Algebraic Graph Theory

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Preface

It is a pleasure for me to acknowledge the help which I have received during the preparation of this book. A preliminary draft of the manuscript was read by Dr R. J. Wilson, and his detailed comments resulted in substantial changes and improvements. I was then fortunate to be able to rely upon the expert assistance of my wife for the production of a typescript. Ideas and helpful criticisms were offered by several friends and colleagues, among them G. de Barra, R. M. Damerell, A. D. Gardiner, R. K. Guy, P. McMullen and J. W. Moon. The general editor of the Cambridge Mathematical Tracts, Professor C. T. C. Wall, was swift and perceptive in his appraisal, and his comments were much appreciated. The staff of the Cambridge University Press maintained their usual high standard of courtesy and efficiency throughout the process of publication.

During the months January–April 1973, when the final stages of the writing were completed, I held a visiting appointment at the University of Waterloo, and my thanks are due to Professor W. T. Tutte for arranging this. In addition, I owe a mathematical debt to Professor Tutte, for he is the author of the two results, Theorems 13.9 and 18.6, which I regard as the most important in the book. I should venture the opinion that, were it not for his pioneering work, these results would still be unknown to this day.

NORMAN BIGGS

Waterloo, Canada
March 1973

Contents

1. *Introduction*

This book is concerned with the use of algebraic techniques in the study of graphs. We aim to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs.

The exposition which we shall give is not part of the modern functorial approach to topology, despite the claims of those who hold that, since graphs are one-dimensional spaces, graph theory is merely one-dimensional topology. By that definition, algebraic graph theory would consist only of the homology of 1-complexes. But the problems dealt with in graph theory are more delicate than those which form the substance of algebraic topology, and even if these problems can be generalized to dimensions greater than one, there is usually no hope of a general solution at the present time. Consequently, the algebra used in algebraic graph theory is largely unrelated to the subject which has come to be known as homological algebra.

This book is not an introduction to graph theory. It would be to the reader's advantage if he were familiar with the basic concepts of the subject, for example, as they are set out in the book by R. J. Wilson entitled *Introduction to graph theory*. However, for the convenience of those readers who do not have this background, we give brief explanations of important standard terms. These explanations are usually accompanied by a reference to Wilson's book (in the form [W, p. 99]), where further details may be found. In the same way, some concepts from permutation-group theory are accompanied by a reference [B, p. 99] to the author's book *Finite groups of automorphisms*. Both these books are described fully at the end of this chapter.

A few other books are also referred to for results which may be unfamiliar to some readers. In such cases, the result required is necessary for an understanding of the topic under discussion, so that the reference is given in full, enclosed in square brackets, where it is needed. Other references, of a supplementary nature, are given in parentheses in the form (Smith 1971) or Smith (1971). In such cases, the full reference may be found in the bibliography at the end of the book.

The tract is in three parts, each of which is further subdivided into a number of short chapters. Within each chapter, the major definitions and results are labelled using the decimal system.

The first part deals with the applications of linear algebra and matrix theory to the study of graphs. We begin by introducing the adjacency matrix of a graph; this matrix completely determines the graph, and its spectral properties are shown to be related to properties of the graph. For example, if a graph is regular, then the eigenvalues of its adjacency matrix are bounded in absolute value by the valency of the graph. In the case of a line graph, there is a strong lower bound for the eigenvalues.

Another matrix which completely describes a graph is the incidence matrix of the graph. This matrix represents a linear mapping which, in modern language, determines the homology of the graph; however, the sophistication of this language obscures the underlying simplicity of the situation. The problem of choosing a basis for the homology of a graph is just that of finding a fundamental system of circuits, and we solve this problem by using a spanning tree in the graph. At the same time we study the cutsets of the graph. These ideas are then applied to the systematic solution of network equations, a topic which supplied the stimulus for the original theoretical development.

We then investigate various formulae for the number of spanning trees in a graph, and apply these formulae to several well-known families of graphs. The first part of the book ends with results which are derived from the expansion of certain determinants, and which illuminate the relationship between a graph and the characteristic polynomial of its adjacency matrix.

The second part of the book deals with the problem of colouring the vertices of a graph in such a way that adjacent vertices have different colours. The least number of colours for which such a colouring is possible is called the chromatic number of the graph, and we begin by investigating some connections between this number and the eigenvalues of the adjacency matrix of the graph.

