

E. Arbarello  
M. Cornalba  
P. A. Griffiths  
J. Harris

Volume 267

Grundlehren  
der mathematischen  
Wissenschaften

A Series of  
Comprehensive Studies  
in Mathematics

# Geometry of Algebraic Curves

Volume 1

代数曲线几何 第1卷



Springer

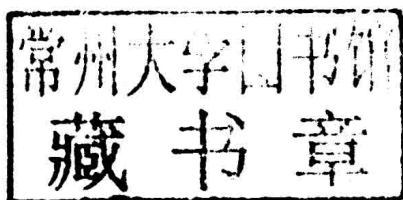
世界图书出版公司  
[www.wpcbj.com.cn](http://www.wpcbj.com.cn)

E. Arbarello  
M. Cornalba  
P. A. Griffiths  
J. Harris

# Geometry of Algebraic Curves

Volume I

With 10 Illustrations



Springer-Verlag  
New York Berlin Heidelberg Tokyo

E. Arbarello  
Dipartimento di Matematica  
Istituto "Guido Castelnuovo"  
Università di Roma "La Sapienza"  
00187 Roma  
Italia

M. Cornalba  
Dipartimento di Matematica  
Università di Pavia  
27100 Pavia  
Italia

P. A. Griffiths  
Office of the Provost  
Duke University  
Durham, NC 27760  
U.S.A.

J. Harris  
Department of Mathematics  
Brown University  
Providence, RI 02912  
U.S.A.

---

AMS Subject Classification: 14-XX

---

Library of Congress Cataloging in Publication Data

Main entry under title:

Geometry of algebraic curves.

(Grundlehren der mathematischen Wissenschaften; 267)

Bibliography: v. 1, p.

Includes index.

1. Curves, Algebraic. I. Arbarello, E. II. Series.

QA565.G46 1984 512'.33 84-5373

© 1985 by Springer-Verlag New York Inc.

All rights reserved. No part of this book may be translated or reproduced in any form without permission from Springer-Verlag, 175 Fifth Avenue, New York, New York 10010, U.S.A.

Reprint from English language edition:

Geometry of Algebraic Curves: Volume I

by Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths

Copyright © 1985, Springer-Verlag Berlin Heidelberg

Springer-Verlag Berlin Heidelberg is a part of Springer Science+Business Media

This reprint has been authorized by Springer Science & Business Media for distribution in China Mainland only and not for export therefrom.

## 图书在版编目 (CIP) 数据

代数曲线几何 = Geometry of algebraic curves. 第 1 卷: 英文/(意)阿尔巴雷洛 (Arbarello, E.) 著. —影印本. —北京: 世界图书出版公司北京公司, 2014. 2  
ISBN 978 - 7 - 5100 - 7590 - 2

I. ①代… II. ①阿… III. ①代数曲线—英文 IV. ①O187.1

中国版本图书馆 CIP 数据核字 (2014) 第 026482 号

---

书 名: Geometry of Algebraic Curves: Volume I  
作 者: E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris  
中译名: 代数曲线几何 第 1 卷  
责任编辑: 高蓉 刘慧

---

出版者: 世界图书出版公司北京公司  
印刷者: 三河市国英印务有限公司  
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)  
联系电话: 010 - 64021602, 010 - 64015659  
电子信箱: kjb@wpcbj.com.cn

---

开 本: 24 开  
印 张: 17  
版 次: 2014 年 5 月  
版权登记: 图字: 01 - 2013 - 8233

---

书 号: 978 - 7 - 5100 - 7590 - 2 定 价: 69.00 元

---

# Grundlehren der mathematischen Wissenschaften 267

*A Series of Comprehensive Studies in Mathematics*

## *Editors*

M. Artin S. S. Chern J. M. Fröhlich E. Heinz  
H. Hironaka F. Hirzebruch L. Hörmander S. Mac Lane  
W. Magnus C. C. Moore J. K. Moser M. Nagata  
W. Schmidt D. S. Scott Ya. G. Sinai J. Tits  
B. L. van der Waerden M. Waldschmidt S. Watanabe

