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Volume 16

# Game Theory and Applications

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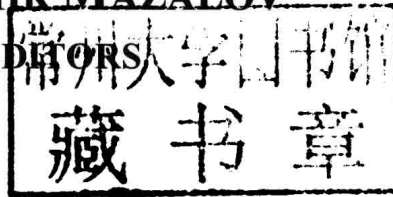
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GAME THEORY AND APPLICATIONS

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## VOLUME 16

LEON PETROSJAN  
AND  
VLADIMIR MAZALOV  
EDITORS



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**GAME THEORY AND APPLICATIONS**

# **GAME THEORY AND APPLICATIONS**

**VOLUME 16**

# GAME THEORY AND APPLICATIONS

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## PREFACE

The workshop “Game theory for finance, social and biological sciences”, held in Warwick 14-17 of April 2010, was organised in the framework of the 2009/2010 EPSRC Symposium on the Mathematics of Complexity Science and Systems Biology (Main organisers Robert MacKay and David Wild). Special attention was given to problems in dynamic games under partial information and to the development of numerical methods for high-dimensional games (there is an increasing focus on this arena as recent theory is leading to solution methods for problems which were heretofore quite intractable). The interdisciplinary aspects touched upon were related to dynamical systems via replicator dynamics, with probability (measure-valued processes), with statistical mechanics (kinetic equation, non-equilibrium behaviour), with max-plus (or tropical, or idempotent) mathematics. Speakers from all over the world included Steven Alpern from LSE, London (How to Patrol a Network against an Unknown Attack), Marianne Akian from INRIA, France (Tropical Polyhedra are Equivalent to Mean Payoff Games), Tibor Antal from Harvard (Who Laughs Last? Perturbation Theory for Games), Tamer Basar from Illinois (Non-Neutral Decision Making in Stochastic Teams and Games), Pierre Bernhard from INRIA-Sophia Antipolis-Méditerranée (A Robust Control Approach to Option Pricing: the Uniqueness Theorem), Constantinos Daskalakis from MIT (The Complexity of Equilibria), Maurizio Falcone from Università di Roma Sapienza (A Constructive Approach to Pursuit-Evasion Games), Sayantal Ghosal from Warwick (P-Stable Equilibrium: Definition and Some Properties), Paul Goldberg from Liverpool (How hard is Competition for Rank?), Sergiu Hart from Jerusalem (Comparing and Measuring Risks), Onesimo Hernandez-Lerma from Cinvestav, Mexico (Overtaking Equilibria for ZeroSum Markov Games), David Gill from Southampton (A Structural Analysis of Disappointment Aversion in a Real Effort Competition), George Mailath from Pennsylvania (A Foundation for Markov Equilibria in Infinite Horizon Perfect Information Games), Leon Petrosyan from St. Petersburg (How to Make the Cooperation Stable?), Vladimir Mazalov from Petrozavodsk (Equilibrium in N-Person Game of “Showcase Showdown”), Krzysztof Szajowski from Wrocław (Stopping Games under Partial Information), George Zaccour from HEC Montreal (Investment Dynamics: Good News Principle), Myrna Wooders from Vanderbilt (Share Equilibrium in Local Public Good Economies) and many other outstanding contributors. The conference

was exclusively marked by a rare natural event: the eruption of an Icelandic volcano, which blocked the functioning of most of the airlines, turning the way back for many participants to an adventurous enterprise. In this volume we publish the review of Martin Shubik “The present and future of game theory” and the contributions presenting extending versions of the talks given at the workshop.

*Leon Petrosjan, Vladimir Mazalov, Vassili Kolokoltsov and William McEneaney*

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## *Chapter 1*

# **A ROBUST CONTROL APPROACH TO OPTION PRICING: THE UNIQUENESS THEOREM**

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## **1. Introduction**

In a series of papers starting with [4], and culminating, so far, with [6, 5, 7] we have developed a probability-free theory of option pricing, both for vanilla options and digital options. The most comprehensive account of this theory is in the unpublished doctoral dissertation of Stéphane Thiery [13]. A rather complete account is to appear in the volume [8].