The algebraic technique for counting the colourings of a graph is founded on a polynomial known as the chromatic polynomial. We first discuss some simple ways of calculating this polynomial, and show how these can be applied in several important cases. Many important properties of the chromatic polynomial of a graph stem from its connection with the family of subgraphs of the graph, and we show how the chromatic polynomial can be expanded in terms of subgraphs. From our first (additive) expansion another (multiplicative) expansion can be derived, and the latter depends upon a very restricted class of subgraphs. This leads to efficient methods for approximating the chromatic polynomials of large graphs.

A completely different kind of expansion relates the chromatic polynomial to the spanning trees of a graph; this expansion has several remarkable features and leads to new ways of looking at the colouring problem, and some new properties of chromatic polynomials.

The third part of the book is concerned with symmetry and regularity properties. A symmetry property of a graph is related to the existence of automorphisms – that is, permutations of the vertices which preserve adjacency. A regularity property is defined in purely numerical terms. Consequently, symmetry properties induce regularity properties, but the converse is not necessarily true.

We first study the elementary properties of automorphisms, and explain the connection between the automorphisms of a graph and the eigenvalues of its adjacency matrix. We then introduce a hierarchy of symmetry conditions which can be imposed on a graph, and proceed to investigate their consequences. The condition that all vertices be alike (under the action of the group of automorphisms) turns out to be rather a weak one, but a slight strengthening of it leads to highly non-trivial conclusions. In fact, under certain conditions, there is an absolute bound to the level of symmetry which a graph can possess.

A new kind of symmetry property, called distance-transitivity, and the consequent regularity property, called distanceregularity, are then introduced. We return to the methods of linear algebra to derive strong constraints upon the existence of graphs with these properties. Finally, these constraints are applied to the problem of finding minimal regular graphs whose valency and girth are given.

At the end of each chapter there are some supplementary results and examples, labelled by the number of the chapter and a letter (as, for example, 9A). The reader is warned that these results are variable in difficulty and in kind. Their presence allows the inclusion of a great deal of material which would otherwise have interrupted the mainstream of the exposition, or would have had to be omitted altogether.

We end this introductory chapter by describing the few ways in which we differ from the terminology of Wilson's book.

In this book, a *general graph* Γ consists of three things: a finite set $V\Gamma$ of *vertices*, a finite set $E\Gamma$ of *edges*, and an *incidence* relation between vertices and edges. If v is a vertex, e is an edge, and (v, e) is a pair in the incidence relation, then we say that v is *incident* with e, and e is *incident* with v. Each edge is incident with either one vertex (in which case it is a *loop*) or two vertices.

If each edge is incident with two vertices, and no two edges are incident with the same pair of vertices, then we say that Γ is a *simple graph* or briefly, a *graph*. In this case, $E\Gamma$ can be identified with a subset of the set of (unordered) pairs of vertices, and we shall always assume that this identification has been made. We shall deal mainly with graphs (that is, simple graphs), except in Part Two, where it is sometimes essential to consider general graphs.

If v and w are vertices of a graph Γ, and $e = \{v, w\}$ is an edge of Γ, then we say that e *joins* v and w, and that v and w are the *ends* of e. The number of edges of which v is an end is called the *valency* of v.

We consider two kinds of subgraph of a general graph Γ. An *edge-subgraph* of Γ is constructed by taking a subset S of $E\Gamma$ together with all vertices of Γ incident in Γ with some edge belonging to S. A *vertex-subgraph* of Γ is constructed by taking a subset U of $V\Gamma$ together with all edges of Γ which are incident in Γ only with vertices belonging to U. In both cases the incidence relation in the subgraph is inherited from the incidence relation in Γ. We shall use the notation $\langle S \rangle_{\Gamma}$, $\langle U \rangle_{\Gamma}$ for these subgraphs, and usually, when the context is clear, the subscript reference to Γ will be omitted.

Further new terminology and notation will be defined when it is required.

Basic references

R. J. Wilson. *Introduction to graph theory* (Oliver and Boyd, Edinburgh, 1972).

N. L. Biggs. *Finite groups of automorphisms*, London Math. Society Lecture Notes Series, No. 6 (Cambridge University Press, 1971).

PART ONE

Linear algebra in graph theory

2. *The spectrum of a graph*

We begin by defining a matrix which will play an important role in many parts of this book. We shall suppose that Γ is a graph whose vertex-set $V\Gamma$ is the set $\{v_1, v_2, \ldots, v_n\}$; as explained in Chapter 1, we shall take $E\Gamma$ to be a subset of the set of unordered pairs of elements of $V\Gamma$. If $\{v_i, v_j\}$ is an edge, then we say that v_i and v_j are *adjacent*.