## *Managing Editors*

M. Berger B. Eckmann S. R. S. Varadhan

## Grundlehren der mathematischen Wissenschaften

*A Series of Comprehensive Studies in Mathematics*

*A Selection*

180. Landkof: Foundations of Modern Potential Theory
181. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications I
182. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications II
183. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications III
184. Rosenblatt: Markov Processes, Structure and Asymptotic Behavior
185. Rubinowicz: Sommerfeldsche Polynommethode
186. Handbook for Automatic Computation. Vol. 2. Wilkinson/Reinsch: Linear Algebra
187. Siegel/Moser: Lectures on Celestial Mechanics
188. Warner: Harmonic Analysis on Semi-Simple Lie Groups I
189. Warner: Harmonic Analysis on Semi-Simple Lie Groups II
190. Faith: Algebra: Rings, Modules, and Categories I
191. Faith: Algebra II, Ring Theory
192. Malcev: Algebraic Systems
193. Pólya/Szegő: Problems and Theorems in Analysis I
194. Igusa: Theta Functions
195. Berberian: Baer\*-Rings
196. Athreya/Ney: Branching Processes
197. Benz: Vorlesungen über Geometrie der Algebren
198. Gaal: Linear Analysis and Representation Theory
199. Nitsche: Vorlesungen über Minimalflächen
200. Dold: Lectures on Algebraic Topology
201. Beck: Continuous Flows in the Plane
202. Schmetterer: Introduction to Mathematical Statistics
203. Schoeneberg: Elliptic Modular Functions
204. Popov: Hyperstability of Control Systems
205. Nikolskii: Approximation of Functions of Several Variables and Imbedding Theorems
206. André: Homologie des Algèbres Commutatives
207. Donoghue: Monotone Matrix Functions and Analytic Continuation
208. Lacey: The Isometric Theory of Classical Banach Spaces
209. Ringel: Map Color Theorem
210. Gihman/Skorohod: The Theory of Stochastic Processes I
211. Comfort/Negrepointis: The Theory of Ultrafilters
212. Switzer: Algebraic Topology—Homotopy and Homology
213. Shafarevich: Basic Algebraic Geometry
214. van der Waerden: Group Theory and Quantum Mechanics
215. Schaefer: Banach Lattices and Positive Operators
216. Pólya/Szegő: Problems and Theorems in Analysis II
217. Stenström: Rings of Quotients
218. Gihman/Skorohod: The Theory of Stochastic Process II
219. Duvant/Lions: Inequalities in Mechanics and Physics
220. Kirillov: Elements of the Theory of Representations
221. Mumford: Algebraic Geometry I: Complex Projective Varieties
222. Lang: Introduction to Modular Forms
223. Bergh/Löfström: Interpolation Spaces. An Introduction
224. Gilbarg/Trudinger: Elliptic Partial Differential Equations of Second Order

*Continued after Index*

**To the memory of Aldo Andreotti**





## Preface

In recent years there has been enormous activity in the theory of algebraic curves. Many long-standing problems have been solved using the general techniques developed in algebraic geometry during the 1950's and 1960's. Additionally, unexpected and deep connections between algebraic curves and differential equations have been uncovered, and these in turn shed light on other classical problems in curve theory. It seems fair to say that the theory of algebraic curves looks completely different now from how it appeared 15 years ago; in particular, our current state of knowledge represents a significant advance beyond the legacy left by the classical geometers such as Noether, Castelnuovo, Enriques, and Severi.

