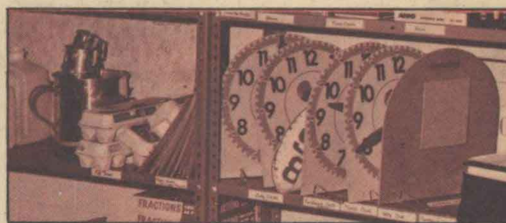
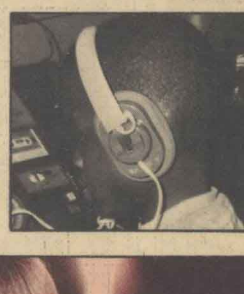
Teaching school mathematics

W. Servais and T. Varga

A Unesco source book





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Preface

Curriculum development considered in its widest sense has been recognized as the main generating force not only for quality in education but also for fostering the ability to assimilate changes, especially those due to the rapid scientific and technical evolution now in progress. This book is designed to assist those involved in curriculum planning and development by providing information on new approaches and experiences, on new methods and techniques in the teaching of mathematics - a subject of considerable concern in curriculum-reform programmes.

The introduction of a new curriculum involves a change in the very purposes of education and in teaching practices. Thus curriculum has different meanings in various contexts and at different working levels. What is offered here is a survey of new approaches to teaching mathematics, including both practice and theory - from examples of classroom application to the problems of identifying the processes by which mathematics learning takes place. For those readers on the operational level there is provided a selection of syllabuses, a survey of available teaching aids and materials, and a reference appendix to the fundamental concepts of the new mathematics. A comprehensive list for further reading is to be found in the bibliography. Thus it is hoped that the reader, whether working in mathematical education as a curriculum planner, teacher-educator or practising teacher, will find material of interest. Despite the increasing importance of the primary stage in the reform movement, the present book focuses on the secondary level where the mathematics curriculum reform started and has been more readily applicable.

As with the teaching of any science, the teaching of mathematics must necessarily keep pace with advances in the field of mathematics itself. Towards this end, various national and international professional organizations are making sustained efforts to promote the fundamental reforms necessary for the improvement of school mathematics teaching. Mention is made here of one of Unesco's activities in this field, a meeting of experts, the International Symposium on School Mathematics Teaching, convened in Budapest by the Hungarian National Commission for Unesco with the participation of the Organization. At this meeting recommendations were formulated concerning mathematics curricula and the furthering of mathematics teaching to serve as international guidelines. Reference to the Budapest recommendations – a valid and far-reaching statement of objectives for the reform – is made throughout the book.

To prepare this book Unesco approached two experts in the field of mathematics teaching: Willy Servais of the Institut Supérieur de Pédagogie and Préfet des Études at the Athenée Provincial du Centre in Morlanwelz, Belgium, and Tamas Varga, Research Officer of the National Institute of Education in Budapest. Mr Servais and Mr Varga served as consultant editors for this volume, but also wrote major portions of the text, contributing in the areas of their special interest and concern. They turned to the following leading mathematicians and mathematics educators for contributions: Matts Håstad, Secretary of the Nordic Committee for the Modernization of School Mathematics; Anna Zofia Krygowska, Professor of Mathematics and Mathematical Education, Teacher Training College, Cracow; Geoffrey Matthews, Project Organizer of the Nuffield Foundation Mathematics Teaching Project; E. A. Peel, Professor of Educational Psychology, Department of Education, University of Birmingham; and Angelo Pescarini, teacher of mathematics in the secondary schools of Ravenna. These contributions reflect the different experiences and approaches being tried in curricula and methods. To present adequately the current situation in mathematics teaching, the manuscript was submitted to other specialists and educators whose comments and suggestions were incorporated in the book.

Any opinions expressed herein are those of the editors and the authors and do not necessarily reflect the views of Unesco.

It is planned to follow this first title with other guides to the teaching of the basic sciences: biology, chemistry and physics. Science teaching has proved to be a field which most countries are anxious to develop and certainly it represents a marked priority within the national scheme of educational development.