The main claims of that new approach are, on the one hand, the possibility of constructing a consistent theory of hedging portfolios with either continuous or discrete time trading paradigms, the former being the limit of the latter for vanishing time steps, with one and the same (continuous time) market model, and, on the other hand, to accommodate transaction costs and closing costs in a natural way, with a nontrivial hedging portfolio.

It may also be argued that although it seems somewhat un-natural, still our market model implies much less knowledge about the future market prices than the classical probabilistic Samuelson model, used in the Black and Scholes theory. A discussion of the strengths and weaknesses of the new approach, as well as of related contributions in the literature, mostly [1] and [10], can be found in [7].

The reference [13] stresses that the last missing item is a uniqueness theorem for the viscosity solution of a particular, highly degenerate, Isaacs Differential Quasi Variational Inequality (DQVI). In the article [7], we got around that difficulty by resorting to a refined form of Isaacs' verification theorem. However, on the one hand, this relies on the true, but unpublished, fact that the viscosity condition implies satisfaction of our old "corner conditions" [3], and on the other hand, it is much less satisfactory than directly proving that uniqueness.

In this article, we sketch the overall context and prove the uniqueness sought. Notice, however, that the present proof does not account for the discontinuous payment digital option, while that of [3] can be extended to that case, thanks to the concept of barrier.

## 2. Modelization

### 2.1. Option Pricing

Our problem relates to an economy with a fixed, known, riskless interest rate  $\rho$ . In a classical fashion, all monetary values will be assumed to be expressed in end-time value computed at that fixed riskless rate, so that, without loss of generality, the riskless rate can be taken as (seemingly) zero.

We consider a financial derivative called *option* characterized by

- an exercise time, or initial maturity,  $T > 0$ ,
- an underlying security, such as a stock or currency, whose *price* on the market is always well defined. This price at time  $t$  is usually called  $S(t)$ . As indicated above, we shall use instead its end-time price  $u(t) = e^{\rho(T-t)}S(t)$ ,
- a closure payment  $M(u(T))$ . Typical instances are  $M(u) = \max\{u - K, 0\}$  (for a given *exercise price*  $K$ ) for a vanilla call, or  $M(u) = \max\{K - u, 0\}$  for a vanilla put.

### 2.2. Market

We share with Roorda, Engwerda, and Schumacher [12, 11] the view that a market model is a set  $\Omega$  of possible price trajectories, and we borrow from them the name of *interval model* for our model. It is defined by two real numbers  $\tau^- < 0$  and  $\tau^+ > 0$ , and  $\Omega$  is the set of all absolutely continuous functions  $u(\cdot)$  such that for any two time instants  $t_1$  and  $t_2$ ,

$$e^{\tau^-(t_2-t_1)} \leq \frac{u(t_2)}{u(t_1)} \leq e^{\tau^+(t_2-t_1)}. \quad (2.1)$$

The notation  $\tau^\varepsilon$  will be used to handle both  $\tau^+$  and  $\tau^-$  at a time. Hence, in that notation, it is understood that  $\varepsilon \in \{-, +\}$ , sometimes identified to  $\{-1, +1\}$ . We shall also let  $(\tau_\star, \tau^\star) = (\min_\varepsilon |\tau^\varepsilon|, \max_\varepsilon |\tau^\varepsilon|)$ .

We shall make use the equivalent characterization

$$\dot{u} = \tau u, \quad u(0) = u_0, \quad \tau \in [\tau^-, \tau^+]. \quad (2.2)$$

In that formulation,  $\tau(\cdot)$  is a measurable function, which plays the role of the “control” of the market. We shall let  $\Psi$  denote the set of measurable functions from  $[0, T]$  into  $[\tau^-, \tau^+]$ . It is equivalent to specify a  $u(\cdot) \in \Omega$  or a  $(u(0), \tau(\cdot)) \in \mathbb{R}^+ \times \Psi$ . This is an a priori unknown time function. The concept of nonanticipative strategies embodies that fact.