These books give a presentation of one of the central areas of this recent activity; namely, the study of linear series on both a fixed curve (Volume I) and on a variable curve (Volume II). Our goal is to give a comprehensive and self-contained account of the extrinsic geometry of algebraic curves, which in our opinion constitutes the main geometric core of the recent advances in curve theory. Along the way we shall, of course, discuss applications of the theory of linear series to a number of classical topics (e.g., the geometry of the Riemann theta divisor) as well as to some of the current research (e.g., the Kodaira dimension of the moduli space of curves).

A brief description of the contents of the various chapters is given in the Guide for the Reader. Here we remark that these volumes are written in the spirit of the classical treatises on the geometry of curves, such as Enriques-Chisini, rather than in the style of the theory of compact Riemann surfaces, or of the theory of algebraic functions of one variable. Of course, we hope that we have made our subject understandable and attractive to interested mathematicians who have studied, in whatever manner, the basics of curve theory and who have some familiarity with the terminology of modern algebraic geometry.

We now would like to say a few words about that material which is *not* included. Naturally, moduli of curves play an essential role in the geometry of algebraic curves, and we have attempted to give a useful and concrete discussion of those aspects of the theory of moduli that enter into our work. However, we do not give a general account of moduli of curves, and we say nothing about the theory of Teichmüller spaces. Secondly, it is obvious that

theory of abelian varieties and theta functions is closely intertwined with algebraic curve theory, and we have also tried to give a self-contained presentation of those aspects that are directly relevant to our work, there being no pretense to discuss the general theory of abelian varieties and theta functions. Thirdly, again the general theory of algebraic varieties clearly underlies this study; we have attempted to utilize the general methods in a concrete and practical manner, while having nothing to add to the theory beyond the satisfaction of seeing how it applies to specific geometric problems. Finally, arithmetical questions, as well as the recent beautiful connections between algebraic curves and differential equations including the ramifications with the Schottky problem, are not discussed.

These books are dedicated to Aldo Andreotti. He was a man of tremendous mathematical insight and personal wisdom, and it is fair to say that Andreotti's view of our subject as it appears, for example, in his classic paper "On a theorem of Torelli," set the tone for our view of the theory of algebraic curves. Moreover, his influence on the four of us, both individually and collectively, was enormous.

It is a pleasure to acknowledge the help we have received from numerous colleagues. Specifically, we would like to thank Corrado De Concini, David Eisenbud, Bill Fulton, Mark Green, Steve Kleiman, and Edoardo Sernesi for many valuable comments and suggestions. We would also like to express our appreciation to Roy Smith and Harsh Pittie, who organized a conference on Brill–Noether theory in February, 1979 at Athens, Georgia, the notes of which formed the earliest (and by now totally unrecognizable) version of this work.

Also, it is a pleasure to thank Steve Diaz, Ed Griffin, and Francesco Scattone for excellent proofreading.

Finally, our warmest appreciation goes to Laura Schlesinger, Carol Ferreira, and Kathy Jacques for skillfully typing the various successive versions of the manuscript.

## Guide for the Reader

This book is not an introduction to the theory of algebraic curves. Rather, it addresses itself to those who have mastered the basics of curve theory and wish to venture beyond them into more recently explored ground. However, we felt that it would be useful to provide the reader with a condensed account of elementary curve theory that would serve the dual purpose of establishing our viewpoint and notation, and furnish a handy reference for those results which are more frequently used in the main body of our work. This is done in the first chapter; quite naturally, few proofs are given in full or even sketched. One notable exception is provided by the theorem of Riemann, describing the theta divisor of the Jacobian of a curve of genus  $g$  as a translate of the image under the Abel–Jacobi map of the  $(g - 1)$ -fold symmetric product of the curve.

