FUNCTIONAL ANALYSIS

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Preface

Functional analysis, partly because of its many applications, has become a very popular mathematical discipline. My own lectures on the subject have been attended by applied mathematicians, probabilists, classical and numerical analysts, and even an algebraic topologist. This book grew out of my attempts to present the material in a way that was interesting and understandable to people with such diverse backgrounds and professional goals. I have aimed at an audience of professional mathematicians who want to learn some functional analysis, and second-year graduate students who are taking a course in the subject. The only background material needed is what is usually covered in a one-year graduate level course in analysis, and an acquaintance with linear algebra. The book is designed to enable the reader to get actively involved in the development of the mathematics. This can be done by working the starred problems that appear at the end of nearly every section. I often refer to these exercises during subsequent discussions and proofs. Solutions to those starred problems appearing in the introductory chapters (Chapters 1–4) can be found in Appendix A.

The introductory chapters contain the basic facts from the theory of normed spaces. Here the mathematics is developed through the discussion of a sequence of gradually more sophisticated questions. We begin with the most naïve approach of all. In Sections 2 and 3 of Chapter 1, we study finite dimensional normed spaces and ask which of our results are true in the infinite dimensional case. Of course this approach does not lead very far, but it does guide us to some useful facts. In order to carry our discussion of normed spaces further, we take a hint from the history of the subject and learn something about integral equations. This is done in Chapter 2, where we also discuss the Riesz theory of compact operators. A key result in that theory is the theorem associating to each compact operator a pair of complementary subspaces. At this point we inquire into the connection between such pairs of subspaces and continuous projection operators and ask if every closed linear subspace of a normed space has a complement. The discussion of these questions, which occupies some of Chapter 2 and most of Chapter 3, leads us to some very deep theorems. It also exhibits the importance of continuous, linear functionals.

Chapter 4 deals with the weak topology of a normed space, and it also contains an introduction to the theory of locally convex spaces. The latter material is used to prove that a Banach space whose unit ball is compact for the weak topology is reflexive. It is used again in Chapter 5 and in Chapter 7.

One advantage of the approach sketched above is that the important theorems stand out as those which must be appealed to again and again to answer our questions. It should also be mentioned that several of the questions discussed in the text have been the subject of a great many research papers. I have made no attempt to give an account of all of this work. However, the closely related problems of characterizing reflexive Banach spaces and characterizing those Banach spaces that are dual spaces are discussed further in Appendix B.

The last three chapters of the book are independent of one another, and each deals with a special topic. In Chapter 5, John Kelley's elegant proof of the Krien–Milman theorem is presented. That theorem is used to settle the question, Is every Banach space the dual of some other Banach space? (See Chapter 5 for a more precise statement.) Chapter 5 also contains the theorem of Eberlein. I have presented Eberlein's original proof of his famous theorem because I feel that it gives insights into this result not found in more modern proofs. It does not yield the most general result known; but, I feel, the gain in insight is well worth the slight loss in generality. Chapter 6 contains a sample of the interesting, and sometimes surprising, ways that functional analysis enters into discussions of classical analysis. This material can be read immediately after Section 1 of Chapter 4. Distributions are discussed in the last chapter. The Fourier transform is treated early (Section 3) because it requires less machinery than some of the other topics. However, Fourier transforms are not used in any subsequent section. Applications of the theory of distributions to harmonic analysis (Section 3) and to partial differential equations (Section 5e) are also discussed. Readers who are interested only in distributions can read Chapter 7 immediately. They will however occasionally have to go back to Chapter 4 and read some background material.

I would like to take this opportunity to thank Andrea Blum for writing Appendix A. She patiently solved each of these problems and proved that they really can be done. I discussed my ideas for a book with R. P. Boas of Northwestern University and John S. Lomont of the University of Arizona. They each made valuable suggestions, and it is my pleasure now to thank them both. I would also like to thank Louise Fields for the excellent job she did typing the final version of the manuscript.

