DE GRUYTER

Elena Rubei

ALGEBRAIC GEOMETRY

A CONCISE DICTIONARY

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Author
Prof. Dr. Elena Rubei
Università degli Studi di Firenze
Dipartimento di Matematica e Informatica "U. Dini"
Viale Morgagni 67/A
50134 Firenze
Italy
rubei@math.unifi.it

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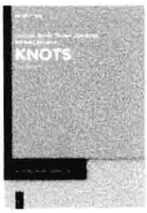

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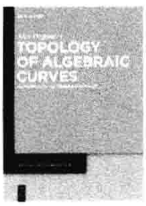

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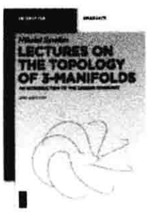

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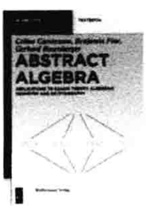

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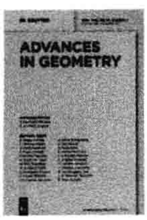

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Preface

This little dictionary of algebraic geometry is intended to be useful mainly to undergraduate and Ph.D. students. For each word listed in the dictionary, we have given the definition, some references, and the main theorems about that term (without the proofs). Some terms from other subjects close to algebraic geometry have also been included. The dictionary has been conceived to help beginners who know some basic facts on algebraic geometry, but not every basic fact, to follow seminars and read papers by giving them basic definitions and statements and providing them with a glimpse of what is nowadays considered to be the basic notions of algebraic geometry. For the sake of simplicity, in some items (for instance algebraic surfaces and Abelian varieties), we have considered only the case of the varieties over the field of complex numbers.

I warmly thank Giorgio Ottaviani for many helpful discussions on algebraic geometry and his invaluable support during all the years I have been in Firenze and, in particular, during the writing of this book. I thank also Roberto Pignatelli for a helpful suggestion.

Firenze, December 2013 Elena Rubei

Notation

$\mathbb{A}^n_K$	We denote by $\mathbb{A}^n_K$ the affine space of dimension n over K.
$A^p(X)$	For any C^∞ manifold X, $A^p(X)$ denotes the vector space of the C^∞ p-forms on X.
$A^{p,q}(X)$	For any almost complex manifold X (see "Almost complex manifolds, holomorphic maps, holomorphic tangent bundles"), $A^{p,q}(X)$ denotes the vector space of the C^∞ (p,q)-forms on X.
$b_i(X,R)$	For any topological space X and ring R, $b_i(X,R)$ (Betti number) denotes the rank of $H_i(X,R)$; it is also denoted by $h_i(X,R)$; see "Singular homology and cohomology".
b.p.f.	The abbreviation b.p.f. stands for "base point free", see "Bundles, fibre -".
$CH^p(X)$	For any algebraic variety X, $CH^p(X)$ denotes the p-th Chow group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
$Cl(X)$	For any algebraic variety X, $Cl(X)$ denotes the divisor class group of X; see "Divisors".
$C^\infty(X,E)$	For any C^∞ vector bundle E on a C^∞ manifold X, $C^\infty(X,E)$ denotes the set of the C^∞ sections of E; see "Bundles, fibre -".
$\delta_{\alpha,\beta}$	$\delta_{\alpha,\beta}$ stands (as usual) for the Kronecker delta.
(D)	For any Cartier divisor D, (D) denotes the line bundle associated to D; see "Divisors".
f_*, f^*, f^{-1}	See "Pull-back and push-forward of cycles", "Direct and inverse image sheaves", "Singular homology and cohomology".
$\|D\|, \mathcal{L}(D)$	For any divisor D, we denote by $\|D\|$ the complete linear system associated to D, see "Linear systems"; see also "Linear systems" for the definition of $\mathcal{L}(D)$.
φ_L	For any line bundle L on a variety X, φ_L denotes the map associated to L; see "Bundles, fibre -".
g^r_d	g^r_d denotes a linear system on a Riemann surface of degree d and projective dimension r; see "Linear systems".
$G(k,V), G(k,\mathbb{P})$	For any vector space V, $G(k,V)$ denotes the Grassmannian of k-planes in V, see "Grassmannians"; analogously, for any projective space $\mathbb{P}$, $G(k,\mathbb{P})$ denotes the Grassmannian of projective k-planes in $\mathbb{P}$.
$H_i(X,R), h_i(X,R)$	For any topological space X and any ring R, $H_i(X,R)$ denotes the i-th homology module of X with coefficients in R (see "Singular homology and cohomology"); $h_i(X,R)$ denotes its rank; $h_i(X,R)$ is also denoted by $b_i(X,R)$ (Betti number)
$H^i(X,R), h^i(X,R)$	For any topological space X and any ring R, $H^i(X,R)$ denotes the i-th cohomology module of X with coefficients in R (see "Singular homology and cohomology"); $h^i(X,R)$ denotes its rank.