### 2.3. Portfolio

A (hedging) portfolio will be composed of an amount  $v$  (in end-time value) of underlying stock, and an amount  $y$  of riskless *bonds*, for a total worth of  $w = v + y$ . In the normalized (or end-value) representation, the bonds are seemingly with zero interest.

### 2.3.1. Buying and Selling

We let  $\xi(t)$  be the buying rate (a sale if  $\xi(t) < 0$ ), which is the trader's control. Therefore we have, in continuous time,

$$\dot{v} = \tau v + \xi. \quad (2.3)$$

However, there is no reason to restrict the buying/selling rate, so that there is no bound on  $\xi$ . To avoid mathematical ill-posedness, we explicitly admit "infinite" buying/selling rate in the form of instantaneous block buy or sale of a finite amount of stock at time instants chosen by the trader together with the amount. Thus the control of the trader also involves the choice of finitely many time instants  $t_k$  and trading amounts  $\xi_k$ , and the model must be augmented with

$$v(t_k^+) = v(t_k^-) + \xi_k, \quad (2.4)$$

meaning that  $v(\cdot)$  has a jump discontinuity of size  $\xi_k$  at time  $t_k$ . Equivalently, we may keep formula (2.3) but allow for impulses  $\xi_k \delta(t - t_k)$  in  $\xi(\cdot)$ .

We shall therefore let  $\xi(\cdot) \in \Xi$ , the set of real time functions (or rather distributions) defined over  $[0, T]$  which are the sum of a measurable function  $\xi^c(\cdot)$  and a finite number of weighted translated Dirac impulses  $\xi_k \delta(t - t_k)$ .

### 2.3.2. Transaction Costs

We assume that there are transaction costs. In this paper, we assume that they are proportional to the transaction amount. But we allow for different proportionality coefficients for a buy or a sale of underlying stock. Hence let  $C^+$  ( $-C^-$ ) be the cost coefficient for a buy (sale), so that the cost of a transaction of amount  $\xi$  is  $C^\varepsilon \xi$  with  $\varepsilon = \text{sign}(\xi)$ . We have chosen  $C^-$  negative, so that, as it should, that formula always gives a positive cost.

We shall use the convention that when we write  $C^\varepsilon(\text{expression})$ , and except if otherwise specified, the symbol  $\varepsilon$  in  $C^\varepsilon$  stands for the sign of the *expression*. We shall also let  $(C_*, C^*) = (\min_\varepsilon |C^\varepsilon|, \max_\varepsilon |C^\varepsilon|)$ .

Our portfolio will always be assumed *self-financed*; i.e., the sale of one of the commodities, underlying stock or riskless bonds, must exactly pay for the buy of the other one *and* the transaction costs. It is a simple matter to see that the worth  $w$  of the portfolio then obeys

$$\forall t \in (t_{k-1}, t_k), \quad \dot{w} = \tau w - C^\varepsilon \xi^c, \quad w(t_{k-1}) = w(t_{k-1}^+), \quad (2.5)$$

between two jump instants, and at jump instants,

$$w(t_k^+) = w(t_k^-) - C^{\varepsilon_k} \xi_k. \quad (2.6)$$

This is equivalent to

$$\begin{aligned} w(t) &= w(0) + \int_0^t (\tau(s)v(s) - C^\varepsilon \xi(s)) ds \\ &= w(0) + \int_0^t (\tau(s)v(s) - C^\varepsilon \xi^c(s)) ds - \sum_{k|t_k \leq t} C^{\varepsilon_k} \xi_k. \end{aligned} \quad (2.7)$$

## 2.4. Hedging

### 2.4.1. Strategies

The initial portfolio is to be created at step 0. As a consequence the seller's price is obtained taking  $v(0) = 0$ . Then, formally, admissible hedging strategies will be functions  $\varphi : \Omega \rightarrow \Xi$  which enjoy the property of being nonanticipative:

$$\forall (u_1(\cdot), u_2(\cdot)) \in \Omega \times \Omega, \quad [u_1|_{[0,t]} = u_2|_{[0,t]}] \Rightarrow [\varphi(u_1(\cdot))|_{[0,t]} = \varphi(u_2(\cdot))|_{[0,t]}].$$

(It is understood here that the restriction of  $\delta(t - t_k)$  to a closed interval not containing  $t_k$  is 0, and its restriction to a closed interval containing  $t_k$  is an impulse.)