Remarks on Notation. The chapters are divided into sections. If in a discussion, in say Chapter 4, I want to refer to Theorem 1 in Section 3 of that same chapter, I write "Section 3, Theorem 1." If in that same discussion I want to refer to Theorem 1 in Section 3 of some other chapter, say Chapter 1, then rather than say "Theorem 1 of Section 3 to Chapter I," I simply write "Section 1.3, Theorem 1."

Contents

CHAPTER 1

Preliminaries

Our treatment of functional analysis begins with a long discussion of normed vector spaces. The two most important classes of normed spaces are the Hilbert spaces and the Banach spaces. Although every Hilbert space is a Banach space, the two classes are always treated separately. The concept of a Hilbert space has its origin in the papers of David Hilbert on the theory of integral equations, and it is well known that Hilbert was attracted to this area by the pioneering work of I. Fredholm. Hilbert space theory, by which I mean not only the study of these spaces but also of the continuous, linear operators on them, is one of the most important branches of functional analysis. However, to do the subject justice, we would have to double the length of this book and so, except for an occasional remark, we shall say no more about it.

Stefan Banach was not the first mathematician to investigate the spaces that now bear his name. He did, however, make many important contributions to their study. His book, "Theory of Linear Operations" [2], contains much of what was known about these spaces at the time of its publication (1932), and some of the deepest results in the book are due to Banach himself. The strange title is explained in the preface. Banach writes: "The theory of operations, created by V. Volterra, has as its object the study of functions defined on infinite dimensional spaces." He goes on to discuss the importance of this theory and some of its applications. Now Volterra is recognized, along with Fredholm, as one of the founders of the modern theory of integral equations, and it was undoubtedly his work in this area that led him to the theory of operations.

One fact is clear from our epigrammatic sketch of the history of the theory of normed spaces, and that is that a large portion of this theory has its roots in the study of integral equations. This fact is worth keeping in mind.

1. Norms on a Vector Space

We shall begin our formal discussion of normed spaces here. In all that follows R and C will denote the field of real numbers and the field of complex numbers, respectively. We shall always assume that these two fields have their standard, metric, topologies, and all of the vector spaces that we consider will be defined over one or the other of these two fields. Sometimes it is not necessary to specify over which of these fields a certain vector space is defined. In that case we shall speak of a vector space over K.

Definition 1. Let E be a vector space over the field K. A nonnegative, real-valued function p on E is said to be a norm on E if:

(a) $p(x + y) \leq p(x) + p(y)$ for all x, and y in E;
(b) $p(\alpha x) = |\alpha| p(x)$ for all x in E and all α in K;
(c) $p(x) = 0$ if, and only if, $x = 0$.

If p is a norm on E it is customary to denote, for each x in E, the number $p(x)$ by $\|x\|$. A vector space E on which a norm is defined will be called a normed space. If we want to emphasize the norm, say $\|\cdot\|$, on E, we shall speak of the normed space $(E, \|\cdot\|)$.

There is a natural metric associated with $(E, \|\cdot\|)$; we define the distance between any two points x and y of E to be $\|x - y\|$. This metric gives us a topology on E that we call the norm topology of $(E, \|\cdot\|)$, or the topology induced on E by $\|\cdot\|$. Whenever we speak of, say, a norm convergent sequence in E, or a convergent sequence in $(E, \|\cdot\|)$, we mean a sequence of points of E that is convergent for the metric topology induced on E by $\|\cdot\|$. Similarly, we shall speak of norm compact subsets of E (or compact subsets of $(E, \|\cdot\|)$), of norm continuous functions on E, etc.

The plane (i.e., the vector space over R of all ordered pairs of real numbers) with $\|(x, y)\|$ defined to be the square root of $x^2 + y^2$ is, perhaps, the most familiar example of a normed space. More generally, for any fixed, positive integer n, the vector space over K of all ordered n-tuples of elements of K (we shall call it K^n) can be given a norm by defining $\|(x_1, x_2, \ldots, x_n)\|$ to be the square root of $\sum_{j=1}^{n} |x_j|^2$. This will be called the Euclidean norm on K^n.

