ANNALS OF DISCRETE MATHEMATICS

54

Submodular Functions and Electrical Networks

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SUBMODULAR FUNCTIONS AND ELECTRICAL NETWORKS

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1997
ELSEVIER
AMSTERDAM • LAUSANNE • NEW YORK • OXFORD • SHANNON • TOKYO

ELSEVIER SCIENCE B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

ISBN: 0 444 82523 1

This book is printed on acid-free paper.

Printed in The Netherlands

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கல்வி கரையில; கற்பவர் நாள்சில;
மெல்ல நினைக்கிற் பிணிபல; தெள்ளிதின்
ஆராயந் தமைவுடைய கற்பவே நீரொழியப்
பாலுண் குருகிற்றெரிந்து.

–Nāladi
circa 800 A.D.

Learning is a shoreless sea; the learner's days are few;
Prolonged study is beset with a thousand ills;
With clear discrimination learn what's meet for you
Like swan that leaves the water, drinks the milk.

Preface

This book has grown out of an attempt to understand the role that the topology of an electrical network plays in its **efficient** analysis. The approach taken is to transform the problem of solving a network with a given topology, to that of solving another with a different topology (and same devices), but with additional inputs and constraints. An instance of this approach is network analysis by multiport decomposition - breaking up a network into multiports, solving these in terms of port variables and finally imposing the port connection conditions and getting the complete solution. The motivation for our approach is that of building more efficient circuit simulators, whether they are to run singly or in parallel. Some of the ideas contained in the book have already been implemented - BITSIM, the general purpose circuit simulator built at the VLSI Design Centre, I.I.T. Bombay, is based on the 'topological hybrid analysis' contained in this book and can further be adapted to use topological decomposition ideas.

Many combinatorial optimization problems arise naturally when one adopts the above approach, particularly the hybrid rank problem and its generalizations. The theory required for the solution of these problems was developed by electrical engineers parallel to, and independent of, developments taking place in the theory of matroids and submodular functions. Consider, for instance, the work of Kishi and Kajitani, Iri, Ohtsuki et al in the late 60's on principal partition and its applications, independent of Edmonds' work on matroid partitions (1965). There is a strong case for electrical network topologists and submodular function theorists being aware of each others' fields. It is hoped that the present book would fill this need.

The topological network analysis that we have considered is to be distinguished from the kind of work exemplified by 'Kirchhoff's Third Law' which has been discussed in many books published in the 60's (eg. the book by Seshu and Reed [Seshu+Reed61]). In the 70's much interesting work in this area was done by Iri, Tomizawa, Recski and others using the 'generality assumption' for linear devices. Details may be found, for instance, in Recski's book [Recski89]. In the present book devices play a very secondary role. Mostly we manipulate only Kirchhoff's Laws.

Submodular functions are presented in this book adopting the 'elementary combinatorial' as opposed to the 'polyhedral' approach. Three things made us decide in favour of the former approach.

- It is hoped that the book would be read by designers of VLSI algorithms. In order to be convincing, the algorithms presented would have to be fast. So very general algorithms based on the polyhedral approach are ruled out.
- The polyhedral approach is not very natural to the material on Dilworth truncation.
- There is an excellent and comprehensive monograph, due to S.Fujishige, on the polyhedral approach to submodular functions; a book on polyhedral combinatorics including submodular functions from A.Schrijver is long awaited.

In order to make the book useful to a wider audience, the material on electrical networks and that on submodular functions are presented independently of each other. A final chapter on the hybrid rank problem displays the link. An area which can benefit by algorithms based on submodular functions is that of CAD for VLSI - particularly for building partitioners. Some space has therefore been devoted to partitioning in the chapter on Dilworth truncation.

The book is intended primarily for self study - hence the large number of problems with solutions. However, most of the material has been tested in the class room. The network theory part has been used for many years for an elective course on 'Advanced Network Analysis' - a third course on networks taken by senior undergraduates at the EE Dept, I.I.T. Bombay. The submodular function part has been used for special topics courses on combinatorics taken by doctoral students in Maths and Computer Science. This material can be covered in a semester if the students have a prior background in elementary graphs and matroids, leaving all the starred sections and relegating details and problems to self study.

It is a pleasure to acknowledge the author's indebtedness to his many colleagues, teachers and friends and to express his heartfelt gratitude.