The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne’s book *Algebraic Geometry*. Occasionally, however, we have been compelled to make use of relatively more advanced results, such as the theory of base change or the Grothendieck–Riemann–Roch theorem. Our policy, in this situation, has been to give complete statements and adequate references to the existing literature when the results are first used. The main exception to this rule is provided by Chapter II, which contains a down-to-earth and utilitarian presentation, with complete proofs, of the first and second fundamental theorems of invariant theory for the general linear group, the local structure of determinantal varieties, and their global enumerative properties such as Porteous’ formula. The main reason for this exception is, of course, that the varieties of special divisors, which form one of the main objects of study in this book, have a natural determinantal structure and many of their properties are essentially direct consequences of general facts about determinant varieties. We also felt that most readers would prefer a unified account of the results rather than a sequence of references to scattered sources in the literature.

The first two chapters are thus of a preliminary nature, and most readers will probably want to use them primarily for reference purposes. The main theme of the book, that is, the study of special divisors and the extrinsic geometry of curves, is introduced in Chapter III. Here will be found, beside

elementary facts such as Clifford's theorem (which are discussed here and not in Chapter I simply because they are close in nature to some of the results that will be encountered in later chapters), Castelnuovo's description of extremal curves, Noether's theorem, and the theorems of Enriques-Babbage and Petri on the canonical ideal.

Chapter IV is of a foundational nature. In it the varieties of special divisors and linear series on a fixed curve—the main characters of this book—are defined, and the functors they represent are identified. This is where the results of Chapter II are first applied in a systematic way. Although containing no major results of independent interest, except for Martens' improvement of Clifford's theorem and its subsequent refinement by Mumford, this chapter is, in a sense, the cornerstone on which most of the later chapters rest.

In Chapter V the main theorems of Brill-Noether theory are stated and illustrated by means of examples drawn from low-genus cases. In fact most of the theorems are proved by *ad hoc* arguments, for genus up to six.

Chapter VI is probably the most geometric in nature and collects many of the central results about the geometry of the theta divisor of a Jacobian. The main topics touched upon are Riemann's singularity theorem and its generalization by Kempf, Andreotti's proof of the Torelli theorem for curves, and Andreotti and Mayer's approach to the Schottky problem via the heat equation for the theta function.

Chapter VII contains the proofs of some of the results stated in Chapter V, notably those of the existence and connectedness theorems, and of the enumerative formulas for the classes of the varieties of special linear series and divisors. The enumerative geometry of these varieties, and related ones, is further investigated in the eighth and final chapter of this volume.

The second volume will contain an exposition of the fundamentals of deformation theory and of the main properties of the moduli space of curves, the proof of the remaining results of Brill-Noether theory, a presentation of the basic properties of the varieties of special linear series on a moving curve with special attention to series of dimension one and two (that is, to Hurwitz spaces and varieties of plane curves), and a proof of the theorem that the moduli space of curves of sufficiently high genus is of general type.

# List of Symbols

$\chi$  1  
 $p_a(C)$  1  
 $g(C)$  1  
 $h^i(V, \mathcal{F})$  1  
 $\text{mult}_p(D)$  2  
 $\text{deg}(D)$  2  
 $\text{Div}(C)$  2  
 $\mu_p$  2  
 $(\phi)$  2  
 $\text{Res}_p$  2  
 $\mathcal{L}(D)$  3  
 $D \geq 0$  3  
 $D \sim 0$  4  
 $|D|$  4  
 $g_d^r$  4  
 $\phi_{\mathcal{G}}$  5  
 $\phi_L$  6  
 $\phi_{|D|}$  6  
 $K_C$  7  
 $\Theta_C$  7  
 $l(D)$  7  
 $i(D)$  7  
 $r(D)$  7  
 $f^*(D)$  8  
 $\phi(D)$  12  
 $\bar{D}$  12  


---

 $p_1, \dots, p_d$  12  
 $J(C)$  17  
 $u$  17, 18  
 $\text{Div}^d(C)$  18  
 $C_d$  18  
 $\text{Pic}(C)$  18  
 $\mathcal{H}_g$  22  
 $\mathcal{A}_g$  22

