CHAOS

and

NONLINEAR DYNAMICS

An Introduction
for Scientists and Engineers

Robert C. Hilborn
Department of Physics
Amherst College

New York Oxford
Oxford University Press
1994

Oxford University Press

Oxford New York Toronto
Delhi Bombay Calcutta Madras Karachi
Kuala Lumpur Singapore Hong Kong Tokyo
Nairobi Dar es Salaam Cape Town
Melbourne Auckland Madrid

and associated companies in
Berlin Ibadan

Published by Oxford University Press, Inc.
200 Madison Avenue, New York, New York 10016

Library of Congress Cataloging-in-Publication Data
Hilborn, Robert C.
Chaos and nonlinear dynamics :
an introduction for scientists and engineers /
by Robert C. Hilborn.
p. cm. Includes bibliographical references and index.
ISBN 0-19-505760-0
1. Chaotic behavior in systems. 2. Quantum chaos.
3. Dynamics. 4. Nonlinear theories. I. Title.
Q172.5.C45H55 1993 003.7—dc20 92-40059

9 8 7 6 5 4 3 2 1

Printed in the United States of America
on acid-free paper

Preface

In the last 10 years, the threads of chaos and nonlinear dynamics have spread across the scientific disciplines like a spider's intricate web. Chaos and nonlinear dynamics have provided new theoretical and conceptual tools that allow us to capture, understand, and link together the surprisingly complex behaviors of simple systems—the type of behavior called *chaos*—in essentially every field of contemporary science.

The universality of chaos is both intriguing and puzzling. What is it about the behavior of a convecting fluid, for example, that makes its transition from simple, regular behavior, to complex, chaotic behavior both qualitatively and quantititatively identical to the way an oscillating electrical circuit makes the same kind of transition? The theory of chaos clearly needs to be based on the fundamental laws of physics, chemistry, and biology. In a sense, however, the theory needs to transcend those laws to explain the *universality* of chaos. New ideas, new language, and new ways of reasoning about complex behavior are needed. These ideas, this language, and those modes of reasoning are what chaos theory in particular and nonlinear dynamics in general provide.

The study of nonlinear dynamics is by no means complete, but the field has now matured to the point that it makes sense to bring together in one book the essential elements. The foundations of the theory seem to be firmly in place, and the outlines of the final structure to be erected upon this foundation can now be discerned. This book provides an introduction to chaos and nonlinear dynamics for scientists and engineers who have little or no previous experience with the field. I have assumed no background other than some familiarity with introductory college-level physics and with calculus through elementary differential equations. After completing this book, the reader should be ready to grapple with the current literature in chaos.

Most of us who have been actively engaged in the study of chaos have learned the "tricks of the trade" in a rather piecemeal fashion: some abstract concepts from nonlinear mathematics, some stability theory from the engineers and most of our new ideas from the research literature. When I wanted to introduce my students to chaos, no book provided all of the needed background. Thus, I have written this book both to bring together the essentials of the field and to provide the background for those scientists, engineers, and students who want to discover what the excitement in chaos is all about.

Historically, the study of chaos is strongly rooted in the mathematical study of nonlinear dynamics, going back, at least, to the pioneering work of Henri Poincaré, the noted French mathematician (1854–1912). This heritage has bequeathed to chaos jargon that is (some would say excessively) mathematical in nature. In this book I have approached chaos from the point of view of a scientist who wants to describe and understand the complex behavior of real systems. Thus, I have chosen to introduce the concepts of chaos as descriptors of the behavior of actual systems rather than as abstract mathematical ideas. As these descriptions are refined, we recapture, I hope with more physical intuition and insight, the mathematical basis of those concepts.

Nonlinear dynamics and chaos, like most of contemporary physical science and engineering, is intimately tied to mathematics. To apply the concepts of nonlinear dynamics to her or his field, a scientist, engineer, economist, social scientist or physician must come to grips with at least some of the formalism and quantitative formulations of nonlinear dynamics. The concepts without the quantification are fruitless; likewise, quantification without the guide of concepts is blind number shuffling. Both aspects are necessary. Anyone wishing to make use of nonlinear dynamics must be willing to make an investment of time and energy to master some of the formalism. I have designed this book to provide an introduction to what I believe are the key parts of that formalism.