$H^i(X, \mathcal{E})$	For any sheaf of Abelian groups $\mathcal{E}$ on a topological space X, $H^i(X, \mathcal{E})$ denotes the i-th cohomology group of $\mathcal{E}$, see "Sheaves"; $h^i(X, \mathcal{E})$ denotes its rank; if E is a holomorphic vector bundle on a complex manifold X or an algebraic vector bundle on an algebraic variety X, we sometimes write $H^i(X, E)$ instead of $H^i(X, \mathcal{O}(E))$.
$\chi(X)$	For any topological space X, $\chi(X)$ denotes the Euler–Poincaré characteristic of X, i.e., the sum $\sum_{i=1,\dots,\infty}(-1)^i b_i(X, \mathbb{Z})$, when it is defined.
$\chi(X, \mathcal{E})$	For any sheaf of Abelian groups $\mathcal{E}$ on a topological space X, $\chi(X, \mathcal{E})$ denotes the Euler–Poincaré characteristic of $\mathcal{E}$, i.e., $\sum_{i=1,\dots,\infty}(-1)^i h^i(X, \mathcal{E})$, when it is defined.
K_X, ω_X	For any complex manifold or smooth algebraic variety X, K_X denotes the canonical bundle and ω_X the canonical sheaf, i.e., $\mathcal{O}(K_X)$; see "Canonical bundle, canonical sheaf".
$M_{\mathcal{F}}, \mathcal{F}_M$	See "Serre correspondence".
$M(m \times n, K)$	We denote by $M(m \times n, K)$ the space of the $m \times n$ matrices with entries in K.
nef	The abbreviation nef stands for "numerically effective"; see "Bundles, fibre -".
$NS^p(X)$	For any algebraic variety X, $NS^p(X)$ denotes the p-th Neron–Severi group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
$\mathcal{O}_X, \mathcal{O}_X(s)$	If X is a complex manifold, $\mathcal{O}_X$ (or simply $\mathcal{O}$) denotes the sheaf of holomorphic functions; if X is an algebraic variety, it denotes the sheaf of the regular functions; more generally, it denotes the structure sheaf of a ringed space; see "Space, ringed -"; for the definition of $\mathcal{O}_X(s)$, see "Hyperplane bundles, twisting sheaves".
$\mathcal{O}(E), \Omega^p(E)$	Let E be an algebraic vector bundle on an algebraic variety or a holomorphic vector bundle on a complex manifold; then $\mathcal{O}(E)$ denotes the sheaf of the (regular, resp. holomorphic) sections of E; see "Bundles, fibre -". We denote $\Omega^p \otimes \mathcal{O}(E)$ by $\Omega^p(E)$.
Ω^p	For any complex manifold X, Ω^p denotes the sheaf of the holomorphic p-forms; for any algebraic variety, Ω^p denotes the sheaf of the regular p-forms; see "Zariski tangent space, differential forms, tangent bundle, normal bundle".
$\mathbb{P}^n_K$	We denote by $\mathbb{P}^n_K$ the projective space of dimension n over K.
$p_a(X), p_g(X), P_i(X)$	The symbols $p_a(X)$, $p_g(X)$ and $P_i(X)$ denote respectively the arithmetic genus, the geometric genus and the i-th plurigenus of X (for X variety or manifold); see "Genus, arithmetic, geometric, real, virtual -", "Plurigenera".
$Pic(X)$	For any algebraic variety X, $Pic(X)$ denotes the Picard group of X; see "Equivalence, algebraic, rational, linear -, Chow, Neron–Severi and Picard groups".
PID	PID stands for "principal ideal domain", i.e., for an integral domain such that every ideal is principal.
$\pi_i(X, x)$	For any topological space X and any $x \in X$, $\pi_i(X, x)$ denotes the i-th fundamental group of X at the basepoint x; see "Fundamental group".

