岭南学术文库第二辑　主编：舒元　陈平

SEPARATED CONTINUOUS LINEAR PROGRAMMING AND ITS EXTENSIONS

可分离的连续线性规划及其推广模型

Xiaoqing Wang

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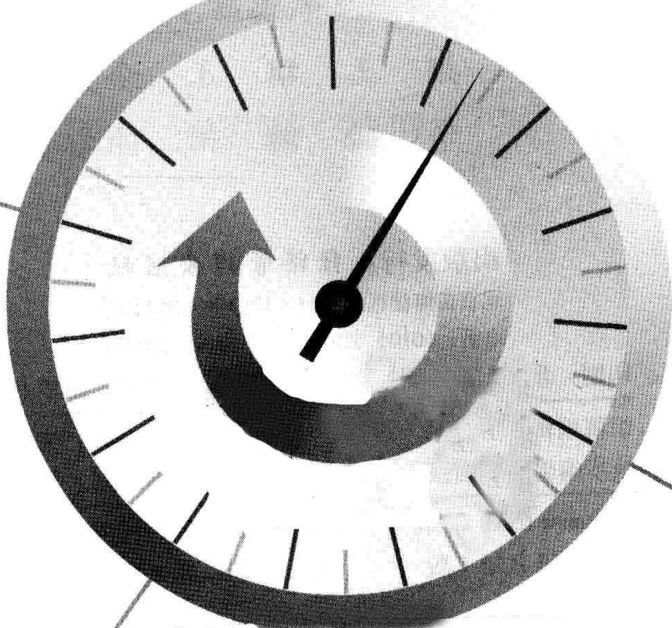

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DEDICATION

To the memory of my grandparents, to my parents, to my husband and daughter

Preface

A Linear Programming (LP) problem is an optimization problem with a linear objective function and linear constraints. This type of problems is very important in the theory and practice of optimization. There are several reasons for this: first, the linear programming problem in a finite-dimensional space is both elegant and complete. It can be solved efficiently by simplex method, interior-point method, etc. In fact, size of the linear programming problem is not an issue as long as it is finite. Second, we can get very useful information, both on the optimal solution and on the problem itself, by conducting the sensitivity analysis. Because of this, many nonlinear real-world problems are formulated as linear programming problems, in spite of the loss of accuracy that this involves. Third, the linear programming has an extremely wide range of industrial applications. For these reasons, LP has been pushed and extended to an ever-broadening frontier. The subject of this book is one of such frontier extensions, the so-called Separated Continuous Linear Programming (SCLP) and its extensions: Generalized Separated Continuous Linear Programming (GSCLP), Separated Continuous Conic Programming (SCCP) and its generalized version (Generalized Separated Continuous Conic Programming (GSCCP)).

Anderson (1978) introduced the SCLP to model the job-shop scheduling problem. Later on, many researchers, including Anderson and his colleagues, conducted extensive research on SCLP, mainly in the duality theory and the solution method. In spite of its wide range of applications, SCLP still is a kind of problem which is difficult to solve in general. In recent years, SCLP has attracted considerable attention in the research field of stochastic networks. The multi-class stochastic network is a system consisting of different classes (types) of jobs which need to be processed and a set of servers which process the jobs. Jobs arrive to the system randomly or according to some probability distribution. Each server can process one or more classes of jobs and the processing time for each server to process one job is different for different class of jobs. The jobs in the same class have the same characteristics such as arrival rate, service requirements, etc. After one job is processed in one server, it may leave the network instantaneously or may become a job of another class and go to another server for processing. The multi-class stochastic network is a very useful model for many real systems. For example, it can model the job-shop operation in which different types of parts need to be processed by one or more machines before the final products can be produced. It can model the economic system in which different assets are used to produce the goods needed in the market. It can also model the communication systems in which different data need to be delivered by using different bandwidth. Also, it can model the road network in a city in which the running speeds of different vehicles are different. For each multi-class stochastic network, there is a corresponding deterministic fluid network, which takes only the first-order data (means and rates) from the stochastic model and assumes that the jobs circulating in the network are continuous flows instead of discrete units. With appropriate scaling, the fluid network is a limit of the stochastic network, in the sense of strong law of large numbers (refer to, e. g. Chen and Yao, 2001). Furthermore, the fluid model has played a central role in studying the stability of stochastic networks (Dai, 1995). Because of these developments, the real-time control (dynamic scheduling) of a stochastic network, which is itself a quite intractable problem, can be turned into the control of a corresponding fluid network, the problem of which takes exactly the form of SCLP.