He was introduced to electrical network theory by Professors R.E.Bedford and K.Shankar of the EE Dept., I.I.T. Bombay, and to graph theory by Professor M.N.Vartak of the Dept. of Maths, I.I.T. Bombay. Professor Masao Iri, formerly of the University of Tokyo, now of the University of Chuo, has kept him abreast of the developments in applied matroid theory during the last two decades and has also generously spared time to comment on the viability of lines of research.

He has benefited through interaction with the following: Professors S.D.Agashe, P.R.Bryant, A.N.Chandorkar, M.Chandramouli, C.A.Desoer, A.Diwan, S.Fujishige, P.L.Hammer, M.V.Hariharan, Y.Kajitani, M.V. Kamath, M.S.Kamath, E.L.Lawler, K.V.V. Murthy, T.Ozawa, S.Patkar, S.K.Pillai, P.G.Poonacha, G.N.Revankar, S.Roy, S.C.Sahasrabudhe, P.C.Sharma, M.Sohoni, V.Subbarao, N.J.Sudarshan, V.K.Tandon, N.Tomizawa, P.P.Varaiya, J.M.Vasi.

The friends mentioned below have critically read parts of the manuscript: S.Batterywala, A.Diwan, N.Jayanthi, S.Patkar, P.G.Poonacha and the '96 batch students of the course 'Advanced Network Analysis'. But for Shabbir Batterywala's assistance (technical, editorial, software consultancy), publication of this book would have been delayed by many months.

Mr Z.A.Shirgaonkar has done the typing in Latex and Mr R.S.Patwardhan has drawn the figures.

The writing of this book was supported by a grant (HN/EE/TXT/95) from the C.D.P., I.I.T. Bombay.

The author is grateful to his mother Lalitha Iyer, wife Jayanthi and son Hari for their continued encouragement and support.

Note to the Reader

This book appears too long because of two reasons:

- it is meant for self study - so contains a large number of exercises and problems with solutions.
- it is aimed at **three** different types of readers:
  - Electrical engineers interested in topological methods of network analysis.
  - Engineers interested in submodular function theory
  - Researchers interested in the link between electrical networks and submodular functions.

To shorten the book for oneself it is not necessary to take recourse to drastic physical measures. During first reading all starred sections, starred exercises and problems may be omitted. If the reader belongs to the first two categories mentioned above, she would already find that only about two hundred pages have to be read.

Sections, exercises and problems have been starred to indicate that they are not necessary for a first reading. Length of the solution is a fair indicator of the level of difficulty of a problem - star does not indicate level of difficulty. There are only a handful of routine (drill type) exercises. Most of the others require some effort. Usually the problems are harder than the exercises.

Many of the results, exercises, problems etc. in this book are well known but cannot easily be credited to any one author. Such results are marked with a '(k)'.

Electrical Engineers interested in topological methods

Such readers should first brush up on linear algebra (say first two chapters of the book by Hoffman and Kunze [Hoffman+Kunze72]), read a bit of graph theory (say the chapter on Kirchhoff's laws in the book by Chua et al [Chua+Desoer+Kuh87] and the first four chapters of the book by Narsingh Deo [Narsingh Deo74]) and then read chapters 2 to 8. The chapter on graphs contains material on contraction and restriction which is not easily available in textbooks on circuit theory, but which is essential for an understanding of subsequent chapters. So this chapter should be read carefully, particularly since it is written tersely. The chapter on matroids is optional. The chapter on electrical networks should be easy reading but scanning it is essential since it fixes some notation used subsequently and also because it contains material motivating subsequent chapters, e.g. multiport decomposition. The next three chapters contain whatever the book has to say on topological network analysis.

Engineers interested in submodular functions

Such readers should read Chapters 2 to 4 and Chapters 9 to 13 and the first four sections of Chapter 14. If the reader is not interested in matroids he may skip material (chapters, sections, exercises, examples) dealing with them without serious loss of continuity. This would mean he would have to be satisfied with bipartite graph based instances of the general theory. The key chapter for such a reader is Chapter 9. This is tersely written-so should be gone through carefully.

Researchers interested in the link between submodular functions and electrical networks

The key chapter for such a reader is Chapter 14. To read the first four sections of this chapter the reader has to be familiar with Chapters 5, 6, 7 from the electrical networks part and the unstarred sections of the chapters on submodular functions. If he has some prior familiarity with submodular functions and electrical networks it is possible to directly begin reading the chapter picking up the required results on submodular functions as and when they are referred to in the text. To read the last section of the chapter, familiarity with Chapter 8 is required.