$q(X)$	For any complex manifold or algebraic variety X, $q(X)$ denotes the irregularity of X; see "Irregularity".
$R^q f_*$, R^q_f	See "Direct and inverse image sheaves"
$sat(I)$	For any ideal I, $sat(I)$ denotes the saturation of I; see "Saturation".
Σ_d	For any $d \in \mathbb{N}$, we denote by Σ_d the group of the permutations on d elements.
$Spec(R)$, $Proj(S)$	For any ring R, $Spec(R)$ denotes its spectrum, see "Schemes". See "Schemes" also for the definition of $Proj(S)$ for S graded ring.
$S \times_B S'$	The symbol $S \times_B S'$ denotes the fibred product of S and S' over B; see "Fibred product".
$\cdot$	The symbol $\cdot$ denotes the intersection of cycles; see "Intersection of cycles". Sometimes it is omitted.
$\sqrt{J}$	For any ideal J in a ring R, we denote by $\sqrt{J}$ the radical of J, i.e., $$\sqrt{J} = \{x \in R \mid \exists n \in \mathbb{N} - \{0\} \text{ s.t. } x^n \in J\}.$$
$UV, U+V, (U:V)$	Let R be a commutative ring and U and V be two ideals in R. We define UV to be the ideal $\left\{\sum_{i=1,\dots,k} u_i v_i \mid k \in \mathbb{N},\ u_i \in U,\ v_i \in V\right\}$. Moreover, we define $U+V = \{u+v \mid u \in U\ v \in V\}$, and $(U:V) := \{x \in R \mid xV \subset U\}$.
$V^\vee$	For any vector space V, we denote by $V^\vee$ its dual space.
$\sqcup$	$\sqcup$ denotes the disjoint union.
$f:(X,S) \to (Y,T)$	If $S \subset X$ and $T \subset Y$, the notation $f:(X,S) \to (Y,T)$ stands for a map $f: X \to Y$ such that $f(S) \subset T$.

Note. The end of the definitions, theorems, and propositions is indicated by the symbol □.

A

Abelian varieties. See "Tori, complex - and Abelian varieties".

Adjunction formula. ([72], [93], [107], [129], [140]). Let X be a complex manifold or a smooth algebraic variety and let Z be a submanifold, respectively a smooth closed subvariety. We have

$$K_Z = K_X|_Z \otimes det\, N_{Z,X},$$

where $N_{Z,X}$ is the normal bundle and K_X and K_Z are the canonical bundles respectively of X and Z (see "Canonical bundle, canonical sheaf", "Zariski tangent space, differential forms, tangent bundle, normal bundle").
If, in addition, Z is a hypersurface, the formula becomes

$$K_Z = K_X|_Z \otimes (Z)|_Z,$$

where (Z) is the bundle associated to the divisor Z, since, in this case the bundle $N_{Z,X}$ is the bundle given by Z (see "Bundles, fibre -" for the definition of bundles associated to divisors).

Albanese varieties. ([93], [163], [166]). The Albanese variety is a generalization of the Jacobian of a compact Riemann surface (see "Jacobians of compact Riemann surfaces") for manifolds of higher dimension.
Let X be a compact Kähler manifold of dimension n (see "Hermitian and Kählerian metrics" and "Hodge theory"). The Albanese variety of X is the complex torus (see "Tori, complex - and Abelian varieties")

$$Alb(X) := \frac{H^0(X, \Omega^1)^\vee}{j(H_1(X, \mathbb{Z}))},$$

where $j : H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1)^\vee$ is defined by $j(\alpha) = \int_\alpha$ for any $\alpha \in H_1(X, \mathbb{Z})$.
The Albanese map

$$\mu : X \to Alb(X)$$

is defined in the following way: we fix a point P_0 of X (base point) and we define

$$\mu(P) = \int_{P_0}^{P}$$

for any $P \in X$, where $\int_{P_0}^{P}$ is the integral along a path joining P_0 and P (thus, obviously, it defines a linear function on $H^0(X, \Omega^1)$ only up to elements of $j(H_1(X, \mathbb{Z}))$). If we choose a basis $\omega_1, \dots, \omega_k$ of $H^0(X, \Omega^1)$ we can describe μ in the following way:

$$\mu(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_k \right).$$

For any compact Kähler manifold X of dimension n, the Albanese variety of X is isomorphic to the n-th intermediate Griffiths' Jacobian of X. The Albanese variety of a smooth complex projective algebraic variety is an Abelian variety, that is, it can be embedded in a projective space. See "Jacobians, Weil and Griffiths intermediate -", "Tori, complex - and Abelian varieties".