At the time of writing this book, there exist some results for SCLP, both in the duality theory and the solution methods. However, the existing results are not satisfactory in the sense that they did not provide a clear picture of the properties of SCLP. The main objectives of this book are to survey the existing results, and more importantly, introduce the work we have done recently which gives some insight into properties of SCLP.

The rest of the book is organized as follows: In Chapter 2, we review the literature on the work related to SCLP, including the duality theory and solution methods. In Chapter 3, we present our results for SCLP including the duality theory and an approximation algorithm. An example is given to illustrate the application of our algorithm to solve SCLP. In Chapter 4, we extend our results to GSCLP. In Chapter 5, we extend our result to SCCP. We also give a brief review on conic programming and two applications of SCCP. In Chapter 6, we further extend our results to GSCCP. Finally, we summarize the findings and the results contained in this book and point out possible future research directions in Chapter 7.

This book could be used as the textbook for a short course on SCLP and its extensions. It can also be used with other books for a course on infinite-dimensional LP. The prerequisites for the book are a preliminary knowledge on optimization theory in n-dimensional real space (and in particular, linear programming), convex analysis, linear algebra and functional analysis.

Much of this book has come out of my PhD dissertation in the Chinese University of Hong Kong. The remainder is work done in Sun Yat-sen University. I would like to acknowledge the crucial support provided by my PhD supervisors, Profs. Shuzhong Zhang and David D. Yao during my stay in the Chinese University of Hong Kong. I am also grateful for Lingnan (University) College, Sun Yat-sen University for the facilities and the stimulating environment provided.

Xiaoqing Wang

Contents

Chapter 1 Introduction

Linear Programming (LP) is probably the most successful mathematical model in terms of its extremely wide range of industrial applications and its superb speed and capacity in solving problems of very large size. For this reason, LP has been pushed and extended to an ever-broadening frontier. The subject of this book is one of such frontier extensions, the so-called Separated Continuous Linear Programming (SCLP) and its extensions: Generalized Separated Continuous Linear Programming (GSCLP), Separated Continuous Conic Programming (SCCP) and its generalized version (Generalized Separated Continuous Conic Programming (GSCCP)).

Before we introduce SCLP, let us first explain what is Continuous Linear Programming (CLP).

Bellman (1957) introduced the following problem to model the "bottle-neck process" in economics which he called CLP:

$$
\begin{aligned}
(CLP)\ \max\quad & \int_0^T c(t)'u(t)\,dt \\
\text{s. t.}\quad & B(t)u(t) + \int_0^t K(s,\ t)u(s)\,ds \leqslant b(t), \\
& u(t) \geqslant 0,\ t \in [0,\ T],
\end{aligned}
$$

where $u(t)$ is the decision variable. The superscript " ' " here denotes the transpose operation for a vector or a matrix.

We see that CLP is very similar to LP, the differences being that the decision variables in the CLP problem are functions of time instead of discrete vectors. Also, the number of constraints in the CLP problem are infinite whereas the number of constraints in the LP problem are finite.

Although CLP can model a lot of problems in practice, it is difficult to solve. Anderson and Nash (1987) summarized some earlier results on CLP and its applications.

Anderson (1978) introduced the following problem to model the job-shop scheduling problem:

$$
\begin{aligned}
\min\quad & \int_0^T c(t)'u(t)\,dt \\
\text{s. t.}\quad & \int_0^t Gu(s)\,ds \leqslant a(t), \\
& Hu(t) \leqslant b(t),
\end{aligned}
$$

$u(t) \geqslant 0, t \in [0, T]$,

here $u(t)$ is the decision variable and is assumed to be bounded measurable function, $b(t)$, $c(t)$ are bounded measurable functions and $a(t)$ is an absolutely continuous function. This is a special case of CLP and termed by Anderson as separated continuous linear programming, abbreviated as SCLP. The word "separated" here refers to the fact that there are two kinds of constraints in SCLP: the constraints involving integration and the instantaneous constraints (Anderson and Nash (1987)).

SCLP has a lot of applications, but it is still difficult to solve in its most general setting. There exists a rich literature on duality theory and algorithms for solving various forms of SCLP.