I believe that many scientists and their students are most comfortable with the traditional differential equations approach to the study of dynamics, and the standard undergraduate training in science and introductory calculus provides ample practice in solving (in closed form) and interpreting linear differential equations. Thus, I have begun the discussion from that point of view. However, the analytic results, which dominate the traditional approach to the study of linear systems, quickly become useless when dealing with nonlinear systems. Thereore, I gradually introduce the methods used to describe and think about these nonlinear systems. These methods generally require less formal mathematical manipulation than do the traditional analytic methods, but they force us to do more thinking.

I mentioned above the background assumed of my readers: familiarity with college-level introductory physics and an acquaintance with calculus through elementary differential equations. Certain portions of the book do indeed stretch or even exceed those prerequisites. I have marked the title of those sections with an asterisk. The reader who feels that those sections exceed her or his mathematical fortitude or technical background can pick up the key ideas by reading the introductory paragraphs of those sections. There I have tried to give an elementary statement of what the more elaborate mathematics does with more rigor and generality. These more advanced sections are self-contained so that the reader who skips the mathematical treatment can still follow the flow of argument in the remainder of the book.

The results of research in science education agree with the reflections of experienced teachers: students, be they young people or experienced scientists venturing into new fields, must become actively engaged with the material they are attempting to learn. The readers of this book are no exception. To provide for that engagement, I have included some exercises in most sections that should be useful both for classroom instruction and for the reader who wants to tackle chaos directly. At the end of each section are references for further reading and some computer exercises. By working through these exercises, both with paper and pencil and with the computer, and by reading about the applications of nonlinear dynamics to various fields of science, the reader can begin to become engaged with the field.

The first two chapters introduce the key concepts, jargon, and important questions raised by chaos by looking at three simple systems that exhibit chaotic behavior: a simple electrical circuit, a model of biological population dynamics, and a set of differential equations modeling fluid convection. These examples were chosen because they show nearly the full spectrum of chaotic behavior, but they are sufficiently simple so that the basic science behind each of them can be easily understood. The surprise is that these simple systems exhibit exceedingly complex behavior. Simplicity of structure does not guarantee simplicity of behavior. By comparing the chaotic behavior of these systems, we recognize both qualitative and quantitative similarities. These similarities are quantified by the numbers first "discovered" by Feigenbaum, and I discuss carefully how well these numbers describe actual systems.

The remainder of the book then tackles the problem of building a theory of chaotic behavior. The key conceptual tool is the description of a system's behavior in state space, a geometrical construction similar to the phase space description familiar from classical mechanics and statistical mechanics. Poincaré sections further simplify the description of the dynamical behavior and allow conceptually simple, but analytically powerful means of classifying the types of dynamical behavior. We make contact with the mathematical scheme of iterated maps from these constructions. I spend some time developing the theory of these maps since they have been important in the historical development of the theory of chaos, and they elegantly and simply illustrate many of the fundamental types of chaotic behavior. Geometrical notions lead to a classification of the so-called routes to chaos and an understanding of how the system's behavior evolves as its environment, as described by suitable parameters, changes. I discuss in some detail the quasi-periodic, intermittency, and crisis routes to chaos.

Although the initial discussion focuses on dissipative systems—systems that "run down" unless provided with an external source of energy—since these systems are better models of most of the real world, I have included an introduction to the chaotic behavior of Hamiltonian systems—systems whose energy is conserved. Chaotic Hamiltonian systems are important theoretically and are crucial for an understanding of "quantum chaos," a subject I treat in the last chapter.

I then turn to the problem of describing chaos quantitatively. Introducing the notions of Lyapunov exponents, fractal dimensions, and various kinds of correlation exponents, I show how each of these quantifiers can be determined, at least in principle, from an analysis of a time series of sampled values of some dynamical variable of the system. All of these quantifiers are related, and some obey universal scaling laws, which tell how they vary as the system becomes more chaotic. I present a case history of the computation of the widely used correlation dimension to illustrate some of the pitfalls of quantifying chaos.

