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Geometry of Algebraic Curves

Volume 1

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A Selection

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To the memory of Aldo Andreotti

Preface

In recent years there has been enormous activity in the theory of algebraic curves. Many long-standing problems have been solved using the general techniques developed in algebraic geometry during the 1950's and 1960's. Additionally, unexpected and deep connections between algebraic curves and differential equations have been uncovered, and these in turn shed light on other classical problems in curve theory. It seems fair to say that the theory of algebraic curves looks completely different now from how it appeared 15 years ago; in particular, our current state of knowledge represents a significant advance beyond the legacy left by the classical geometers such as Noether, Castelnuovo, Enriques, and Severi.

These books give a presentation of one of the central areas of this recent activity; namely, the study of linear series on both a fixed curve (Volume I) and on a variable curve (Volume II). Our goal is to give a comprehensive and self-contained account of the extrinsic geometry of algebraic curves, which in our opinion constitutes the main geometric core of the recent advances in curve theory. Along the way we shall, of course, discuss applications of the theory of linear series to a number of classical topics (e.g., the geometry of the Riemann theta divisor) as well as to some of the current research (e.g., the Kodaira dimension of the moduli space of curves).

A brief description of the contents of the various chapters is given in the Guide for the Reader. Here we remark that these volumes are written in the spirit of the classical treatises on the geometry of curves, such as Enriques–Chisini, rather than in the style of the theory of compact Riemann surfaces, or of the theory of algebraic functions of one variable. Of course, we hope that we have made our subject understandable and attractive to interested mathematicians who have studied, in whatever manner, the basics of curve theory and who have some familiarity with the terminology of modern algebraic geometry.

We now would like to say a few words about that material which is *not* included. Naturally, moduli of curves play an essential role in the geometry of algebraic curves, and we have attempted to give a useful and concrete discussion of those aspects of the theory of moduli that enter into our work. However, we do not give a general account of moduli of curves, and we say nothing about the theory of Teichmüller spaces. Secondly, it is obvious that theory of abelian varieties and theta functions is closely intertwined with algebraic curve theory, and we have also tried to give a self-contained presentation of those aspects that are directly relevant to our work, there being no pretense to discuss the general theory of abelian varieties and theta functions. Thirdly, again the general theory of algebraic varieties clearly underlies this study; we have attempted to utilize the general methods in a concrete and practical manner, while having nothing to add to the theory beyond the satisfaction of seeing how it applies to specific geometric problems. Finally, arithmetical questions, as well as the recent beautiful connections between algebraic curves and differential equations including the ramifications with the Schottky problem, are not discussed.

These books are dedicated to Aldo Andreotti. He was a man of tremendous mathematical insight and personal wisdom, and it is fair to say that Andreotti's view of our subject as it appears, for example, in his classic paper "On a theorem of Torelli," set the tone for our view of the theory of algebraic curves. Moreover, his influence on the four of us, both individually and collectively, was enormous.

It is a pleasure to acknowledge the help we have received from numerous colleagues. Specifically, we would like to thank Corrado De Concini, David Eisenbud, Bill Fulton, Mark Green, Steve Kleiman, and Edoardo Sernesi for many valuable comments and suggestions. We would also like to express our appreciation to Roy Smith and Harsh Pittie, who organized a conference on Brill–Noether theory in February, 1979 at Athens, Georgia, the notes of which formed the earliest (and by now totally unrecognizable) version of this work.

Also, it is a pleasure to thank Steve Diaz, Ed Griffin, and Francesco Scattone for excellent proofreading.

Finally, our warmest appreciation goes to Laura Schlesinger, Carol Ferreira, and Kathy Jacques for skillfully typing the various successive versions of the manuscript.