General Introduction

Prepared by T. Varga

0.1 The reform of mathematics teaching

Towards the middle of this century a vast international movement emerged, aimed at making profound changes in mathematical education. Groups, centres, projects and societies were formed with the object of reshaping school mathematics. Research mathematicians, psychologists, teachers and educationists all became active in the reform both of the content and of the methods of teaching. Individuals and centres joined forces, both within and between countries, and experimental courses were set up.

What are the motives underlying the reform?

One of the possible answers is the conflict between *demands* and *accomplishment*.

The science of mathematics is expanding rapidly; school mathematics lags behind by centuries. Social and technical progress depend more and more on up-to-date mathematics in an increasing range of professions. This is because mathematics is becoming a more flexible tool than it ever was in many fields of life and culture, old and new alike. Computers, as results and promoters of the progress in mathematics and technology, deserve a special mention. Their rapid spread is an important component of the process of mathematization.

Under the pressure of such factors, vocational training and higher education are overburdened with mathematical topics. This pressure is transmitted to schools which are required to modernize their mathematical training.

Yet there is an opposite pressure, resulting from the lack of ability. For the the majority of pupils even the present curriculum seems to be over full and too difficult. There is much evidence that the mathematical thinking of most pupils does not reach a very high level. Much of the effort these pupils would need to continue their mathematical studies could certainly be put to better use.

The contrast between demands and accomplishment appears to be irreconcilable. Those who meet different facets of the problem come to quite opposite conclusions: one would increase the subject matter, another would decrease it; one would move the teaching of certain topics to lower levels, another towards higher grades. Some recent curricular reforms seem to be the resultants of these two forces, without much attempt to find the roots of the problem.

The awareness that more thorough changes are needed, not just the insertion or exclusion or shift of topics, marks a crucial turning point. The whole curriculum, from the very beginning, is seen to need revision both in content and sequence. Teaching methods must also be re-examined. The reform stands or falls by the teachers; their training and re-training, at every level, is fundamental.

Knowing what is needed would not itself have led to the development of the reform movement. The recognition of new possibilities gave it momentum, and instilled the hope that real advances, not half measures, might be reached in the field of mathematics teaching. This optimism has been fostered both by an appreciation of recent developments in mathematics and psychology, and by actual teaching experience.

In this century mathematics moved away from what is generally called school mathematics. This is one fact. Another is that its core became in a sense more integrated, more coherent and thereby more suitable for building a new 'school mathematics'.

Psychology, too, has made its contribution. In particular, genetic child psychology and various learning theories have led to results that pay a dividend in the learning of mathematics, especially by young children.

It is not only the mere theoretical development in mathematics or in psychology which has given rise to optimism, but the experiences to which they have led. This has helped to dispel the sceptical view according to which understanding mathematics is the privilege of a select few endowed with peculiar innate abilities. This view is readily accepted by those who understand mathematics (for they are, then, an *élite*) and also by those who do not (for then they cannot be blamed for it). Most reformers are none the less convinced that the ability to attain a high level of mathematical culture is within the reach of human beings in general, not of only a select company. This conviction is rooted in the teaching experience mentioned above and has led to further experiment.

From these experiments new ideas and principles are now taking shape, related partly to a new system of school mathematics and its teaching, partly to the strategy of the reform movement, and partly to problems of realizing the reform in practice. Some of these ideas and principles are set out below.

0.2 New and old

In speaking of the reform of mathematics teaching the words 'new' or 'modern' are often used, in this work as elsewhere, as terms of praise. The words 'old' or 'traditional' are accordingly used in a pejorative sense. Such labels should not suggest that a topic is to be rejected solely because it is not recent, or another preferred because it is recent (or labelled as recent). No successful and durable reform may be conceived without a reasonable knowledge of both the old and the new in this field, their evaluation, the integration of what has been found valuable and the rejection of what has become worthless in them.

