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Series Editor: Michel Chipot

Thomas C. Sideris

Ordinary Differential Equations and Dynamical Systems

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Dedicated to my father on the ocassion of his 90th birthday

Preface

Many years ago, I had my first opportunity to teach a graduate course on ordinary differential equations at UC, Santa Barbara. Not being a specialist, I sought advice and suggestions from others. In so doing, I had the good fortune of consulting with Manoussos Grillakis, who generously offered to share his lovely notes from John Mallet-Paret's graduate course at Brown University. These notes, combined with some of my own home cooking and spiced with ingredients from other sources, evolved over numerous iterations into the current monograph.

In publishing this work, my goal is to provide a mathematically rigorous introduction to the beautiful subject of ordinary differential equations to beginning graduate or advanced undergraduate students. I assume that students have a solid background in analysis and linear algebra. The presentation emphasizes commonly used techniques without necessarily striving for completeness or for the treatment of a large number of topics. I would half-jokingly subtitle this work as "ODE, as told by an analyst."

The first half of the book is devoted to the development of the basic theory: linear systems, existence and uniqueness of solutions to the initial value problem, flows, stability, and smooth dependence of solutions upon initial conditions and parameters. Much of this theory also serves as the paradigm for evolutionary partial differential equations. The second half of the book is devoted to geometric theory: topological conjugacy, invariant manifolds, existence and stability of periodic solutions, bifurcations, normal forms, and the existence of transverse homoclinic points and their link to chaotic dynamics. A common thread throughout the second part is the use of the implicit function theorem in Banach space. Chapter 5, devoted to this topic, serves as the bridge between the two halves of the book.

A few features (or peculiarities) of the presentation include:

- a characterization of the stable, unstable, and center subspaces of a linear operator in terms of its exponential,
- a proof of smooth dependence using the implicit function theorem,
- a simple proof of the Hartman-Grobman theorem by my colleague, Michael Crandall,
- treatment of the Hopf bifurcation using both normal forms and the Liapunov-Schmidt method,

- a treatment of orbital stability of periodic orbits using the Liapunov-Schmidt method, and
- a complete proof of the existence of transverse homoclinic points for periodic perturbations of Newton's Equation.

I am most grateful to Prof. Grillakis for sharing his notes with me, and I thank both Profs. Grillakis and Mallet-Paret for their consent to publish my interpretation of them. I also thank Prof. Michel Chipot, the Series Editor, for encouraging me to publish this monograph.

Santa Barbara, July 2013 Thomas C. Sideris

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Chapter 1
Introduction

The most general nth order ordinary differential equation (ODE) has the form

$$F(t, y, y', \dots, y^{(n)}) = 0,$$

where F is a continuous function from some open set $\Omega \subset \mathbb{R}^{n+2}$ into $\mathbb{R}$. An n times continuously differentiable real-valued function $y(t)$ is a solution on an interval I if

$$F(t, y(t), y'(t), \dots, y^{(n)}(t)) = 0, \quad t \in I.$$

A necessary condition for existence of a solution is the existence of points $p = (t, y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+2}$ such that $F(p) = 0$. For example, the equation

$$(y')^2 + y^2 + 1 = 0$$

has no (real) solutions, because $F(p) = y_2^2 + y_1^2 + 1 = 0$ has no real solutions.

If $F(p) = 0$ and $\frac{\partial F}{\partial y_{n+1}}(p) \neq 0$, then locally we can solve for y_{n+1} in terms of the other variables by the implicit function theorem

$$y_{n+1} = G(t, y_1, \dots, y_n),$$

and so locally we can write our ODE as

$$y^{(n)} = G(t, y, y', \dots, y^{(n-1)}).$$

This equation can, in turn, be written as a first order system by introducing additional unknowns. Setting

$$x_1 = y, \; x_2 = y', \; \dots, \; x_n = y^{(n-1)},$$

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we have that

$$x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x_{n-1}' = x_n,\ x_n' = G(t, x_1, \ldots, x_n).$$

Therefore, if we define n-vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad f(t,x) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ G(t, x_1, \ldots, x_{n-1}, x_n) \end{bmatrix}$$

we obtain the equivalent first order system

$$x' = f(t, x). \tag{1.1}$$

The point of this discussion is that there is no loss of generality in studying the first order system (1.1), where $f(t, x)$ is a continuous function (at least) defined on some open region in $\mathbb{R}^{n+1}$.

A fundamental question that we will address is the existence and uniqueness of solutions to the initial value problem (IVP)

$$x' = f(t, x), \quad x(t_0) = x_0,$$

for points (t_0, x_0) in the domain of $f(t, x)$. We will then proceed to study the qualitative behavior of such solutions, including periodicity, asymptotic behavior, invariant structures, etc.

In the case where $f(t, x) = f(x)$ is independent of t, the system is called *autonomous*. Every first order system can be rewritten as an autonomous one by introducing an extra unknown. If

$$z_1 = t,\ z_2 = x_1,\ \ldots,\ z_{n+1} = x_n,$$

then from (1.1) we obtain the equivalent autonomous system

$$z' = g(z), \quad g(z) = \begin{bmatrix} 1 \\ f(z) \end{bmatrix}.$$

Suppose that $f(x)$ is a continuous map from an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$. We can regard a solution $x(t)$ of an autonomous system

$$x' = f(x), \tag{1.2}$$

as a curve in $\mathbb{R}^n$. This gives us a geometric interpretation of (1.2). If the vector $x'(t) \neq 0$, then it is tangent to the solution curve at $x(t)$. The Eq. (1.2) tells us what the value of this tangent vector must be, namely, $f(x(t))$. So if there is one and only one solution through each point of U, we know just from the Eq. (1.2) its tangent direction at every point of U. For this reason, $f(x)$ is called a *vector field* or *direction field* on U.

The collection of all solution curves in U is called the *phase diagram* of $f(x)$. If $f \neq 0$ in U, then locally, the curves are parallel. Near a point $x_0 \in U$ where $f(x_0) = 0$, the picture becomes more interesting.

A point $x_0 \in U$ such that $f(x_0) = 0$ is called, interchangably, a *critical point*, a *stationary point*, or an *equilibrium point* of f. If $x_0 \in U$ is an equilibrium point of f, then by direct substitution, $x(t) = x_0$ is a solution of (1.2). Such solutions are referred to as *equilibrium* or *stationary* solutions.

To understand the phase diagram near an equilibrium point we are going to attempt to approximate solutions of (1.2) by solutions of an associated *linearized system*. Suppose that x_0 is an equilibrium point of f. If $f \in C^1(U)$, then Taylor expansion about x_0 yields

$$f(x) \approx Df(x_0)(x - x_0),$$

when $x - x_0$ is small. The linearized system near x_0 is

$$y' = Ay, \quad A = Df(x_0).$$

An important goal is to understand when y is a good approximation to $x - x_0$. Linear systems are simple, and this is the benefit of replacing a nonlinear system by a linearized system near a critical point. For this reason, our first topic will be the study of linear systems.