Recent research has emphasized that these simple descriptors are really only average quantities and that actual chaotic systems show a distribution or spectrum of values for each of these. For the case of fractal dimensions, these chaotic systems are described by what are called "multifractals." There is some indication that these distributions are themselves universal. An elegant thermodynamic formulation of chaotic behavior then leads naturally to a description of these distributions.

The penultimate chapter looks at systems with sufficient spatial extent to exhibit interesting patterns. In recent years there has been dramatic progress in understanding the physics of pattern formation and the related behavior called spatiotemporal chaos. The connection between pattern formation and chaotic dynamical behavior is outlined. I believe that this chapter is the only treatment of pattern formation at this level of presentation. Pattern formation is certainly one of the most important aspects of nonlinear science. Newcomers to the field should certainly be made aware of the fundamental issues.

Unfortunately, fluid dynamics and transport phenomena have all but disappeared from the standard undergraduate physics curriculum in the United States. Thus, I have included a brief introduction to these subjects in Chapter 11. The treatment is hardly exhaustive and provides just enough background so the reader can appreciate the fundamental issues in pattern formation and dissipative structures.

In the final chapter, I turn to a discussion of the problem of chaos and quantum mechanics. Quantum systems show peculiar behavior when their classical (non-quantum) counterparts display chaos. However, there are good reasons to believe that "pure" chaos cannot occur in quantum systems. What then is the connection between chaos and this peculiar behavior? Along with speculations about quantum chaos, I discuss the relationship between chaotic behavior and the more general notions of "complexity" as well as the import of chaos both for the technical development of science and its philosophical implications.

I have included several appendices, which gather together for convenient reference some of the technical background needed for understanding chaos and nonlinear dynamics. Fourier analysis and bifurcation theory are crucial in many aspects of nonlinear dynamics. Appendices A and B provide brief introductions to these subjects. In Appendix C, I present the details of the development of the now legendary Lorenz Model, starting from the fundamental equations of fluid flow and thermal energy diffusion. Appendix D gives an introduction to the scientific research literature on chaos. Appendix E contains the listings of some simple computer programs to illustrate the dynamics of the logistic map model. These programs can serve as useful examples to guide you in creating your own computer programs.

Let me describe the structure of the book. Each chapter is broken down into sections. Equations and exercises are numbered according to the section in which they occur. For example, Eq. (3.4-1) is the first equation in Chapter 3, Section 4. Section numbers are indicated on the top of even-numbered pages. Figures are numbered consecutively in each chapter. References to books are given with the author's name and the year of publication in square brackets, for example, [Gleick, 1987]. New concepts and terms of technical jargon are set in bold italics where they first appear. Double-line boxes set off important results and definitions. Technical comments and asides are indented.

References to papers and articles are cited by giving the initial letters of the family names of the first three authors, or the first three letters of a single author's family name, and the last two digits of the year of publication. For example, (HIL88) refers to a paper by Hilborn published in 1988. I trust that this citation method gives more information than just a numerical reference to a citation at the end of the chapter, without burdening the text with footnotes containing the full citation. The complete reference citations are given at the end of the first chapter in which that reference appears. All references cited are gathered together in an alphabetical listing at the end of the book.

Throughout the book I have relied on simple (usually mathematical) models to illustrate new concepts. Some readers may find this approach frustrating. They may want more discussion of actual applications. I have described applications in several sections, but much of the narrative rides on these simple models. There are two reasons for this. First, these simple models provide us with well-controlled and well-defined "laboratories" for trying out and exploring the many unfamiliar and, in some cases, new concepts of nonlinear dynamics. We can ignore for the moment all the complexities and approximations associated with systems in the "real world." Second, many of the features associated with complex behavior in nonlinear dynamics are in fact independent of the details of the system being investigated. Hence, we can use what we learn about the dynamics of simple mathematical models to help us at least categorize and describe and often understand the behavior of lasers, heart beats, and convecting fluids. The use of these simple models is part and parcel of the methodology of nonlinear dynamics.