Comments on Notation

Sometimes, instead of numbering equations, key statements etc., we have marked them with symbols such as $(*), (**), (\surd)$. These marks are used over and over again and have validity only within a local area such as a paragraph, a proof or the solution to a problem.

In some cases, where there is no room for confusion, the same symbol denotes different objects. For instance, usually B denotes a bipartite graph. But in Chapter 4, B denotes a base of a matroid- elsewhere a base is always denoted by b. The symbol E is used for the edge set of a graph, in particular a bipartite graph. But $E(X), X \subseteq V(\mathcal{G})$ denotes the set of edges with both endpoints within X, while $E_L(X), X \subseteq V_L$, in the case of a bipartite graph, denotes the set of all vertices adjacent only to vertices in X.

We have often used brackets to write two statements in one.
Example: We say that set X is **contained** in Y (**properly contained in** Y), if every element of X is also a member of Y (every element of X is a member of Y

and $X \neq Y$) and denote it by $X \subseteq Y (X \subset Y)$.
This is to be read as the following two statements.

i. We say that set X is **contained** in Y, if every element of X is also a member of Y and denote it by $X \subseteq Y$.

ii. We say that set X is **properly contained in** Y, if every element of X is a member of Y and $X \neq Y$ and denote it by $X \subset Y$.

List of Commonly Used Symbols

Sets, Partitions, Partial Orders

$\{e_1, e_2, \ldots, e_n\}$	*set whose elements are* $e_1, e_2, \ldots, e_n$
$\{x_i : i \in I\}$	*set whose members are* $x_i,\ i \in I$
$(x_i : i \in I)$	*a family (used only in Chapters* 2 *and* 11)
$x \in X$	*element* x *belongs to set* X
$x \notin X$	*element* x *does not belong to set* X
$\forall x$ *or* $\forall\ x$	*for all elements* x
$\exists x$	*there exists an element* x
$X \subseteq Y$	*set* X *is contained in set* Y
$X \subset Y$	*set* X *is properly contained in set* Y
$X \cup Y$	*union of sets* X *and* Y
$X \cap Y$	*intersection of sets* X *and* Y
$X \uplus Y$	*disjoint union of sets* X *and* Y
$\bigcup_{i=1}^{n} X_i$	*union of the sets* X_i
$\biguplus_{i=1}^{n} X_i$	*disjoint union of the sets* X_i
$X - Y$	*set of elements in* X *but not in* Y
$\bar{X}$	*complement of* X
$X \times Y$	*cartesian product of sets* X *and* Y
$X \oplus Y$	*direct sum of sets* X *and* Y
2^S	*collection of subsets of* S
$\mid X \mid$	*size of the subset* X
$(P, \preceq)$	*preorder on* P
$(P, \leq)$	*partial order on* P
Π	*partition* Π
Π_N	*partition that has* N *as a block and all blocks except* N *as singletons*
$\mathcal{P}_X$	*collection of all partitions of* X
$\Pi \leq \Pi'$	*partition* Π *is finer than* Π'
$\Pi \vee \Pi'$	*finest partition coarser than both* Π *and* Π'
$\Pi \wedge \Pi'$	*coarsest partition finer than both* Π *and* Π'

Functions,Set Functions and Operations on Functions

$f(\cdot)$	*function* $f(\cdot)$
$f/Z(\cdot), f(\cdot)$ *on* S	*restriction of* $f(\cdot)$ *to* $Z \subseteq S$
$gf(X), g \circ f(X)$	$g(f(X))$
$(f_1 \oplus f_2)(\cdot)$	*direct sum of functions* $f_1(\cdot)$ *and* $f_2(\cdot)$
$f_{fus \cdot \Pi}(\cdot), f(\cdot)$ *on* 2^S,	*fusion of* $f(\cdot)$ *relative to* Π *i.e.*, $f_{fus.\Pi}(X_f)$ $\equiv f(\bigcup_{T \in X_f} T), X_f \subseteq \Pi$
$f/\mathbf{X}(\cdot), f(\cdot)$ *on* 2^S	*restriction of* $f(\cdot)$ *to* $2^X, X \subseteq S$ *(usually called) restriction of* $f(\cdot)$ *to* X
$f \diamond \mathbf{X}(\cdot), f(\cdot)$ *on* 2^S	*contraction of* $f(\cdot)$ *to* X $f \diamond \mathbf{X}(Y) \equiv f((S-X) \cup Y) - f(S-X)$
$f^d(\cdot), f(\cdot)$ *on* 2^S	*contramodular dual of* $f(\cdot)$ $f^d(X) \equiv f(S) - f(S-X)$
$f^*(\cdot), f(\cdot)$ *on* 2^S	*comodular dual of* $f(\cdot)$ *(with respect to weight function* $\alpha(\cdot)$*)* $f^*(X) \equiv \alpha(X) - (f(S) - f(S-X))$
$P_f, f(\cdot)$ *on* 2^S	*polyhedron associated with* $f(\cdot)$ $\mathbf{x} \in P_f$ *iff* $x(X) \leq f(X) \quad \forall X \subseteq S$
$P_f^d, f(\cdot)$ *on* 2^S	*dual polyhedron associated with* $f(\cdot)$ $\mathbf{x} \in P_f^d$ *iff* $x(X) \geq f(X) \quad \forall X \subseteq S$