Algebras. We say that A is an algebra over a ring R if it is an R-module and a ring with unity (with the same sum) and, for all $a, b \in A$ and $r \in R$, we have

$$r(ab) = (ra)b = a(rb).$$

Algebraic groups. ([27], [126], [228], [235], and references in "Tori, complex - and Abelian varieties"). An algebraic group is a set A that is both an algebraic variety (see "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps") and a group and the two structures are compatible, that is, the map

$$A \times A \to A,$$
$$(x, y) \mapsto xy^{-1}$$

is a morphism between algebraic varieties.

Structure theorem for algebraic groups (Chevalley's theorem). Let A be an algebraic group over an algebraically closed field. Then there exists a (unique) normal affine subgroup N such that A/N is an Abelian variety. □

Definition. We say that an algebraic group is **reductive** if all its representations are completely reducible (see "Representations"). □

Almost complex manifolds, holomorphic maps, holomorphic tangent bundles. ([121], [147], [192], [251]). An **almost complex manifold** is the data of

- a C^∞ manifold M;
- a C^∞ section J of the bundle $TM_{\mathbb{R}} \otimes TM_{\mathbb{R}}^\vee$ (where $TM_{\mathbb{R}}$ is the real tangent bundle) such that, if we see J as a map

  $$J : C^\infty(M, TM_{\mathbb{R}}) \to C^\infty(M, TM_{\mathbb{R}})$$

  (where $C^\infty(M, TM_{\mathbb{R}})$ is the vector space of the C^∞ sections of $TM_{\mathbb{R}}$), we have that

  $$J^2 = -I,$$

  where I is the identity map; in other words, for every $P \in M$, the linear map $J_P : T_P M_{\mathbb{R}} \to T_P M_{\mathbb{R}}$ induced on the real tangent space of M at P is such that

  $$J_P^2 = -I.$$

We can extend J_P to $T_PM_{\mathbb{C}} := T_PM_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. We define the **holomorphic tangent space** of M at P to be the i-eigenspace of J_P; we denote it by $T_P^{1,0}M$ and we denote the holomorphic tangent bundle, i.e., the bundle whose fibre in P is $T_P^{1,0}M$, by $T^{1,0}M$. We define the **antiholomorphic tangent space** of M at P to be the $-i$-eigenspace of J_P; we denote it by $T_P^{0,1}M$ and we denote the antiholomorphic tangent bundle, i.e., the bundle whose fibre in P is $T_P^{0,1}M$, by $T^{0,1}M$. Thus

$$T_PM_{\mathbb{C}} = T_P^{1,0}M \oplus T_P^{0,1}M.$$

Obviously a **complex manifold** (see "Manifolds") is an almost complex manifold: if $\{z_\alpha = x_\alpha + iy_\alpha\}_\alpha$ are the coordinates on a coordinate open subset, we can define J by

$$J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha}, \qquad J\left(\frac{\partial}{\partial y_\alpha}\right) = -\frac{\partial}{\partial x_\alpha}$$

for any α. Thus, in the case of a complex manifold,

$$T_P^{1,0}M = \left\langle \left(\frac{\partial}{\partial z_\alpha}\right)_P \right\rangle_\alpha \qquad T_P^{0,1}M = \left\langle \left(\frac{\partial}{\partial \overline{z}_\alpha}\right)_P \right\rangle_\alpha,$$

where

$$\frac{\partial}{\partial z_\alpha} := \frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}, \qquad \frac{\partial}{\partial \overline{z}_\alpha} := \frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}.$$

We define $\{dx_\alpha, dy_\alpha\}$ to be the dual basis of $\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}$. Let

$$dz_\alpha := dx_\alpha + idy_\alpha$$

and

$$d\overline{z}_\alpha := dx_\alpha - idy_\alpha.$$

Observe that $dz_\alpha(\frac{\partial}{\partial z_\beta}) = 2\delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ is the Kronecker delta.