In practice, we shall find optimal hedging strategies made of a jump at initial time, followed by a state feedback law  $\xi(t) = \phi(t, u(t), v(t))$ .

We shall call  $\Phi$  the set of admissible trading strategies.

### 2.4.2. Closing Costs

The idea of a hedging portfolio is that at exercise time, the writer is going to close off its position after abiding by its contract, buying or selling some of the underlying stock according to the necessity. We assume that it sustains proportional costs on this final transaction. We allow for the case where these costs would be different from the running transaction costs because compensation effects might lower them and also allow for the case without closing costs just by making their rate 0. Let therefore  $c^+ \leq C^+$  and  $-c^- \leq -C^-$  be these rates.

It is a simple matter to see that, in order to cover both cases where the buyer does or does not exercise its option, the portfolio worth at final time should be  $N(u, v)$ , given for a call and a closure in kind by

$$N(u, v) = \max\{c^\varepsilon(-v), u - K + c^\varepsilon(u - v)\},$$

where the notation convention for  $c^\varepsilon(\text{expression})$  holds. We expect that on a typical optimum hedging portfolio for a call,  $0 \leq v(T) \leq u(T)$ . Hence

$$N(u, v) = \max\{-c^-v, u - K + c^+(u - v)\}. \quad (2.8)$$

In the case of a put, where  $-u(T) \leq v(T) \leq 0$ , we need to replace the above expression by

$$N(u, v) = \max\{-c^+v, K - u - c^-(u + v)\}. \quad (2.9)$$

The case of a closure in cash is similar but leads to less appealing mathematical formulas in later developments. The details can be found in [5].

### 2.4.3. Hedging Portfolio

An initial portfolio  $(v(0), w(0))$  and an admissible trading strategy  $\varphi$ , together with a price history  $u(\cdot)$ , generate a dynamic portfolio. We set the following.

**Definition 2.1.** *An initial portfolio  $(v(0) = 0, w(0) = w_0)$  and a trading strategy  $\varphi$  constitute a hedge at  $u_0$  if for any  $u(\cdot) \in \Omega$  such that  $u(0) = u_0$  (equivalently, for any admissible  $\tau(\cdot)$ ), the dynamic portfolio thus generated satisfies*

$$w(T) \geq N(u(T), v(T)). \quad (2.10)$$

Now, we may use (2.7) at time  $T$  to rewrite this:

$$\forall \tau(\cdot) \in \Psi, \quad N(u(T), v(T)) + \int_0^T (-\tau(t)v(t) + C^\varepsilon \xi(t)) dt - w_0 \leq 0.$$

This in turn is clearly equivalent to

$$w_0 \geq \sup_{\tau(\cdot) \in \Psi} \left[ N(u(T), v(T)) + \int_0^T (-\tau(t)v(t) + C^\varepsilon \xi(t)) dt \right].$$

We further set the following.

**Definition 2.2.** *The seller's price of the option at  $u_0$  is the worth of the cheapest hedging portfolio at  $u_0$ .*

The seller's price at  $u_0$  is therefore

$$P(u_0) = \inf_{\varphi \in \Phi} \sup_{\tau(\cdot) \in \Psi} \left[ N(u(T), v(T)) + \int_0^T (-\tau(t)v(t) + C^\varepsilon \xi(t)) dt \right], \quad (2.11)$$

where it is understood that  $v(0) = 0$  and that  $\xi(\cdot) = \varphi(u_0, \tau(\cdot))$ .