Here is an example. Comenius (Jan Amos Komensky) revolted centuries ago against the verbal, memorizing way of teaching that survived from the Middle Ages (when it was justified by lack of printing machines). Experience, said Comenius, should be the starting point. Yet because of the inertia of education his principles have not yet been put into practice: the exposition of ready-made knowledge and the overemphasis on verbal memory as opposed to experience and understanding are all too frequent today, even in mathematics teaching. In this respect the principles of Comenius are still 'new' and 'modern'.

0.3 The reform of content and teaching method

Every balanced reform project seeks to modernize both content and teaching methods. Under given conditions one or other may be more important or more urgent, yet 'both in mutual assistance' is usually considered as a better policy than 'either . . . or . . .'. Routine content automatically entails routine ways of presentation.

Content can be analysed into content proper ('What to teach?') and its inner organization, this latter being most closely related to teaching methods. Similarly, teaching methods can be analysed into presentation of the subject matter (e.g. the use of graphical devices or models) and the organization of classroom work, the former being most closely related to the content.

Table 1

Content		*Teaching methods*	
Content proper (What to teach?)	Organization of the content	Presentation of the content	Organization of classroom work

Some reform trends pay more attention to content and others to teaching methods. The view that the reform of the content is more important than that of the teaching methods is rarely stated explicitly. It is, however, implicit in many publications and schedules. Wherever the reform moves mainly from above (from authorities) downwards, there is a tendency to emphasize content. The content – what to teach – can easily be imposed upon the teacher. Less easy to control is the way it is organized into the curriculum, still less its presentation. Least susceptible of all is the most properly pedagogical activity of organizing children's work. But all of these can be communicated from person to person, like an epidemic. This is an indispensable counterpart, or a desirable alternative, to the introduction of new curricula by higher authorities. If new and more appropriate curricula have been introduced and they still do not pay the expected dividends, the trouble may lie with person-to-person communication of the ingredients.

0.4 Mathematics as a tool and as an autonomous science

In learning mathematics, as in learning a language, use is the best starting point. Few are interested in the structure (mathematical or grammatical) of what they have not experienced in use. If they have, the growing appreciation of the structure is fundamental in promoting correct and efficient use.

Some are anxious lest school mathematics should become by reform too theoretical instead of more practical, as if these two aspects exclude rather than strengthen each other.

Look at mathematical logic. Some decades ago this seemed to be one of the most remote mathematical disciplines to which the rest of mathematics was sometimes opposed as 'concrete mathematics'. Suddenly it has turned out that apart from being theoretical it is also extremely practical. A basic knowledge of logic is, for example, indispensable in understanding and using computers. Logic is now seen as fitting well into the school curriculum and adding much to its theoretical and practical value.

Other topics, elements of which may have a similar twin effect on school mathematics, include mathematical analysis, linear algebra, probability and statistics, information theory and game theory.

To produce this twin effect, it is not enough to have such topics represented in the curriculum. Teachers are needed who themselves think both in terms of applications and in terms of pure mathematics and who can transfer those ways of thinking and of doing to children.

The following example may help to bring home the point. If we, as teachers, suggest to children a useful heuristic rule such as: 'In order to determine three unknowns try to find three conditions', then we train them in the spirit of applications and, of course, heuristic applications. There is no theorem of pure mathematics to the effect that three conditions are either sufficient or necessary to determine three unknowns. Even if we restrict conditions to equations, there is none. Further strong restrictions are needed to guarantee absolute validity to such a statement. In mathematics the proper word for absolute validity is: validity – both in pure and in applied mathematics. Mathematics is a whole; only the aspects differ. From the point of view of applications our primary concern is not validity, at least not to the same extent as in pure mathematical thinking. The time element comes to the fore. We run a risk. But we must know we run a risk. Teachers and pupils, applied mathematicians and academic mathematicians alike must clearly distinguish between heuristic rule and theorem, plausible reasoning and proof.

The traditional teacher has a horror of speaking in inexact terms. But he is not always able to be exact. This makes him inclined to blur the frontier between what is a theorem or a proof and what is not. The clear distinction between these is not exclusive to pure mathematics. An awareness of this distinction enables us to treat the subject in both its theoretical and practical aspects.

