Problems.

For Instructors

Complete Solutions Manual (Warren S. Wright) provides worked-out solutions to all the problems in the text.

Lab Experiments for Differential Equations (Dennis G. Zill/Warren S. Wright) contains an assortment of computer lab experiments with differential equations.

Software

ODE Solver: Numerical Procedures for Ordinary Differential Equations (Thomas Kiffe/William Rundel), for IBM and compatibles and for Macintosh, is a software package that presents tabular and graphical representations of the output for the various numerical methods. No programming is required.

Grapher (Steve Scarborough), for the Macintosh, is a versatile collection of graphing utilities which can plot graphs of rectangular equations, parametric equations, polar equations, interpolating polynomials, series, and direction fields. The program provides an ODE solver for first-order differential equations and systems of two first-order differential equations.

Computer Programs for BASIC, FORTRAN, and Pascal (C. J. Knickerbocker), for IBM and compatibles and for Macintosh, contains listings of computer programs for many of the numerical methods in the text.

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who were generous enough to take time out of their busy schedules to supply the interesting new essays on modeling applications.

Finally, a personal note. Those with sharp eyes may have noticed that the familiar PWS lion logo is absent from the spine of the book. While I look forward to working with the personnel at Brooks/Cole, a different branch of the parent company ITP, I can't help thinking about the many good people I have been fortunate to meet, work with, and, yes, fight with over the last twenty years at PWS. So to all in production, marketing, and editorial—especially to Barbara Lovenvirth, my *de facto* editor—I wish you godspeed and good luck. Thank you for another—the final—job well done.

Dennis G. Zill
Los Angeles

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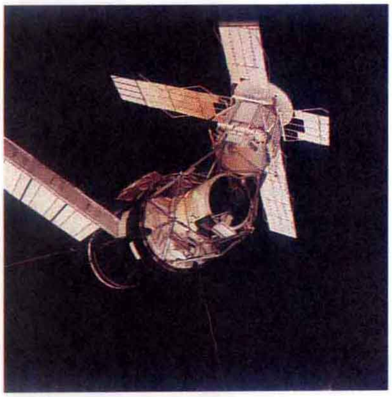

CHAPTER

1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

INTRODUCTION

The words *differential* and *equations* certainly suggest solving some kind of equation that contains derivatives. Just as students in a course in algebra and trigonometry spend a good amount of time solving equations such as $x^2 + 5x + 4 = 0$ for the *variable x*, in this course we wish to solve differential equations such as $y'' + 2y' + y = 0$ for the unknown *function y*. But before you start solving anything, you must learn some of the basic definitions and terminology of the subject.

1.1 DEFINITIONS AND TERMINOLOGY

■ *Ordinary and partial differential equations* ■ *Order of an equation*
■ *Linear and nonlinear equations* ■ *Solution of a differential equation*
■ *Explicit and implicit solutions* ■ *Trivial solution* ■ *Family of solutions*
■ *Particular solution* ■ *General solution* ■ *Systems of differential equations*

A Differential Equation In calculus you learned that the derivative dy/dx of a function $y = \phi(x)$ is itself another function of x found by some appropriate rule. For example, if $y = e^{x^2}$ then $dy/dx = 2xe^{x^2}$. Replacing e^{x^2} by the symbol y then gives

$$\frac{dy}{dx} = 2xy. \quad \textbf{(1)}$$

The problem we face in this course is not "Given a function $y = \phi(x)$, find its derivative." Rather, our problem is "If we are given a differential equation such as (1), is there some way or method by which we can find the unknown function $y = \phi(x)$?"

DEFINITION 1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

Differential equations are classified by **type, order,** and **linearity.**

Classification by Type If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be an **ordinary differential equation (ODE).** For example,

$$\frac{dy}{dx} + 10y = e^x \quad \text{and} \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

are ordinary differential equations. An equation involving the partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE).** For example,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

are partial differential equations.