Vectors and Matrices

$\mathcal{F}, \Re, \mathcal{C}, \Re_+$	*field* $\mathcal{F}$, *real field, complex field,* *set of nonnegative reals*
$\sum x_i$	*summation of elements* x_i
$\mathbf{f}$	*vector* $\mathbf{f}$
$\mathcal{V}$	*vector space* $\mathcal{V}$
$\mathcal{V}^\perp$	*vector space complementary orthogonal to* $\mathcal{V}$
$\mathbf{x}_1 \oplus \mathbf{x}_2$	*direct sum of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *(vector obtained by* *adjoining components of vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*)*
$\mathcal{V}_S \oplus \mathcal{V}_T, S \cap T = \emptyset$	*direct sum of* $\mathcal{V}_S$ *and* $\mathcal{V}_T$ *(obtained by* *collecting all possible direct sums of vectors* *in* $\mathcal{V}_S$ *and* $\mathcal{V}_T$*)*

$dim(\mathcal{V}), r(\mathcal{V})$	*dimension of vector space* $\mathcal{V}$
$d(\mathcal{V}, \mathcal{V}')$	$r(\mathcal{V} + \mathcal{V}') - r(\mathcal{V} \cap \mathcal{V}')$
$\mathbf{A}(i,j)$	i, j^{th} *entry of matrix* $\mathbf{A}$
$\mathbf{A}^T$	*transpose of matrix* $\mathbf{A}$
$\mathbf{A}^{-1}$	*inverse of matrix* $\mathbf{A}$
$< \mathbf{f}, \mathbf{g} >$	*dot product of vectors* $\mathbf{f}, \mathbf{g}$
$\mathcal{R}(\mathbf{A})$	*row space of* $\mathbf{A}$
$\mathcal{C}(\mathbf{A})$	*column space of* $\mathbf{A}$
$det(\mathbf{A})$	*determinant of* $\mathbf{A}$

Graphs and Vector Spaces

$\mathcal{G}$	*graph* $\mathcal{G}$
$V(\mathcal{G})$	*vertex set of* $\mathcal{G}$
$E(\mathcal{G})$	*edge set of* $\mathcal{G}$
t	*a tree*
f	*a forest*
$\bar{t}$	*cotree* $(E(\mathcal{G}) - t)$ *of* $\mathcal{G}$
$\bar{f}$	*coforest* $(E(\mathcal{G}) - f)$ *of* $\mathcal{G}$
$L(e, f)$	$f -$ *circuit of* e *with respect to* f
$B(e, f)$	$f -$ *cutset of* e *with respect to* f
$r(\mathcal{G})$	*rank of* $\mathcal{G}$ (= *number of edges in a forest of* $\mathcal{G}$)
$\nu(\mathcal{G})$	*nullity of* $\mathcal{G}$ (= *number of edges in a coforest of* $\mathcal{G}$)
$\mathcal{G}openT$	*graph obtained from* $\mathcal{G}$ *by opening and removing edges* T
$\mathcal{G}shortT$	*graph obtained from* $\mathcal{G}$ *by shorting and removing edges* T
$\mathcal{G} \cdot T$	*graph obtained from* $\mathcal{G}open(E(\mathcal{G}) - T)$ *by removing isolated vertices, restriction of* $\mathcal{G}$ *to* T
$\mathcal{G} \times T$	*graph obtained from* $\mathcal{G}short(E(\mathcal{G}) - T)$ *by removing isolated vertices, contraction of* $\mathcal{G}$ *to* T