In this book, we consider the following SCLP:

$$(SCLP) \quad \max \quad \int_0^T ((\gamma + (T-t)c)'u(t) + d'x(t))\,dt$$

$$\text{s. t.} \quad \int_0^t Gu(s)\,ds + Fx(t) \leqslant \alpha + at,$$

$$Hu(t) \leqslant b,$$

$$u(t), x(t) \geqslant 0, \ t \in [0, T],$$

where $u(t)$, $x(t)$ are decision variables and are assumed to be bounded measurable functions.

This SCLP was first formulated by Weiss (2008). Comparing with the SCLP formulated by Anderson, the above problem is different in that the coefficient of $x(t)$ in the first constraint and the objective function is more general than those of Anderson's model. On the other hand, this problem restricts $a(t)$, $c(t)$ in Anderson's model to linear functions and $b(t)$ to be constant.

SCLP has in recent years attracted considerable attention in the research field of stochastic networks. The multi-class stochastic network is a system consisting of different classes (types) of jobs which need to be processed and a set of servers which process the jobs. Jobs arrive to the system randomly or according to some probability distribution. Each server can process one or more classes of jobs and the processing time for each server to process one job is different for different class of jobs. The jobs in the same class have the same characteristics such as arrival rate, service requirements, etc. After one job is processed in one server, it may leave the network instantaneously or may become a job of another class and go to another server for processing. The multi-class stochastic network is a very useful model for many real systems. For example, it can model the job-shop operation in which different types of parts need to be processed by one or more machines before the final products can be produced. It can model the economic system in which different assets are used to produce the goods needed in the market. It can also model the communication systems in which different data need to be delivered by using different bandwidth. Also, it can model the road network in a city in which the running speeds of different vehicles are different. For each multi-class stochastic network, there is a corresponding deterministic fluid network, which takes only the first-order data (means and rates) from the stochastic model and assumes that the jobs circulating in the network are continuous flows instead of discrete units. With appropriate scaling, the fluid network is a limit of the stochastic network, in the sense of strong law of large numbers (refer to, e. g. Chen and Yao (2001)). Furthermore, the fluid model has played a central role in studying the stability of stochastic networks (Dai (1995)). Because of these developments, the real-time control (dynamic scheduling) of a stochastic network, which is itself a quite intractable problem, can be turned into the control of a corresponding fluid network, the problem of which takes exactly the form of SCLP.

Let us re-write (*SCLP*) in the following equivalent format using slack variables:

$$
\begin{aligned}
\max \quad & \int_0^T ((\gamma + (T-t)c)'u(t) + d'x(t))\,dt \\
\text{s. t.} \quad & \int_0^t Gu(s)\,ds + Fx(t) + y(t) = \alpha + at, \\
& Hu(t) + z(t) = b, \\
& u(t),\ z(t),\ x(t),\ y(t) \geqslant 0,\ t \in [0,\ T].
\end{aligned}
$$

The first constraint of this SCLP can be re-written as:

$$
(F \quad I)\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = a - Gu(t).
$$

This equation can be viewed as the system dynamics in a control system. So (*SCLP*) can be viewed as a linear optimal control problem with state positivity constraints (Pullan (1995)). Without state positivity constraints, this problem can be solved by the maximum principle (Pontryagin et al. (1962)). However, in the presence of state constraints, the form of the maximum principle gets substantially more complicated and more importantly, it does not fully characterize the optimality.

Motivated by the abundant applications of SCLP, this book is devoted to the topic. Our approach is based on a simple discretization idea. We use even partition of time interval $[0, T]$ to get the discretizations for (*SCLP*) and its dual. By exploiting the relationship among (*SCLP*), its dual and their discretization problems, we are able to derive the duality results for (*SCLP*) and design an approximation algorithm to solve the problem.

Our algorithm has the following characteristics:

(i) We do not presume SCLP and its dual have optimal solutions.

(ii) Our algorithm is in fact a polynomial-time approximation (PTA) scheme.

(iii) The solution obtained is a feasible solution for SCLP with a guaranteed error bound in the objective value as compared to the optimal value.

(iv) The trade-off between the quality of the solution and the computational effort is explicit.