Remark. There is an isomorphism

$$T_PM_{\mathbb{R}} \cong T_P^{1,0}M$$

given by $x \mapsto \frac{1}{2}(x - iJ(x))$ (observe that, through such isomorphism, $\frac{\partial}{\partial x_\alpha}$ goes to $\frac{1}{2}\frac{\partial}{\partial z_\alpha}$). Analogously, there is an isomorphism

$$T_PM_{\mathbb{R}} \cong T_P^{0,1}M$$

given by $x \mapsto \frac{1}{2}(x + iJ(x))$. □

For any almost complex manifold M, let $A^{p,q}(M)$ be the space of the C^∞ (p,q)-forms on M.

Newlander–Nirenberg theorem. Let (M, J) be an almost complex manifold of (real) dimension $2n$. The almost complex structure is induced by a complex structure if and only if one of the following conditions holds:

(1) for all $A, B \in C^\infty(M, T^{1,0}M)$, we have $[A, B] \in C^\infty(M, T^{1,0}M)$ (where $[A, B]$ stands for $A \circ B - B \circ A$);
(2) for all $A, B \in C^\infty(M, T^{0,1}M)$, we have $[A, B] \in C^\infty(M, T^{0,1}M)$;
(3) for all $\alpha \in A^{0,1}(M)$, we have $d\alpha \in A^{1,1}(M) \oplus A^{0,2}(M)$ and for all $\alpha \in A^{1,0}(M)$, we have $d\alpha \in A^{1,1}(M) \oplus A^{2,0}(M)$;
(4) for all $\alpha \in A^{p,q}(M)$, we have $d\alpha \in A^{p+1,q}(M) \oplus A^{p,q+1}(M)$ for every $p, q \in \{0, \dots, n\}$;
(5) for all $X, Y \in C^\infty(M, TM_{\mathbb{R}})$, we have

$$[X, Y] + J[X, JY] + J[JX, Y] - [JX, JY] = 0$$

(the first member of the equality is called Nijenhuis tensor). □

Definition. Let M be a complex manifold. We can decompose

$$d : A^{p,q}(M) \longrightarrow A^{p+1,q}(M) \oplus A^{p,q+1}(M)$$

as

$$d = \partial + \bar{\partial},$$

where

$$\partial : A^{p,q}(M) \longrightarrow A^{p+1,q}(M),$$

$$\bar{\partial} : A^{p,q}(M) \longrightarrow A^{p,q+1}(M)$$

are the compositions of d respectively with the projections

$$A^{p+1,q}(M) \oplus A^{p,q+1}(M) \to A^{p+1,q}(M),$$

$$A^{p+1,q}(M) \oplus A^{p,q+1}(M) \to A^{p,q+1}(M).$$ □

Let $f : M \to N$ be a C^∞ map between two complex manifolds. By extending the differential

$$df_P^{\mathbb{R}} : T_P M_{\mathbb{R}} \to T_P N_{\mathbb{R}}$$

by $\mathbb{C}$-linearity, we get a map

$$df_P^{\mathbb{C}} : T_P M_{\mathbb{C}} \to T_P N_{\mathbb{C}}.$$

We say that f is **holomorphic** if one of the following equivalent conditions holds:
(i) for every component $f^j = f_1^j + i f_2^j$ of f in local coordinates of M and N, we have $\frac{\partial f_1^j}{\partial x_\alpha} = \frac{\partial f_2^j}{\partial y_\alpha}$ and $\frac{\partial f_2^j}{\partial x_\alpha} = -\frac{\partial f_1^j}{\partial y_\alpha}$;
(ii) $\frac{\partial f}{\partial \bar{z}_\alpha} = 0$ for every $\alpha = 1, \dots, dim(M)$, where $\{z_\alpha\}_\alpha$ are local coordinates of M;
(iii) $df^{\mathbb{R}} \circ J^M = J^N \circ df^{\mathbb{R}}$ (where J^M and J^N denote the operators J on M and N respectively), i.e., the differential of f is "$\mathbb{C}$-linear" for the complex structures given by J;

(iv) $df^{\mathbb{C}}(T^{1,0}M) \subseteq T^{1,0}N$;

(v) $df^{\mathbb{C}}(T^{0,1}M) \subseteq T^{0,1}N$.

Ample and very ample. See "Bundles, fibre -" (or "Divisors").

Anticanonical. See "Fano manifolds".

Arithmetically Cohen–Macaulay or arithmetically Gorenstein. See "Cohen–Macaulay, Gorenstein, (arithmetically -,-)".