Classification by Order The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\underset{\text{second-order}\ \downarrow}{\frac{d^2y}{dx^2}} + 5\underset{\downarrow\ \text{first-order}}{\left(\frac{dy}{dx}\right)^3} - 4y = e^x$$

is a second-order ordinary differential equation. Since the differential equation $(y - x)\,dx + 4x\,dy = 0$ can be put into the form

$$4x\frac{dy}{dx} + y = x$$

by dividing by the differential dx, it is an example of a first-order ordinary differential equation.

A general nth-order ordinary differential equation is often represented by the symbolism

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{2}$$

In general discussions in this text we shall assume that an nth-order differential equation (2) can be solved for the highest derivative $y^{(n)}$, that is,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

Classification as Linear or Nonlinear A differential equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ is said to be **linear** when f is a linear function of $y, y', \ldots, y^{(n-1)}$. This means that an equation is linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

From this last equation we see the two characteristic properties of linear differential equations:

(*i*) The dependent variable y and all its derivatives are of the first degree; that is, the power of each term involving y is 1.
(*ii*) Each coefficient depends only on the independent variable x.

Functions of y such as $\sin y$ or functions of the derivatives of y such as $e^{y'}$ cannot appear in a linear equation. A differential equation that is not linear is said to be **nonlinear.** The equations

$$(y - x)\,dx + 4x\,dy = 0, \qquad y'' - 2y' + y = 0, \qquad x^3\frac{d^3y}{dx^3} - 4x\frac{dy}{dx} + 6y = e^x$$

are linear first-, second-, and third-order ordinary differential equations, respectively. On the other hand,

$$\underset{\substack{\text{coefficient}\\ \text{depends on } y\\ \downarrow}}{(1 + y)y' + 2y = e^x}, \qquad \underset{\substack{\text{nonlinear}\\ \text{function of } y\\ \downarrow}}{\frac{d^2y}{dx^2} + \sin y = 0}, \qquad \underset{\substack{\text{power not 1}\\ \downarrow}}{\frac{d^4y}{dx^4} + y^2 = 0}$$

are nonlinear first-, second-, and fourth-order ordinary differential equations, respectively.

Solutions As mentioned before, one of our goals in this course is to solve, or find **solutions** of, differential equations.

DEFINITION 1.2 **Solution of a Differential Equation**

Any function ϕ defined on some interval I, which when substituted into a differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an ordinary differential equation (2) is a function ϕ that possesses at least n derivatives and

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \quad \text{for all } x \text{ in } I.$$

We say that $y = \phi(x)$ *satisfies* the differential equation. The interval I could be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on. For our purposes, we shall also assume that a solution ϕ is a real-valued function.

EXAMPLE 1 **Verification of a Solution**

Verify that $y = x^4/16$ is a solution of the nonlinear equation

$$\frac{dy}{dx} = xy^{1/2}$$

on the interval $(-\infty, \infty)$.

SOLUTION One way of verifying that the given function is a solution is to write the differential equation as $dy/dx - xy^{1/2} = 0$ and then see, after substituting, whether the sum $dy/dx - xy^{1/2}$ is zero for every x in the interval. Using

$$\frac{dy}{dx} = 4\frac{x^3}{16} = \frac{x^3}{4} \qquad \text{and} \qquad y^{1/2} = \left(\frac{x^4}{16}\right)^{1/2} = \frac{x^2}{4},$$

we see that $$\frac{dy}{dx} - xy^{1/2} = \frac{x^3}{4} - x\left(\frac{x^4}{16}\right)^{1/2} = \frac{x^3}{4} - \frac{x^3}{4} = 0$$

for every real number. Note that $y^{1/2} = x^2/4$ is, by definition, the nonnegative square root of $x^4/16$. ■

EXAMPLE 2 **Verification of a Solution**

The function $y = xe^x$ is a solution of the linear equation

$$y'' - 2y' + y = 0$$

on the interval $(-\infty, \infty)$. To show this, we compute

$$y' = xe^x + e^x \qquad \text{and} \qquad y'' = xe^x + 2e^x.$$

Observe that

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$$

for every real number. ■

Not every differential equation that we write necessarily has a solution. You are encouraged to think about this and then solve Problem 51 in Exercises 1.1.