It is inspirational that most of our results for SCLP can be extended to GSCLP, SCCP and GSCCP in the following forms:

$$(GSCLP)\quad \max \int_0^T (c(T-t)'u(t) + d(T-t)'x(t))\,dt$$

$$\text{s. t.}\quad a(t) - \int_0^t Gu(s)\,ds - Fx(t) \geqslant 0,$$

$$b(t) - Hu(t) \geqslant 0,$$

$$u(t) \geqslant 0, x(t) \geqslant 0,\ t \in [0, T],$$

$$(SCCP)\quad \max \int_0^T ((\gamma + (T-t)c)'u(t) + d'x(t))\,dt$$

$$\text{s. t.}\quad \alpha + ta - \int_0^t Gu(s)\,ds - Fx(t) \in K_1,$$

$$b - Hu(t) \in K_2,$$

$$u(t) \in K_3,\ x(t) \in K_4,$$

and

$$(GSCCP)\quad \max \int_0^T (c(T-t)'u(t) + d(T-t)'x(t))\,dt$$

$$\text{s. t.}\quad a(t) - \int_0^t Gu(s)\,ds - Fx(t) \in K_1,$$

$$b(t) - Hu(t) \in K_2,$$

$$u(t) \in K_3,\ x(t) \in K_4,$$

where K_i, $i = 1, 2, 3, 4$, are closed convex cones and $u(t)$, $x(t)$ are bounded measurable functions, $a(t)$, $c(t)$ are piecewise linear functions, $b(t)$, $d(t)$ are piecewise constant functions. To our knowledge, this is the first attempt to formulate SCCP and GSCCP and solve them.

The rest of the book is organized as follows: In Chapter 2, we review the literature on the work related to SCLP, including the duality theory and solution methods. In Chapter 3, we present our results for SCLP including the duality theory and an approximation algorithm. An example is given to illustrate the application of our algorithm to solve SCLP. In Chapter 4, we extend our results to GSCLP. In Chapter 5, we extend our result to SCCP. We also give a brief review on conic programming and two applications of SCCP. In Chapter 6, we further extend our results to GSCCP. Finally, we summarize the findings and the results contained in this book and point out possible future research directions in Chapter 7.

Chapter 2 Literature review

As with LP, research on SCLP can be roughly divided into two categories: duality theory and solution methods. Since they are closely related to each other, we will review the results in these two categories as an integrated whole. Among all the existing work done on SCLP, The result in Weiss (2008) is significant because the algorithm he proposed can produce an exact optimal solution for SCLP in finite number of "SCLP" pivot steps whereas all other methods either produce an approximate solution or converge to the optimal solution in infinite steps. We will give a detailed review on his work in this chapter. The reader is suggested to have the following questions in mind when reading this chapter in order to get a clear picture of the work being surveyed: One, when does SCLP have an optimal solution? Two, what is the form of the optimal solution? Three, how to get the optimal solution?

Recall that Anderson introduced the following SCLP, which we denote as (*SCLP1*) here:

$$
\begin{aligned}
(SCLP1)\ \min\quad & \int_0^T c(t)'u(t)\,dt \\
\text{s.t.}\quad & \int_0^t Gu(s)\,ds \leqslant a(t), \\
& Hu(t) \leqslant b(t), \\
& u(t) \geqslant 0,\ t \in [0, T],
\end{aligned}
$$

where $u(t)$, $b(t)$, $c(t)$ are bounded measurable functions and $a(t)$ is an absolutely continuous function.

Anderson (1978, 1980) also conducted pioneering work on the solution methods for (*SCLP1*). Since the simplex method is so successful to solve the LP problem, he tried to extend it to (*SCLP1*). To use the simplex method, it is necessary to extend such concepts as "extreme point", "dual variable", "pivot" to (*SCLP1*).

Using essentially the same method as LP, Anderson and Nash (1987) derived a dual problem of (*SCLP1*), denoted by (*SCLP1* *) as below:

$$
\begin{aligned}
(SCLP1^*)\ \max\quad & -\int_0^T a(t)'p(t)\,dt - \int_0^T b(t)'q(t)\,dt \\
\text{s.t.}\quad & c(t) + \int_t^T G'p(s)\,ds + H'q(t) \geqslant 0, \\
& p(t) \geqslant 0, q(t) \geqslant 0, t \in [0, T].
\end{aligned}
$$

Anderson, Nash and Perold (1983) pointed out: (i) A feasible solution of ($SCLP1$) is an extreme point solution if and only if the columns of $K = \begin{pmatrix} G & I & 0 \\ H & 0 & I \end{pmatrix}$ indexed by the support of this feasible solution are linear independent for almost all t in $[0, T]$ (the support of a feasible solution is a set of indices of non-zero components of this feasible solution). (ii) If the feasible region for ($SCLP1$) is non-empty and bounded, then ($SCLP1$) is solvable and there exists an extreme point optimal solution for ($SCLP1$). (iii) Suppose the feasible region for ($SCLP1$) is non-empty and bounded, if $a(t)$, $c(t)$ are piecewise linear, with $a(t)$ also absolutely continuous, $b(t)$ is piecewise constant, then ($SCLP1$) has an optimal solution in which $u(t)$ is piecewise constant on $[0, T]$.