It is easy to see that, if there is one norm on a vector space E, then there are infinitely many of them; for if $\|\cdot\|$ is a norm on E then so is $\|\cdot\|_\lambda$ where, for the fixed positive real number λ and each x in E, we define $\|x\|_\lambda$ to be $\lambda\|x\|$. There may be other norms on $(E, \|\cdot\|)$ besides those that can be obtained from $\|\cdot\|$ in this way. On the vector space R^2, for example, we have defined $\|(x, y)\|$ to be the square root of $x^2 + y^2$. But we could also define a norm on this space by taking $\|(x, y)\|_1$ to be $|x| + |y|$, or by taking $\|(x, y)\|_2$ to be the maximum of the numbers $|x|$, $|y|$. Now in the applications of the theory of normed spaces one is often concerned with a family of continuous, linear operators on a specific normed space. Here we are using the term "linear operator" to mean a linear map from a vector space into itself. Clearly the set of all continuous, linear operators on a given normed space is determined by the topology on that space and not by the particular norm that induces that topology. So it seems reasonable to say that two norms on a vector space are equivalent if they induce the same topology on the space. This idea is worth further discussion.

Given a normed space $(E, \|\cdot\|)$ and a point x_0 in E, the map $\phi(x) = x + x_0$ is clearly onto, one-to-one, and continuous. It also has a continuous inverse: $\phi^{-1}(x) = x - x_0$. So ϕ is a homeomorphism from E with the norm topology onto itself. Since the point x_0 is arbitrary, this means that the neighborhoods of any point of E are just translates of the neighborhoods of zero. Hence we may compare the topologies induced on E by two different norms by just comparing the systems of neighborhoods of zero in these two topologies.

Definition 2. Let $(E, \|\cdot\|)$ be a normed space. The set $\{x \text{ in } E \mid \|x\| \le 1\}$ is called the unit ball of E. We shall denote it by $\mathscr{B}_1$ or $\mathscr{B}_1(\|\cdot\|)$.

If $r\mathscr{B}_1$ is taken to mean $\{rx \mid x \text{ in } \mathscr{B}_1\}$, then any neighborhood of zero contains some set in the family $\{r\mathscr{B}_1 \mid r \text{ in } R, r > 0\}$.

Definition 3. Let E be a vector space over K and let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on E. We shall say that $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$, and we shall write $\|\cdot\|_1 \le \|\cdot\|_2$, if there is a positive number λ such that $\lambda\mathscr{B}_1(\|\cdot\|_2) \subset \mathscr{B}_1(\|\cdot\|_1)$. We shall say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, and we shall write $\|\cdot\|_1 \equiv \|\cdot\|_2$, if we have both $\|\cdot\|_2 \le \|\cdot\|_1$ and $\|\cdot\|_1 \le \|\cdot\|_2$.

Suppose that $\|\cdot\|_1$, $\|\cdot\|_2$ are two norms on E with $\|\cdot\|_1 \le \|\cdot\|_2$. Let λ be a positive number such that $\lambda\mathscr{B}_1(\|\cdot\|_2) \subset \mathscr{B}_1(\|\cdot\|_1)$. For any nonzero vector x in E we have $\|\lambda x\|x\|_2^{-1}\|_1 \le 1$. Hence $\lambda\|x\|_1 \le \|x\|_2$ for all nonzero elements of E, and clearly this holds for the zero vector also. Thus we can state: *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space E are equivalent iff there are positive constants λ and μ such that $\lambda\|x\|_1 \le \|x\|_2 \le \mu\|x\|_1$ for every x in E.*

The identity map on a vector space E, I_E, is defined by the equation $I_E x = x$ for all x in E. This map is, of course, an isomorphism from the vector space E onto itself. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on E, then the discussion above shows that $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ iff the map I_E is continuous from $(E, \|\cdot\|_2)$ onto $(E, \|\cdot\|_1)$, and that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ iff I_E is a homeomorphism between these two spaces.