Explicit and Implicit Solutions You should be familiar with the terms *explicit* and *implicit functions* from your study of calculus. Because some methods for solving differential equations lead directly to these two forms, solutions of differential equations can be further distinguished as either explicit solutions or implicit solutions. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate. We have already seen in our initial discussion that $y = e^{x^2}$ is an explicit solution of $dy/dx = 2xy$. In Examples 1 and 2, $y = x^4/16$ and $y = xe^x$ are explicit solutions of $dy/dx = xy^{1/2}$ and $y'' - 2y' + y = 0$, respectively. Note that in Examples 1 and 2 each differential equation possesses the constant solution $y = 0$, $-\infty < x < \infty$. An explicit solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution.** A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (2) on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I. In other words, $G(x, y) = 0$ defines the function ϕ implicitly.

EXAMPLE 3 **Verification of an Implicit Solution**

The relation $x^2 + y^2 - 4 = 0$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{3}$$

on the interval $-2 < x < 2$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 - \frac{d}{dx}4 = \frac{d}{dx}0 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0.$$

Solving the last equation for the symbol dy/dx gives (3). In addition, you should verify that the functions $y_1 = \sqrt{4 - x^2}$ and $y_2 = -\sqrt{4 - x^2}$ satisfy the relation (in other words, $x^2 + y_1^2 - 4 = 0$ and $x^2 + y_2^2 - 4 = 0$) and are solutions of the differential equation on $-2 < x < 2$. ■

Any relation of the form $x^2 + y^2 - c = 0$ *formally* satisfies (3) for any constant c. However, it is understood that the relation should always make sense in the real number system; thus, for example, we cannot say that $x^2 + y^2 + 4 = 0$ is an implicit solution of the equation. (Why not?)

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

More Terminology The study of differential equations is similar to that of integral calculus. A solution is sometimes referred to as an **integral** of the equation, and its graph is called an **integral curve** or a **solution curve.** When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, when solving a first-order differential equation $F(x, y, y') = 0$, we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions, called a **one-parameter family of solutions.** When solving an nth-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This simply means that a single differential equation can possess an infinite number of solutions corresponding to the unlimited choices for the parameter(s). A solution of a differential equation that is free of arbitrary parameters is called a **particular solution.** For example, by direct substitution we can show that any function in the one-parameter family $y = ce^{x^2}$, where c is an arbitrary constant, also satisfies equation (1). The original solution $y = e^{x^2}$ corresponds to $c = 1$ and thus is a particular solution of the equation. Figure 1.1 shows some of the integral curves in this family. The trivial solution $y = 0$, corresponding to $c = 0$, is also a particular solution of (1).

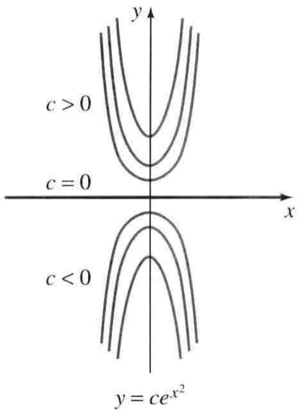

FIGURE 1.1

EXAMPLE 4 Particular Solutions

The function $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the linear second-order equation $y'' - y = 0$. Some particular solutions are $y = 0$ ($c_1 = c_2 = 0$), $y = e^x$ ($c_1 = 1, c_2 = 0$), and $y = 5e^x - 2e^{-x}$ ($c_1 = 5, c_2 = -2$). ■

In all the preceding examples we have used x and y to denote the independent and dependent variables, respectively. But in practice these two variables are represented by many different symbols. For example, we could denote the independent variable by t and the dependent variable by x.

EXAMPLE 5 Using Different Symbols

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants, are solutions of the differential equation

$$x'' + 16x = 0.$$

For $x = c_1 \cos 4t$ the first two derivatives with respect to t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting x'' and x then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0.$$

In like manner, for $x = c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_2 \sin 4t + 16(c_2 \sin 4t) = 0.$$

Finally, it is easy to verify that the linear combination of solutions, or two-parameter family, $x = c_1 \cos 4t + c_2 \sin 4t$ is a solution of the given equation. ■