Guide for the Reader

This book is not an introduction to the theory of algebraic curves. Rather, it addresses itself to those who have mastered the basics of curve theory and wish to venture beyond them into more recently explored ground. However, we felt that it would be useful to provide the reader with a condensed account of elementary curve theory that would serve the dual purpose of establishing our viewpoint and notation, and furnish a handy reference for those results which are more frequently used in the main body of our work. This is done in the first chapter; quite naturally, few proofs are given in full or even sketched. One notable exception is provided by the theorem of Riemann, describing the theta divisor of the Jacobian of a curve of genus g as a translate of the image under the Abel–Jacobi map of the $(g - 1)$-fold symmetric product of the curve.

The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne's book *Algebraic Geometry*. Occasionally, however, we have been compelled to make use of relatively more advanced results, such as the theory of base change or the Grothendieck–Riemann–Roch theorem. Our policy, in this situation, has been to give complete statements and adequate references to the existing literature when the results are first used. The main exception to this rule is provided by Chapter II, which contains a down-to-earth and utilitarian presentation, with complete proofs, of the first and second fundamental theorems of invariant theory for the general linear group, the local structure of determinantal varieties, and their global enumerative properties such as Porteous' formula. The main reason for this exception is, of course, that the varieties of special divisors, which form one of the main objects of study in this book, have a natural determinantal structure and many of their properties are essentially direct consequences of general facts about determinant varieties. We also felt that most readers would prefer a unified account of the results rather than a sequence of references to scattered sources in the literature.

The first two chapters are thus of a preliminary nature, and most readers will probably want to use them primarily for reference purposes. The main theme of the book, that is, the study of special divisors and the extrinsic geometry of curves, is introduced in Chapter III. Here will be found, beside elementary facts such as Clifford's theorem (which are discussed here and not in Chapter I simply because they are close in nature to some of the results that will be encountered in later chapters), Castelnuovo's description of extremal curves, Noether's theorem, and the theorems of Enriques–Babbage and Petri on the canonical ideal.

Chapter IV is of a foundational nature. In it the varieties of special divisors and linear series on a fixed curve—the main characters of this book—are defined, and the functors they represent are identified. This is where the results of Chapter II are first applied in a systematic way. Although containing no major results of independent interest, except for Martens' improvement of Clifford's theorem and its subsequent refinement by Mumford, this chapter is, in a sense, the cornerstone on which most of the later chapters rest.

In Chapter V the main theorems of Brill–Noether theory are stated and illustrated by means of examples drawn from low-genus cases. In fact most of the theorems are proved by *ad hoc* arguments, for genus up to six.

Chapter VI is probably the most geometric in nature and collects many of the central results about the geometry of the theta divisor of a Jacobian. The main topics touched upon are Riemann's singularity theorem and its generalization by Kempf, Andreotti's proof of the Torelli theorem for curves, and Andreotti and Mayer's approach to the Schottky problem via the heat equation for the theta function.

Chapter VII contains the proofs of some of the results stated in Chapter V, notably those of the existence and connectedness theorems, and of the enumerative formulas for the classes of the varieties of special linear series and divisors. The enumerative geometry of these varieties, and related ones, is further investigated in the eighth and final chapter of this volume.

The second volume will contain an exposition of the fundamentals of deformation theory and of the main properties of the moduli space of curves, the proof of the remaining results of Brill–Noether theory, a presentation of the basic properties of the varieties of special linear series on a moving curve with special attention to series of dimension one and two (that is, to Hurwitz spaces and varieties of plane curves), and a proof of the theorem that the moduli space of curves of sufficiently high genus is of general type.

List of Symbols

Note. Throughout this book, if V is a vector space (resp. if E is a vector bundle) we will denote by $\mathbb{P}V$ (resp. $\mathbb{P}E$) the space of one-dimensional subspaces of V (resp. of the fibers of E); thus

$$\mathbb{P}V = \mathrm{Proj}(\bigoplus \mathrm{Sym}^n V^*).$$

More generally, if C is a cone, $\mathbb{P}C$ will stand for its projectivization. Similarly, by $G(k, V)$ (resp. $G(k, E)$) we will mean the space of k-dimensional subspaces of V (resp. of the fibers of E).

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