Definition 4. Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two normed spaces over the same field. Let T be an isomorphism from E_1 onto E_2. We shall say that T is a topological isomorphism if it is a homeomorphism from $(E_1, \|\cdot\|_1)$ onto $(E_2, \|\cdot\|_2)$.

We note that two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a vector space E are equivalent iff the identity map is a topological isomorphism from $(E, \|\cdot\|_1)$ onto $(E, \|\cdot\|_2)$.

EXERCISES 1

A number of useful facts, facts that will be referred to later on in the text, are scattered among the exercises. Any problem that is referred to later on is marked with a star. The number of starred problems will decrease as the material gets more difficult.

*1. Let $(E, \|\cdot\|)$ be a normed space, let $\{x_n\}$ be a sequence of points of E, and suppose that this sequence converges to a point x_0 of E for the norm topology. Show that $\lim\|x_n\| = \|x_0\|$.

*2. Let E, F be two normed spaces over the same field and let u be a linear map from E into F.
   (a) Show that u is continuous on E iff it is continuous at zero.
   (b) Show that u is continuous on E iff there is a constant M such that $\|u(x)\| \le M$ for all x in the unit ball of E. Hint: If u is continuous at zero but no such M exists, then for each positive integer n we can find a point x_n in E such that $\|x_n\| \le 1$ and $\|u(x_n)\| \ge n$.
   (c) Show that u is continuous on E iff there is a constant M such that $\|u(x)\| \le M\|x\|$ for all x in E.

3. We have defined three different norms on the vector space R^2. Sketch the unit ball of each of these three normed spaces. Show that any two of our three norms are equivalent.

*4. Let E, F be two normed spaces over the same field and let u be a topological isomorphism from E onto F. Denote the norm on E by $\|\cdot\|_E$ and the norm on F by $\|\cdot\|_F$.
   (a) For each x in E define $|\|x\||$ to be $\|u(x)\|_F$. Show that $|\|\cdot\||$ is a norm on E and that it is equivalent to $\|\cdot\|_E$.
   (b) For each y in F define $|\|y\||$ as follows: Let x be the unique element of E such that $u(x) = y$ and take $|\|y\||$ to be $\|x\|_E$. Show that $|\|\cdot\||$ is a norm on F and that it is equivalent to $\|\cdot\|_F$.
   (c) Suppose now that E is just a vector space over K, F is a normed space over K, and u is an isomorphism from E onto F. The function $|\|\cdot\||$ defined on E as in (a) is still a norm on E. Show that if E is given this norm then u becomes a topological isomorphism from E onto F. If E is a normed space and F is just a vector space then similar remarks can be made about the function defined on F as in (b).

*5. Let E, F be two normed spaces over the same field and let u be a topological isomorphism from E onto F. If $|\cdot|_E$ and $|\cdot|_F$ are norms on E and F, respectively, that are equivalent to the original norms on these spaces, show that u is a topological isomorphism from $(E, |\cdot|_E)$ onto $(F, |\cdot|_F)$.

2. Finite Dimensional Normed Spaces

Before continuing with our general discussion it is instructive to investigate the properties of a special class of normed spaces. We have in mind spaces $(E, \|\cdot\|)$ for which E is a finite dimensional vector space over K. Such spaces do arise in applications.

Recall that a vector space over K is said to be finite dimensional if, for some nonnegative integer n, it has a basis containing n elements; the number n is called the dimension of the space. We allow n to be nonnegative in order to include the trivial vector space, i.e., the vector space over K whose only element is the zero vector. A basis for this space is, by convention, the empty set. Hence the trivial vector space has dimension zero. It is easy to see that any finite dimensional vector space over K can be given a norm. In fact

Theorem 1. Any two norms on a finite dimensional vector space are equivalent.