This book did not appear spontaneously out of (dare I say) the void of chaos. My thinking and writing about chaos and nonlinear dynamics have been influenced by many books. The text by Schuster—see the citation for [Schuster, 1988] at the end of Chapter 1—although it appeared in print after I had begun writing this book, shares much of the same conceptual strategy, but is written at the graduate level in physics.

I want to say a bit about what this book is not (reviewers take note). The history of nonlinear dynamics and chaos is not explored here except by a few references to early developments. These references (and all others in this book) are the ones I and my students have found useful. They do not necessarily point to the original creators or discoverers. Sorting out and understanding this history will take the skills of a disciplined historian of science. Also, this book is not a scholarly monograph on nonlinear dynamics and chaos. I have not proved many (or even most) of the theorems, nor I have explored all of the ramifications of the results stated here. My purpose is frankly (and, I believe, laudably) pedagogical. What I have tried to do is to provide an overview and a series of explanations of what the science of nonlinear dynamics and chaos is all about and what it does and what it (yet) cannot do.

Amherst, Mass.
October, 1992 R.C.H.

Acknowledgments

My writing of this book owes so much to so many that I hesitate to name a few for fear of leaving out many others. But particular credit must go to the undergraduate students at Oberlin College and at Amherst College, who have collectively over the years stimulated, challenged, and enriched my thinking and understanding of nonlinear dynamics. Among these students, special credit must go to Kenny Blum, Sharon Ross, Simon Kaplan, Clay Johnston, Ryan Wallach, Richard Rubenstein, Allen Rogel, and John Watson, who have worked with me on many different chaos projects. Jennifer Eden gave a careful and critical reading of the first five chapters.

Discussions and conversations with colleagues are too numerous to list in detail, but I must acknowledge particular debts to Jerry Gollub, Julio Ottino, Neal Abraham, John Delos, Bob Romer, Kannan Jagannathan, and Bob Krotkov. H. Eugene Stanley gave the entire manuscript a careful reading. His many suggestions for improvements are gratefully acknowledged. My agent Patricia Van der Leun and Oxford's Physical Sciences editor Jeffrey Robbins provided much needed encouragement. Copy editor Steve Bedney helped impose some order on my initial manuscript. Finally, without the patience of family and friends, this book would have been considerably more chaotic in both its writing and its final form.

Contents

I

THE PHENOMENOLOGY OF CHAOS

1

Three Chaotic Systems

1.1 Prelude

Chaos is the term used to describe the *apparently* complex behavior of what we consider to be simple, well-behaved systems. Chaotic behavior, when looked at casually, looks erratic and almost random—almost like the behavior of a system strongly influenced by outside, random "noise" or the complicated behavior of a system with many, many degrees of freedom, each "doing its own thing."

The type of behavior, however, that in the last 10 years has come to be called *chaotic* arises in very simple systems (those with only a few active degrees of freedom), which are almost free of noise. In fact, these systems are essentially deterministic; that is, precise knowledge of the conditions of the system at one time allow us, at least in principle, to predict exactly the future behavior of that system. The problem of understanding chaos is to reconcile these apparently conflicting notions: randomness and determinism.

The key element in this understanding is the notion of ***nonlinearity***. We can develop an intuitive idea of nonlinearity by characterizing the behavior of a system in terms of stimulus and response: If we give the system a "kick" and observe a certain response to that kick, then we can ask what happens if we kick the system twice as hard. If the response is twice as large, then the system's behavior is said to be linear (at least for the range of kick sizes we have used). If the response is not twice as large (it might be larger or smaller), then we say the system's behavior is nonlinear. In an acoustic system such as a record, tape, or compact disc player, nonlinearity manifests itself as a distortion in the sound being reproduced. In the next section, we will develop a more formal definition of nonlinearity. The study of nonlinear behavior is called ***nonlinear dynamics***.

Why have scientists, engineers, and mathematicians become intrigued by chaos? The answer to that question has two parts: (1) The study of chaos has provided new conceptual and theoretical tools enabling us to categorize and understand complex behavior that had confounded previous theories; (2) chaotic behavior seems to be universal—it shows up in mechanical oscillators, electrical circuits, lasers, nonlinear optical systems, chemical reactions, nerve cells,