DEFINITION 2.1 The *adjacency matrix* of Γ is the $n \times n$ matrix $\mathbf{A} = \mathbf{A}(\Gamma)$, over the complex field, whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

It follows directly from the definition that $\mathbf{A}$ is a real symmetric matrix, and that the trace of $\mathbf{A}$ is zero. Since the rows and columns of $\mathbf{A}$ correspond to an arbitrary labelling of the vertices of Γ, it is clear that we shall be interested primarily in those properties of the adjacency matrix which are invariant under permutations of the rows and columns. Foremost among such properties are the spectral properties of $\mathbf{A}$.

Suppose that λ is an eigenvalue of $\mathbf{A}$. Then, since $\mathbf{A}$ is real and symmetric, λ is real, and the multiplicity of λ as a root of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ.

DEFINITION 2.2 The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $\mathbf{A}(\Gamma)$, together with their multiplicities as eigenvalues of $\mathbf{A}(\Gamma)$. If the distinct eigenvalues of $\mathbf{A}(\Gamma)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\operatorname{Spec}\Gamma = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

For example, the *complete graph* K_n has n vertices, and each distinct pair are adjacent [W, p. 16]. Thus, the graph K_4 has adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and an easy calculation shows that the spectrum of K_4 is:

$$\operatorname{Spec} K_4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

We shall often refer to the eigenvalues of $\mathbf{A}(\Gamma)$ as the *eigenvalues of* Γ. Also, the characteristic polynomial of $\mathbf{A}(\Gamma)$ will be denoted by $\chi(\Gamma; \lambda)$, and referred to as the *characteristic polynomial of* Γ.

Let us suppose that the characteristic polynomial of Γ is

$$\chi(\Gamma; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \ldots + c_n.$$

Then the coefficients c_i can be interpreted as sums of principal minors of $\mathbf{A}$, and this leads to the following simple result.

Proposition 2.3 *Using the notation given above, we have:*

(1) $c_1 = 0$;

(2) $-c_2$ *is the number of edges of* Γ;

(3) $-c_3$ *is twice the number of triangles in* Γ.

Proof For each $i \in \{1, 2, \ldots, n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of $\mathbf{A}$ which have i rows and columns. Thus:

(1) Since the diagonal elements of $\mathbf{A}$ are all zero, $c_1 = 0$.

(2) A principal minor with two rows and columns, and which has a non-zero entry, must be of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

There is one such minor for each pair of adjacent vertices of Γ, and each has value -1. Hence $(-1)^2 c_2 = -|E\Gamma|$, giving the result.

(3) There are essentially three possibilities for non-trivial principal minors with three rows and columns:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix},$$

and, of these, the only non-zero one is the last (whose value is 2). This principal minor corresponds to three mutually adjacent vertices in Γ, and so we have the required description of c_3. □

These elementary results indicate that the characteristic polynomial of a graph is a typical object of the kind one considers in algebraic theory: it is an algebraic construction which contains graphical information. Proposition 2.3 is just a pointer, and we shall obtain a more comprehensive result on the coefficients of the characteristic polynomial in Chapter 7.

Suppose $\mathbf{A}$ is the adjacency matrix of a graph Γ. Then the set of polynomials in $\mathbf{A}$, with complex coefficients, forms an algebra under the usual matrix operations. This algebra has finite dimension as a complex vector space.

DEFINITION 2.4 The *adjacency algebra* of a graph Γ is the algebra of polynomials in the adjacency matrix $\mathbf{A} = \mathbf{A}(\Gamma)$. We shall denote the adjacency algebra of Γ by $\mathscr{A}(\Gamma)$.

Since every element of the adjacency algebra is a linear combination of powers of $\mathbf{A}$, we can obtain results about $\mathscr{A}(\Gamma)$ from a study of these powers. We define a *walk* of length l in Γ, joining v_i to v_j, to be a finite sequence of vertices of Γ,

$$v_i = u_0, u_1, \ldots, u_l = v_j,$$

such that u_{t-1} and u_t are adjacent for $1 \leqslant t \leqslant l$. (If u_{t-1} and u_{t+1} are distinct, $1 \leqslant t \leqslant l-1$, then we say that the walk is a *path*.)

LEMMA 2.5 *The number of walks of length l in Γ, joining v_i to v_j, is the entry in position (i, j) of the matrix $\mathbf{A}^l$.*