Proof. Let F be a finite dimensional vector space over K and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on F. Choose a basis $x_1, x_2, \ldots, x_n$ for F and define a third norm, $|\cdot|$, as follows: For each x in F there is a unique set of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x = \sum \alpha_j x_j$. Take $|x|$ to be the maximum of the numbers $|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|$. Suppose that each of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ is equivalent to $|\cdot|$. Then there are positive scalars m_1, M_1 and m_2, M_2 such that

$$m_1 |x| \le \|x\|_1 \le M_1 |x| \qquad \text{and} \qquad m_2|x| \le \|x\|_2 \le M_2|x|$$

for each x in F. It follows that

$$(m_2/M_1)\|x\|_1 \le m_2|x| \le \|x\|_2 \le M_2|x| \le (M_2/m_1)\|x\|_1$$

for each x in F, and hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Now let $\|\cdot\|$ denote either $\|\cdot\|_1$ or $\|\cdot\|_2$. We shall show that $\|\cdot\|$ is equivalent to $|\cdot|$. If $x = \sum \alpha_j x_j$ then

$$\|x\| \le \sum |\alpha_j| \, \|x_j\| \le |x| (\sum \|x_j\|).$$

Since $(\sum \|x_j\|)$ is a constant we see that $\|\cdot\|$ is weaker than $|\cdot|$ on F. Let $S = \{x \text{ in } F \mid |x| = 1\}$ and choose a sequence $\{y_k\}$ of points of S such that $\lim \|y_k\| = \inf\{\|x\| \mid x \text{ in } S\}$. For each y_k there are scalars α_{k1}, $\alpha_{k2}, \ldots, \alpha_{kn}$ such that $y_k = \sum \alpha_{kj} x_j$. Since each $y_k \in S$, $|\alpha_{kj}| \le 1$ for $j = 1, 2, \ldots, n$ and every k. These inequalities imply that there is a subsequence of $\{y_k\}$ (call it $\{y_k\}$ also) such that $\lim \alpha_{kj}$ exists and equals, say, α_j for $j = 1, 2, \ldots, n$. Let $y = \sum \alpha_j x_j$ and note that $\{y_k\}$ converges to y for $|\cdot|$, i.e., $\lim |y - y_k| = 0$. It follows that $|y| = 1$ (Exercises 1, problem 1) and hence, in particular, $y \ne 0$.

At this point we make an observation. Let λ be the maximum of the numbers $\|x_j\|$, $j = 1, 2, \ldots, n$. If a positive number ε is given then the elements $x = \sum \beta_j x_j$ and $z = \sum \gamma_j x_j$ of F will satisfy the inequality $\|x - z\| < \varepsilon$, if $|\beta_j - \gamma_j| < \varepsilon/\lambda n$ for each j, i.e., if $|x - z| < \varepsilon/\lambda n$. This fact, together with the remarks contained in the last paragraph, implies that $\lim \|y_k\| = \|y\|$. So $\|y\| = \inf\{\|x\| \mid x \text{ in } S\}$ and since $y \ne 0$ this infimum is positive. Now if x is any nonzero element of F, then $x/|x|$ is in S and hence $\|x\| \ge \|y\| \, |x|$. It follows that $|\cdot|$ is weaker than $\|\cdot\|$ on F.

Corollary 1. If two finite dimensional normed spaces over K have the same dimension, then they are topologically isomorphic.

Proof. It suffices to show that if a normed space $(F, \|\cdot\|)$ over K has (finite) dimension n then it is topologically isomorphic to the space K^n with the Euclidean norm. There is an isomorphism u from F onto K^n. We can use this map to define a new norm on F as follows: For each x in F let $|\|x\||$ be the norm of $u(x)$ in K^n (Exercises 1, problem 4a). When F is given this new norm u becomes a topological isomorphism (Exercises 1, problem 4c). However, Theorem 1 shows that $|\|\cdot\||$ is equivalent to $\|\cdot\|$ on F. Hence u is a topological isomorphism from $(F, \|\cdot\|)$ onto K^n (Exercises 1, problem 5).

Theorem 1 has two other useful corollaries. In order to state them we need some more terminology. The proofs of these corollaries will be left to the reader (see problem 1 below). We have already noted that a normed space $(E, \|\cdot\|)$ has an associated metric. If E, with this metric,