$\Gamma_g$  22  
 $\theta$  23  
 $\Theta$  23  
 $w_d$  25  
 $c_d$  25  
 $\theta$  25  
 $\kappa$  27  
 $j(C)$  28  
 $\mathcal{M}_g$  28  
 $\Delta_i$  29  
 $\mathcal{E}$  29  
 $\phi^{(k)}$  37  
 $\mathcal{M}(C)$  41  
 $T_p(Y)$  61  
 $\mathcal{T}_p(X)$  61  
 $\mathbb{P}\mathcal{T}_p(X)$  62  
 $X_k(\phi)$  83  
 $-F$  85  
 $E - F$  85  
 $\Delta_{p,q}(a)$  86  
 $\mu_0$  108, 159  
 $\pi(d, r)$  116  
 $C_d^r$  153, 177  
 $W_d^r(C)$  153, 177  
 $G_d^r(C)$  153  
 $\rho$  159  
 $\mu_{0,w}$  187  
 $\phi_d$  223  
 $V_d$  223  
 $\mathcal{J}_g$  249  
 $Nm$  281  
 $\mathbb{P}E$  304  
 $\mathcal{O}_{\mathbb{P}E}(1)$  305  
 $K(X)$  331

$\text{ch}(E)$	331	$(g(t))_{t_1, \dots, t_n}$	341
$\pi_1$	331	$\Gamma_d(\mathcal{D})$	341
$\text{td}(E)$	332	$V'_d$	345
$\binom{n}{i}$	341	$\mu(a, b, m, i, \beta)$	352
		$\phi_a$	358

**Note.** Throughout this book, if  $V$  is a vector space (resp. if  $E$  is a vector bundle) we will denote by  $\mathbb{P}V$  (resp.  $\mathbb{P}E$ ) the space of one-dimensional subspaces of  $V$  (resp. of the fibers of  $E$ ); thus

$$\mathbb{P}V = \text{Proj}(\bigoplus \text{Sym}^n V^*).$$

More generally, if  $C$  is a cone,  $\mathbb{P}C$  will stand for its projectivization. Similarly, by  $G(k, V)$  (resp.  $G(k, E)$ ) we will mean the space of  $k$ -dimensional subspaces of  $V$  (resp. of the fibers of  $E$ ).

# Contents

Guide for the Reader	xii
List of Symbols	xv
CHAPTER I	
Preliminaries	1
§1. Divisors and Line Bundles on Curves	1
§2. The Riemann–Roch and Duality Theorems	6
§3. Abel’s Theorem	15
§4. Abelian Varieties and the Theta Function	20
§5. Poincaré’s Formula and Riemann’s Theorem	25
§6. A Few Words About Moduli	28
Bibliographical Notes	30
Exercises	31
A. Elementary Exercises on Plane Curves	31
B. Projections	35
C. Ramification and Plücker Formulas	37
D. Miscellaneous Exercises on Linear Systems	40
E. Weierstrass Points	41
F. Automorphisms	44
G. Period Matrices	48
H. Elementary Properties of Abelian Varieties	48
APPENDIX A	
The Riemann–Roch Theorem, Hodge Theorem, and Adjoint Linear Systems	50
§1. Applications of the Discussion About Plane Curves with Nodes	56
§2. Adjoint Conditions in General	57
CHAPTER II	
Determinantal Varieties	61
§1. Tangent Cones to Analytic Spaces	61
§2. Generic Determinantal Varieties: Geometric Description	67
§3. The Ideal of a Generic Determinantal Variety	70
§4. Determinantal Varieties and Porteous’ Formula	83
(i) Sylvester’s Determinant	87
(ii) The Top Chern Class of a Tensor Product	89
(iii) Porteous’ Formula	90
(iv) What Has Been Proved	92