### 3. Solving the Minimax Impulse Control Problem

#### 3.1. The Related DQVI

We are therefore led to the investigation of the impulse control differential game whose dynamics are given by (2.2), (2.3), and (2.4) and the criterion by (2.11). In a classical fashion we introduce its Isaacs value function:

$$W(t, u, v) = \inf_{\varphi \in \Phi} \sup_{\tau(\cdot) \in \Psi} \left[ N(u(T), v(T)) + \int_t^T (-\tau(s)v(s) + C^\varepsilon \xi(s)) ds \right] \quad (3.1)$$

where the dynamics are integrated from  $u(t) = u$ ,  $v(t) = v$ . Hence the seller's price is  $P(u_0) = W(0, u_0, 0)$ .

There are new features in that game, in that, on the one hand, impulse controls are allowed, and hence an Isaacs quasi-variational inequality (or QVI; see Bensoussan and Lions [2]) should be at work, but, on the other hand, impulse costs have a zero infimum. As a consequence, that QVI is degenerate, and no general result is available. In [6], we introduce the so-called Joshua transformation that lets us show the following fact.

**Theorem 3.1.** *The function  $W$  defined by (3.1) is a continuous viscosity solution of the following “differential QVI” (DQVI):*

$$0 = \min \left\{ \frac{\partial W}{\partial t} + \max_{\tau \in [\tau^-, \tau^+]} \tau \left[ \frac{\partial W}{\partial u} u + \left( \frac{\partial W}{\partial v} - 1 \right) v \right], \right. \\ \left. \frac{\partial W}{\partial v} + C^+, \quad - \left( \frac{\partial W}{\partial v} + C^- \right) \right\}, \quad (3.2)$$

$$W(T, u, v) = N(u, v).$$

This PDE in turn lends itself to an analysis, either along the lines of the Isaacs–Breakwell theory through the construction of a field of characteristics for a transformed game as in [6], or using the theory of viscosity solutions and the representation theorem as outlined in [7]. The solution we seek may further be characterized by its behavior at infinity. Yet its uniqueness does not derive from the classical results on viscosity solutions.

### 3.2. Representation

We introduce two functions  $\check{v}(t, u)$ , a representation of the singular manifold, and  $\check{w}(t, u)$ , the restriction of  $W$  to that manifold, handled jointly as

$$\mathcal{V}(t, u) = \begin{pmatrix} \check{v}(t, u) \\ \check{w}(t, u) \end{pmatrix}.$$

That pair of functions is entirely defined by a linear PDE that involves the following two matrices ( $q^-$  and  $q^+$  are defined hereafter in (3.4)):

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \frac{1}{q^+ - q^-} \begin{pmatrix} \tau^+ q^+ - \tau^- q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-) q^+ q^- & \tau^- q^+ - \tau^+ q^- \end{pmatrix},$$

and it seems to play a very important role in the overall theory. Namely,

$$\mathcal{V}_t + \mathcal{T}(\mathcal{V}_u u - \mathcal{S}\mathcal{V}) = 0. \quad (3.3)$$

The definitions of  $q^+$  and  $q^-$ , as well as the terminal conditions at  $T$  for (3.3), depend on the type of option considered. For a simple call or put, and a closure in kind, we have

$$q^-(t) = \max\{(1 + c^-) \exp[\tau^-(T - t)] - 1, C^-\}, \\ q^+(t) = \min\{(1 + c^+) \exp[\tau^+(T - t)] - 1, C^+\}. \quad (3.4)$$

Notice that  $q^\varepsilon = C^\varepsilon$  for  $t \leq t_\varepsilon$ , with

$$T - t_\varepsilon = \frac{1}{\tau^\varepsilon} \ln \frac{1 + C^\varepsilon}{1 + c^\varepsilon}. \quad (3.5)$$

The terminal conditions are given, for a call, by

$$\mathcal{V}^t(T, u) = \begin{cases} \begin{pmatrix} 0 & 0 \end{pmatrix} & \text{if } u < \frac{K}{1 + c^+}, \\ \frac{(1 + c^+)u - K}{c^+ - c^-} \begin{pmatrix} 1 & -c^- \end{pmatrix} & \text{if } \frac{K}{1 + c^+} \leq u < \frac{K}{1 + c^-}, \\ \begin{pmatrix} u & u - K \end{pmatrix} & \text{if } u \geq \frac{K}{1 + c^-} \end{cases} \quad (3.6)$$

and symmetric formulas for a put. (All combinations call/put, closure in kind/in cash, are detailed in [5]). Standard techniques of hyperbolic PDE's let us prove that that equation has a unique solution with these terminal conditions. (See [13]).