In order to see clearly where exactness can or cannot be expected, the distinction between *physical systems* and their *mathematical models** is vital. The idea is old but its consequent realization in schools is a new and promising feature of the new trends.

Certain aspects of a physical system (e.g. a moving body or a statistical sample) can be more or less exactly characterized by mathematical models (e.g. functions). Usually one may choose between models that fit less well to the system but are simpler and more manageable, and others that fit better but are more complicated and more cumbersome. It is usually much harder to find (and harder still to invent) a mathematical model suitable for solving a problem about a physical system and to interpret the result obtained within the mathematical model in terms of the physical system, than to solve just that part of the problem which falls within the mathematical model. The translation from and to the physical system rarely – and then only partially – lends itself to algorithmization; it demands much of sound judgement and intuition. Every teacher of mathematics knows how much more difficult it is to teach the solution of 'word problems' by means of equations than solving the equations themselves, in spite of the fact that word problems in books usually lead to much simpler equations than those extracted from real-life situations and are often little more than straightforward translations of equations.

School mathematics is further detached from real life by an excessive use of simple whole numbers, in order to avoid long and tedious calculations. The best way to get rid of such time wasting is, however, not always by using simple numbers, nor by avoiding numerical data, but by using calculating machines and other devices such as the slide rule.

Such devices, applied in real-life situations, help to develop in children a feeling for order of magnitude and reasonable approximations, and skill in estimation or in the use of rapid rough calculations of numerical results.

All in all, a balance between numerical and non-numerical problems, of widely differing origins, is generally expected from new mathematics teaching.

*The word 'model' is used in mathematics in another, nearly opposite sense as a *mathematical model satisfying a system of axioms*. The point is that a system of axioms is more abstract than its model, but a physical system is less abstract than its mathematical model. If we insist on speaking about models of axiom systems then it would be more reasonable to speak of a physical *model* and a mathematical *system* describing it, than the other way round.

Where to put the emphasis is a matter for consideration and is to be answered according to the situation.

0.5 Mathematics as an art

Many students leave school without ever having felt the beauty of mathematics. More often than not they take the opposite view.

One of the fundamental aims of the present reforms is to help pupils enjoy mathematics, to make them realize its beauty. As a beginning, the fear and anxiety so often raised in them should be removed. Essential to this approach is freedom of expression, arising from playful activity. To realize and enjoy the beauty of mathematics, pupils must be given sufficient opportunity for free, playful, creative activity, where each can bring out his own measure of wit, taste, fantasy, and display thereby his personality.

Students who have a feeling for beauty realize more easily, for example, that mathematics gains in beauty if we put $3^0 = 1$ and $3^{-2} = \frac{1}{9}$ rather than accepting, say $3^0 = 0$ and $3^{-2} = -9$. They find this harmonious and they also find it useful because it fits into and extends the pattern of earlier knowledge. Such examples make them aware of how beauty can show the way towards utility.

An important type of problem developing (and making use of) the sense of beauty is the search for patterns.

Mathematical recreations also have a role in helping pupils to like mathematics. Puzzles in mathematics are somewhat similar to songs in music: short and self-contained, not too ambitious and accessible to many. Some are the personal inventions of creators of mathematics; others are of unknown origin, becoming polished through centuries, emerging here and there in different variants like folk songs. They are also like anecdotes, which often point to deep ideas. Think of their role in the development of topology, probability or logic. Puzzles can be excellent starting points and incentives for deep ideas in school mathematics as in creative mathematics itself.

0.6 Mathematics as a whole

One of the main disadvantages of traditional school mathematics is its piecemeal character; on this there is general agreement. In the new-style curricula, however different they are, unifying tendencies emerge. *Set*, *relation*, *function*, *group*, *vector* and many others are 'unifying concepts'. Their place in the curriculum is the object of much controversy and experiment.

Those who prefer later introduction, because these concepts are too abstract for children, think it is preferable first to meet a number of special cases in order to have a firm base for generalization. Those who are for an early introduction think that arriving at a reasonably general concept through concrete