§5. A Few Applications and Examples	93
Bibliographical Notes	100
Exercises	100
A. Symmetric Bilinear Maps	100
B. Quadrics	102
C. Applications of Porteous' Formula	104
D. Chern Numbers of Kernel Bundles	105
CHAPTER III	
Introduction to Special Divisors	107
§1. Clifford's Theorem and the General Position Theorem	107
§2. Castelnuovo's Bound, Noether's Theorem, and Extremal Curves	113
§3. The Enriques–Babbage Theorem and Petri's Analysis of the Canonical Ideal	123
Bibliographical Notes	135
Exercises	136
A. Symmetric Products of $\mathbb{P}^1$	136
B. Refinements of Clifford's Theorem	137
C. Complete Intersections	138
D. Projective Normality (I)	140
E. Castelnuovo's Bound on $k$ -Normality	141
F. Intersections of Quadrics	142
G. Space Curves of Maximum Genus	143
H. G. Gherardelli's Theorem	147
I. Extremal Curves	147
J. Nearly Castelnuovo Curves	149
K. Castelnuovo's Theorem	151
L. Secant Planes	152
CHAPTER IV	
The Varieties of Special Linear Series on a Curve	153
§1. The Brill–Noether Matrix and the Variety $C_d^r$	154
§2. The Universal Divisor and the Poincaré Line Bundles	164
§3. The Varieties $W_d^r(C)$ and $G_d^r(C)$ Parametrizing Special Linear Series on a Curve	176
§4. The Zariski Tangent Spaces to $G_d^r(C)$ and $W_d^r(C)$	185
§5. First Consequences of the Infinitesimal Study of $G_d^r(C)$ and $W_d^r(C)$	191
Biographical Notes	195
Exercises	196
A. Elementary Exercises on $\mu_0$	196
B. An Interesting Identification	197
C. Tangent Spaces to $W_1(C)$	197
D. Mumford's Theorem for $g_d^2$ 's	198
E. Martens–Mumford Theorem for Birational Morphisms	198
F. Linear Series on Some Complete Intersections	199
G. Keem's Theorems	200
CHAPTER V	
The Basic Results of the Brill–Noether Theory	203
Bibliographical Notes	217
Exercises	218



A. $W_d^1(C)$ on a Curve $C$ of Genus 6	218
B. Embeddings of Small Degree	220
C. Projective Normality (II)	221
D. The Difference Map $\phi_d: C_d \times C_d \rightarrow J(C)$ (I)	223
CHAPTER VI	
The Geometric Theory of Riemann's Theta Function	225
§1. The Riemann Singularity Theorem	225
§2. Kempf's Generalization of the Riemann Singularity Theorem	239
§3. The Torelli Theorem	245
§4. The Theory of Andreotti and Mayer	249
Bibliographical Notes	261
Exercises	262
A. The Difference Map $\phi_d$ (II)	262
B. Refined Torelli Theorems	263
C. Translates of $W_{g-1}$ , Their Intersections, and the Torelli Theorem	265
D. Prill's Problem	268
E. Another Proof of the Torelli Theorem	268
F. Curves of Genus 5	270
G. Accola's Theorem	275
H. The Difference Map $\phi_d$ (III)	276
I. Geometry of the Abelian Sum Map $u$ in Low Genera	278
APPENDIX B	
Theta Characteristics	281
§1. Norm Maps	281
§2. The Weil Pairing	282
§3. Theta Characteristics	287
§4. Quadratic Forms Over $Z/2$	292
APPENDIX C	
Prym Varieties	295
Exercises	303
CHAPTER VII	
The Existence and Connectedness Theorems for $W_d^r(C)$	304
§1. Ample Vector Bundles	304
§2. The Existence Theorem	308
§3. The Connectedness Theorem	311
§4. The Class of $W_d^r(C)$	316
§5. The Class of $C_d^r$	321
Bibliographical Notes	326
Exercises	326
A. The Connectedness Theorem	326
B. Analytic Cohomology of $C_d$ , $d \leq 2g - 2$	328
C. Excess Linear Series	329
CHAPTER VIII	
Enumerative Geometry of Curves	330
§1. The Grothendieck–Riemann–Roch Formula	330