$\mathcal{G}_1 \cong \mathcal{G}_2$	$\mathcal{G}_1$ *is* $2-$ *isomorphic to* $\mathcal{G}_2$
$r(T)$	$r(\mathcal{G} \cdot T)$
$\nu(T)$	$\nu(\mathcal{G} \times T)$
$\mathcal{H}$	*hypergraph* $\mathcal{H}$
$B(V_L, V_R, E)$	*bipartite graph with left vertex set* V_L, *right vertex set* V_R *and edge set* E
$\mathbf{A}$	*(usually) incidence matrix*
$\mathbf{A_r}$	*reduced incidence matrix*
$\mathbf{Q}_f$	*fundamental cutset matrix of forest* f
$\mathbf{B}_f$	*fundamental circuit matrix of forest* f
KCE	*Kirchhoff's current equations*
KCL	*Kirchhoff's current Law*
KVE	*Kirchhoff's voltage equations*
KVL	*Kirchhoff's voltage Law*
$\mathcal{V}_i(\mathcal{G})$	*solution space of* KCE *of* $\mathcal{G}$
$\mathcal{V}_v(\mathcal{G})$	*solution space of* KVE *of* $\mathcal{G}$
$\mathcal{V} \cdot T$	*restriction of vector space* $\mathcal{V}$ *to* T
$\mathcal{V} \times T$	*contraction of vector space* $\mathcal{V}$ *to* T
$\xi(T)$ *for* $\mathcal{V}$	$r(\mathcal{V} \cdot T) - r(\mathcal{V} \times T)$

Flow Graphs

$F(\mathcal{G}) \equiv (\mathcal{G}, \mathbf{c}, s, t)$	*flow graph on graph* $\mathcal{G}$ *with capacity function* $\mathbf{c}$, *source* s, *sink* t
(A, B)	$cut(A, B)$
$c(A, B)$	*capacity of* $cut(A, B)$
$f(A, B)$	*flow across* $cut(A, B)$, *from* A *to* B
$\vert\mathbf{f}\vert$	*value of flow* $\mathbf{f}$
$F(B, \mathbf{c}_L, \mathbf{c}_R)$	*flowgraph associated with bipartite graph* B *with source to left vertex capacity* $\mathbf{c}_L$, *right vertex to sink capacity* $\mathbf{c}_R$ *and (left to right) bipartite graph edge capacity* ∞

Matroids

$\mathcal{M} \equiv (S, \mathcal{I})$	*matroid* $\mathcal{M}$
$\mathcal{I}$	*collection of independent sets*
$\mathcal{M}^*$	*dual of the matroid* $\mathcal{M}$
B	*(only in Chapter 4) base of a matroid*
$L(e, B)$	f *− circuit of* e *with respect to base* B
$B(e, B)$	f *− bond of* e *with respect to base* B
$r(T)$	*rank of the subset* T *in the given matroid*
$r(\mathcal{M})$	*rank of the underlying set of the matroid*
$\nu(T)$	*rank of the subset* T *in the dual of the given matroid*
$\nu(\mathcal{M})$	*rank of the underlying set in the dual matroid*
$\mathcal{M}(\mathcal{G})$	*polygon matroid of the graph* $\mathcal{G}$ *(bases are forests)*
$\mathcal{M}^*(\mathcal{G})$	*bond matroid of the graph* $\mathcal{G}$ *(bases are coforests)*
$\mathcal{M}(\mathcal{V})$	*matroid whose bases are maximal independent columns of a representative matrix of* $\mathcal{V}$
$\mathcal{M}^*(\mathcal{V})$	*dual of* $\mathcal{M}(\mathcal{V})$
$\int(X)$	*span (closure) of the subset* X *in the matroid*
$\mathcal{M} \cdot T$	*restriction of* $\mathcal{M}$ *to* T
$\mathcal{M} \times T$	*contraction of* $\mathcal{M}$ *to* T
$\mathcal{M}_1 \vee \mathcal{M}_2$	*union of matroids* $\mathcal{M}_1$ *and* $\mathcal{M}_2$

Electrical Networks

$\mathbf{v}$	*voltage vector*
$\mathbf{i}$	*current vector*
$\mathcal{N}$	*electrical network*
$\mathcal{N}_{AP}$	*electrical multiport with port set* P *and remaining edge set* A
$\mathcal{E}$	*set of voltage sources in the network*
$\mathcal{J}$	*set of current sources in the network*
R	*resistance, also collection of resistors or 'current controlled voltage' elements in the network*