In [7], we proved the following fact:

**Theorem 3.2.** *The function  $W$  defined by the formula*

$$W(t, u, v) = \check{w}(t, u) + q^\varepsilon(\check{v}(t, u) - v), \quad \varepsilon = \text{sign}(\check{v} - v), \quad (3.7)$$

where  $q^\varepsilon$  is given by formula (3.4) (for a simple call or put), and  $(\check{v} \ \check{w}) = \mathcal{V}^t$  is given by (3.3) and the terminal conditions (3.6) for a call (and symmetrical formulas for a put) is a viscosity solution of (3.1).

If the uniqueness of the viscosity solution can be proved, this implies that formula (3.7) indeed provides the Value of the game problem, and hence solves the pricing problem via  $P(u_0) = W(0, u_0, 0)$ . A huge computational advantage as compared to integrating (3.1).

### 3.3. Discrete Trading

We consider also the case where the trader is only allowed bulk trading (“impulses” in the above setting) at predetermined instants of time  $t_k = kh$ ,  $k = 0, 1, \dots, \mathbb{K}$ , with  $h$  a given time step and  $\mathbb{K}h = T$ . Everything else remains unchanged, in particular the market model. A problem interesting in its own right, and, we shall see, as an approximation to the continuous trading solution.

A similar analysis leads to a discrete Isaacs equation,

$$\begin{aligned} \forall k < \mathbb{K}, \forall (u, v), \\ W_k^h(u, v) &= \min_{\xi} \max_{\tau \in [\tau_h^-, \tau_h^+]} [W_{k+1}^h((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v+\xi) + C^\varepsilon \xi], \\ \forall (u, v), \quad W_{\mathbb{K}}^h(u, v) &= N(u, v). \end{aligned} \quad (3.8)$$

A carefull analysis shows that its solution  $\{W_k^h\}_{k \in \{0, \dots, \mathbb{K}\}}$  can be obtained via the following procedure. Notice first that  $q_\ell^\varepsilon := q^\varepsilon(t_\ell)$  can be computed via the recursion

$$\begin{aligned} q_K^\varepsilon &= c^\varepsilon, \\ q_{k+\frac{1}{2}}^\varepsilon &= (1 + \tau_h^\varepsilon) q_{k+1}^\varepsilon + \tau_h^\varepsilon, \\ q_{k+1}^\varepsilon &= \varepsilon \min \{ \varepsilon q_{k+\frac{1}{2}}^\varepsilon, \varepsilon C^\varepsilon \}. \end{aligned} \quad (3.9)$$

Then, let, for all integer  $\ell$

$$Q_\ell^\varepsilon = \begin{pmatrix} q_\ell^\varepsilon & 1 \end{pmatrix}, \quad \text{and} \quad \mathcal{V}_\ell^h(u) = \begin{pmatrix} \check{v}_\ell^h(u) \\ \check{w}_\ell^h(u) \end{pmatrix}. \quad (3.10)$$

Take  $\check{v}_{\mathbb{K}}^h(u) = \check{v}(T, u)$ ,  $\check{w}_{\mathbb{K}}^h(u) = \check{w}(T, u)$  as given by (3.6) for a call (symmetrically for a put) and

$$\mathcal{V}_k^h(u) = \frac{1}{q_{k+\frac{1}{2}}^+ - q_{k+\frac{1}{2}}^-} \begin{pmatrix} 1 & -1 \\ -q_{k+\frac{1}{2}}^- & q_{k+\frac{1}{2}}^+ \end{pmatrix} \begin{pmatrix} Q_{k+1}^+ \mathcal{V}_{k+1}^h((1+\tau^+)u) \\ Q_{k+1}^- \mathcal{V}_{k+1}^h((1+\tau^-)u) \end{pmatrix}. \quad (3.11)$$