Artinian. See "Noetherian (and Artinian)".

B

Base point free (b.p.f.) See "Bundles, fibre -".

Beilinson's complex. ([5], [23], [63], [207], [209], [137]).

Beilinson's theorem I. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$ (see "Coherent sheaves"). Then there exists a complex of sheaves

$$0 \to L^{-n} \xrightarrow{d_{-n}} L^{-n+1} \xrightarrow{d_{-n+1}} \cdots\cdots\cdots \xrightarrow{d_{n-2}} L^{n-1} \xrightarrow{d_{n-1}} L^n \to 0$$

with $L^k = \oplus_{i,j \text{ s.t. } i-j=k} \Omega^j(j)^{h^i(\mathcal{F}(-j))}$ such that

$$\frac{Ker\ d_k}{Im\ d_{k-1}} = \begin{cases} \mathcal{F} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

Every morphism $\Omega^p(p) \to \Omega^p(p)$ induced by one of the morphisms d_k is zero. □

Beilinson's theorem II. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$. Then there exists a complex of sheaves

$$0 \to L^{-n} \xrightarrow{d_{-n}} L^{-n+1} \xrightarrow{d_{-n+1}} \cdots\cdots\cdots \xrightarrow{d_{n-2}} L^{n-1} \xrightarrow{d_{n-1}} L^n \to 0$$

with $L^k = \oplus_{i,j \text{ s.t. } i-j=k} \mathcal{O}(-j)^{h^i(\mathcal{F}\otimes\Omega^j(j))}$ such that

$$\frac{Ker\ d_k}{Im\ d_{k-1}} = \begin{cases} \mathcal{F} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

Every morphism $\mathcal{O}(-p) \to \mathcal{O}(-p)$ induced by one of the morphisms d_k is zero. □

Bertini's theorem. ([93], [104], [107], [129], [136], [228]). On a smooth quasi-projective algebraic variety X over an algebraically closed field of characteristic 0, the general element of a finite-dimensional linear system (see "Linear systems") is smooth away from the base locus of the system.

Bezout's theorem. ([72], [93], [104], [107], [196]). Let K be an algebraic closed field.

Bezout's theorem. Let V and V' be two projective algebraic varieties of respective dimension m and m' in $\mathbb{P}^n_K$. Suppose that $m + m' \geq n$ and that, for every irreducible component C of $V \cap V'$ and for P general point of C, V and V' are smooth at P and $T_P(\mathbb{P}^n_K) = T_P(V) + T_P(V')$, where T_P denotes the tangent space at P. Then

$$\deg(V \cap V') = \deg(V)\ \deg(V'). \qquad \square$$

(See "Degree of an algebraic subset" for the definition of degree).

By using intersection multiplicities (see "Intersection of cycles"), we can state a stronger result: suppose that V and V' are two projective algebraic varieties of dimension m and m' in $\mathbb{P}^n_K$ with $m + m' \geq n$ and suppose they intersect properly, i.e., the codimension of every irreducible component C of $V \cap V'$ is the sum of the codimension of V and the codimension of V'; then, by using an appropriate definition of intersection multiplicity of V and V' along C, which we denote by $i_C(V, V')$, we have that

$$deg(V)\, deg(V') = \sum_C i_C(V, V')\, deg(C),$$

where the sum runs over all irreducible components C of $V \cap V'$ (see, e.g., [72]).

Bielliptic surfaces. See "Surfaces, algebraic -".

Big. See "Bundles, fibre -" or "Divisors".

Birational. See "Varieties, algebraic -, Zariski topology, regular and rational functions, morphisms and rational maps".

Blowing-up (or σ-process). ([22], [93], [104], [107], [196], [228]). We follow mainly [93] and [104].
Roughly speaking, the blow-up of a manifold along a subvariety is a geometric transformation replacing the subvariety with all the directions pointing out from it. For instance the blow-up of a manifold in a point replaces the point with all the directions pointing out from it.

We define the blow-up of $\mathbb{C}^n$ in a point $P \in \mathbb{C}^n$ as follows. By changing coordinates we can suppose $P = 0$; we define the blow-up of $\mathbb{C}^n$ in 0 as the set

$$Bl_0(\mathbb{C}^n) := \{(x, l) \in \mathbb{C}^n \times \mathbb{P}^{n-1}_{\mathbb{C}} \mid x \in l\}$$