Based on these results, Anderson and Philpott (1989a) developed the so-called continuous-time network simplex algorithm to solve a kind of continuous network program which can be formulated as ($SCLP1$) with piecewise linear $a(t)$ and $c(t)$, with $a(t)$ being absolutely continuous, and $b(t)$ piecewise constant.

This is the first algorithm which was implemented in a computer to solve the ($SCLP1$) problem. Unfortunately, there is no convergence guarantee for this algorithm, and it often produces a sequence of solutions which converge to a sub-optimal solution (Philpott and Craddock (1995)).

Later, Pullan (1993) introduced another dual problem of ($SCLP1$), denoted by ($SCLP1^{*\prime}$) as below:

$$(SCLP1^{*\prime}) \quad \max \quad -\int_0^T d\pi(t)'a(t) - \int_0^T b(t)'q(t)\,dt$$

$$\text{s. t.} \quad c(t) - G'\pi(t) + H'q(t) \geqslant 0,$$

$$q(t) \geqslant 0, \text{a. e. on } [0, T],$$

$$\pi(t) \text{ monotonically increasing and right continuous on } [0, T] \text{ with } \pi(T) = 0.$$

This dual was introduced to avoid the diffculty in the duality theory of SCLP mentioned in Grinold (1968): for ($SCLP1$) and ($SCLP1^{*}$), there exist such instances that either the primal or the dual problem has no feasible or no optimal solution, while the other one has an optimal solution. See also Buie and Abrham (1986) for the similar treatment for the dual of CLP.

Pullan also introduced a new discretization for ($SCLP1$) which is a finite LP problem. (There is another discretization problem for ($SCLP1$) called the "standard discretization" for ($SCLP1$) which is also a finite LP problem. This discretization problem was used before in the context of CLP in, for example, Buie, Abrham (1973) and Tyndall (1965)). Based on the relationship between (*SCLP*1), (*SCLP*1$^{*\prime}$) and the discretization problem he introduced, Pullan proposed an algorithm to solve (*SCLP*1) with piecewise linear $a(t)$, $c(t)$, with $a(t)$ also absolutely continuous, and $b(t)$ piecewise constant (these assumptions on the problem input are the same as those in Anderson and Philpott (1989a)). The algorithm is a discretization-based algorithm. Initially, a feasible solution for (*SCLP*1) is obtained by solving the standard discretization problem for (*SCLP*1) with an initial partition of $[0, T]$. Then, in each iteration, according to the partition of $[0, T]$, the discretization problem for (*SCLP*1) Pullan introduced is formulated and solved. The solution of this discretization is then used together with the existing feasible solution for (*SCLP*1) to construct a new feasible solution for (*SCLP*1) with the strictly improving objective value and the new partition of $[0, T]$. After each iteration, the number of the breakpoints in the new feasible solution for (*SCLP*1) is tripled. This process continues until an optimal solution for (*SCLP*1) is found or the resulting feasible solution for (*SCLP*1) is within a pre-described limit. He also gave some variants of this algorithm in this paper. Anderson and Pullan (1996) proposed a purification algorithm which can produce an extreme point solution for (*SCLP*1) from a feasible solution for (*SCLP*1) with the same or improving objective function value. The preliminary result suggested that the algorithm in Pullan (1993), when combined with this purification algorithm, can produce the exact optimal solution for (*SCLP*1) with finite number of iterations. Later, in Pullan (2000), he proved that his algorithm, without any purification procedures, either terminates in a finite number of iterations with an optimal solution for (*SCLP*1), or the objective values of the resulting feasible iterative solutions for (*SCLP*1) converge to the optimal objective value of (*SCLP*1).

Following Pullan's work, Philpott and Craddock (1995) proposed an algorithm based on the results of Pullan on (*SCLP*1) for solving a problem considered by Anderson and Philpott (1989a) before. That algorithm can also solve (*SCLP*1) considered by Pullan (1993). The method is similar to Pullan's, except that the criterion for adding breakpoints in producing the new feasible solution for (*SCLP*1) is different. In this algorithm, after each iteration, the number of breakpoints in the resulting feasible solution for (*SCLP*1) is at most doubled. It also proved the convergence of the algorithm: the algorithm either terminates