We leave to the reader the tedious, but straightforward, task to check that this is indeed a consistent finite difference scheme for (3.3). This provides our preferred fast algorithm to compute the premiums in our theory. As a matter of fact, we claim the following:

**Theorem 3.3.** *The solution of (3.8) is given by (3.9), (3.10), (3.11), and (3.6) for a call, as*

$$W_k^h(u, v) = \check{w}_k^h(u) + q_k^\varepsilon(\check{v}_k^h(u) - v) = Q_k^\varepsilon \mathcal{V}_k^h(u) - q_k^\varepsilon v, \quad \varepsilon = \text{sign}(\check{v}_k^h(u) - v).$$

Finally, the main theorem of [6], and a central result in that theory, is the following convergence theorem. (Which can very probably be extended to a sequence  $h = T/\mathbb{K}$ ,  $\mathbb{K} \rightarrow \infty \in \mathbb{N}$ .) Let  $W^h(t, u, v)$  be the Value function of the minimax problem where the minimizer is allowed to make an impulse at the initial time  $t$ , and then only at the discrete instants  $t_k$  as defined above. It is an interpolation of the sequence  $\{W_k^h\}$ . (This is the correct definition of  $W^h(t, u, v)$ . It only appears in [13].)

**Theorem 3.4.** *Choose  $h = T \times 2^{-n}$ . As  $n \rightarrow \infty$  in  $\mathbb{N}$ ,  $W^h$  converges uniformly on any compact to a viscosity solution  $W$  of the DQVI (3.2).*

Since (3.11) can be viewed as a finite difference scheme for (3.3), it is clear that this limit  $W$  is the same function  $W$  as given by (3.7). But here again, we need a uniqueness theorem of the viscosity solution of (3.2) to conclude that the Value of the discrete trading problem converges towards that of the continuous trading problem.

### 3.4. Uniqueness

At this point, we know that if the viscosity solution of the DQVI (3.2) can be proved unique, we have both an interesting representation formula (3.7) for the value function of the continuous trading problem, and a fast algorithm (3.11) to approximate it via the Value function of the discrete trading problem.

In order to exploit the technical result of the next section, we need to introduce a modified problem.

Let  $R$  be a fixed positive number, and  $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$  be the region  $u \in [0, R]$ ,  $|v| \leq R$ . For the time being, we consider only problems of option hedging where  $(u(0), v(0)) \in \mathcal{R}$ .

As a consequence, for these problems, and for all  $t \in [0, T]$ , we have  $u(t) \leq Re^{\tau^+ T}$ .

Concerning  $v(\cdot)$ , the control  $\xi$  might send it anywhere in  $\mathbb{R}$ . But we know from the analysis according to the Isaacs-Breakwell theory that the minimizing strategies never create large  $v(t)$ 's. As a matter of fact, let  $W_0$  be the maximum payoff obtained by the strategy  $\varphi = 0$  (after maximization in  $\tau(\cdot)$ ) for any  $(u_0, v_0) \in \mathcal{R}$ . Let  $a$  be a large number, chosen satisfying  $a > 2 \exp(\tau^+ T)[W_0/(RC_\star) + 1]$ , and  $S = aR$ . We claim the following:

**Proposition 3.1.** *Any nonanticipative strategy  $\phi$  that causes  $|v| > S$  is dominated by the strategy  $\phi = 0$ .*

**Proof.** Let  $\zeta$  be a positive number,  $\zeta < C_\star \exp(-\tau^+ T)/4\tau^+$ . Any nonanticipative strategy  $\phi$  may be challenged by the control function generated by the following rule: If  $v(t - \zeta) > S$ , choose  $\tau(t) = \tau^-$  if  $v(t - \zeta) < -R$ ; choose  $\tau(t) = \tau^+$ . Due to the small time delay