

SIXTH EDITION

STRUCTURAL ANALYSIS

A UNIFIED CLASSICAL
AND MATRIX APPROACH



Spon Press

A. GHALI, A.M. NEVILLE and T.G. BROWN

SPON TEXT

STRUCTURAL ANALYSIS

This comprehensive textbook, now in its sixth edition, combines classical and matrix-based methods of structural analysis and develops them concurrently. New solved examples and problems have been added, giving over 140 worked examples and more than 400 problems with answers.

The introductory chapter on structural analysis modeling gives a good grounding to the beginner, showing how structures can be modeled as beams, plane or space frames and trusses, plane grids or assemblages of finite elements. Idealization of loads, anticipated deformations, deflected shapes and bending moment diagrams are presented. Readers are also shown how to idealize real three-dimensional structures into simplified models that can be analyzed with little or no calculation, or by more involved calculations using computers. Dynamic analysis, essential for structures subject to seismic ground motion, is further developed in this edition and in a code-neutral manner. The topic of structural reliability analysis is discussed in a new chapter.

Translated into six languages, this textbook is of considerable international renown, and is widely recommended by many civil and structural engineering lecturers to their students because of its clear and thorough style and content.

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Preface to the sixth edition

We are proud that this book now appears in its sixth edition; it exists also in six translations, the most recent being the Japanese (2003) and the Chinese (2008). This new edition is updated and expanded.

The changes from the fifth edition have been guided by developments in knowledge as well as by reviews and comments of teachers, students, and designers who have been using the earlier editions. There are added chapters and sections, an additional appendix, revisions to existing chapters, and more examples and problems. The answers to all problems are given at the end of the book. Throughout the book, great attention is given to the analysis of three-dimensional spatial structures.

The book starts with a chapter on structural analysis modeling by idealizing a structure as a beam, a plane or a space frame, and a plane or a space truss, a plane grid, or as an assemblage of finite elements. There are new sections on the strut-and-tie models for the analysis of reinforced structures after cracking. There is a discussion of the suitability of these models, forces, and deformations, sketching deflected shapes, and bending moment diagrams, and a comparison of internal forces and deflections in beams, arches and trusses.

The chapter on modeling is followed by a chapter on the analysis of statically determinate structures, intended to provide a better preparation for students. To encourage early use of computers, five of the computer programs described in Appendix L, available from a web site, are mentioned in Chapter 1. These are for the linear analysis of plane and space trusses, plane and space frames, and plane grids. Simple matrix algebra programs, which can perform frequently needed matrix operations, can also be downloaded from the web site. The web site address is:

<http://www.routledge.com/books/Structural-Analysis-isbn9780415774338>

In Chapters 3 to 6 we introduce two distinct general approaches of analysis: the force method and the displacement method. Both methods involve the solution of linear simultaneous equations relating forces to displacements. The emphasis in these four chapters is on the basic ideas in the two methods, without obscuring the procedure by the details of derivation of the coefficients needed to form the equations. Instead, use is made of Appendices B, C, and D, which give, respectively, displacements due to applied unit forces, forces corresponding to unit displacements, and fixed-end forces in straight members due to various loadings. The consideration of the details of the methods of displacement calculation is thus delayed to Chapters 7 to 10, by which time the need for this material in the analysis of statically indeterminate structures is clear. This sequence of presentation of material is particularly suitable when the reader is already acquainted with some of the methods for calculating the deflection of beams. If, however, it is thought preferable first to deal with methods of calculation of displacements, Chapters 7 to 10 should be studied before Chapters 4 to 6; this will not disturb the continuity.

The material presented is both elementary and advanced, covering the whole field of structural analysis. The classical and modern methods of structural analysis are combined in a unified presentation, but some of the techniques not widely used in modern practice have been omitted.

However, the classical methods of column analogy and moment distribution, suitable for hand calculations, continue to be useful for preliminary calculation and for checking computer results; these are presented in Chapter 11. To provide space for new topics needed in modern practice, the coverage of the two methods is shorter compared to the fifth edition. The no-shear moment distribution technique, suitable for frames having many joint translations, has been removed because computers are now commonly employed for such frames.

The methods for obtaining the influence lines for beams, frames, grids and trusses are combined in Chapter 12, which is shorter than the sum of the two chapters in the fifth edition. In Chapter 13, the effects of axial forces on the stiffness characteristics of members of framed structures are discussed and applied in the determination of the critical buckling loads of continuous frames.

Chapter 14 deals with the analysis of shear walls, commonly used in modern buildings. The chapter summarizes the present knowledge, states the simplifying assumptions usually involved, and presents a method of analysis that can be applied in most practical cases.

The provision of outriggers is an effective means of reducing the drift and the bending moments due to lateral loads in high-rise buildings. The analysis of outrigger-braced buildings and the location of the outriggers for optimum effectiveness are discussed in new sections in Chapter 14. The analysis is demonstrated in a solved example of a 50-storey building.

The finite-difference method and, to an even larger extent, the finite-element method are powerful tools, which involve a large amount of computation. Chapter 15 deals with the use of finite differences in the analysis of structures composed of beam elements and extends the procedure to axisymmetrical shells of revolution. The finite-difference method is also used in the analysis of plates. Chapters 16 and 17 are concerned with two- and three-dimensional finite elements. Chapters 21, 22, 16, and 17 can be used, in that order, in a graduate course on the fundamentals of the finite-element method.

Modern design of structures is based on both the elastic and plastic analyses. The plastic analysis cannot replace the elastic analysis but supplements it by giving useful information about the collapse load and the mode of collapse. Chapters 18 and 19 deal with the plastic analysis of framed structures and slabs respectively.

An introduction to structural dynamics is presented in Chapter 20. This is a study of the response of structures to dynamic loading produced by machinery, gusts of wind, blast, or earthquakes. First, free and forced vibrations of a system with one degree of freedom are discussed. This is then extended to multi-degree-of-freedom systems. Several new sections discuss the dynamic analysis of structures subjected to earthquakes.

Some structures, such as cable nets and fabrics, trusses, and frames with slender members, may have large deformations, so that it is necessary to consider equilibrium in the real deformed configurations. This requires the geometric nonlinear analysis treated in Chapter 23, in which the Newton-Raphson's iterative technique is employed. The same chapter also introduces the material-nonlinear analysis, in which the stress-strain relation of the material used in the structure is nonlinear.

Chapter 24, based on the probability theory, is new in this edition. This chapter represents a practical introductory tool for the reliability analysis of structures. The objective is to provide a measure for the reliability or the probability of satisfactory performance of new or existing structures. The most important probability aspects used in Chapter 24 are presented in Appendix M; previous knowledge of probability and statistics is not required. Chapter 24 and Appendix M were written in collaboration with Professor Andrzej S. Nowak of the University of Nebraska, Lincoln, USA. Professor Marc Maes was the first to include reliability of structures in an undergraduate course on structural analysis at the University of Calgary, Canada; the authors are grateful to him for making his lecture notes available.

The techniques of analysis, which are introduced, are illustrated by many solved examples and a large number of problems at the ends of chapters, with answers given at the end of the

book. In this edition, with new solved examples and problems added, there are more than 140 worked examples and more than 400 problems with answers.

No specific system of units is used in most of the examples and problems. However, there is a small number of examples and problems where it was thought advantageous to use actual dimensions of the structure and to specify the magnitude of forces. These problems are set in the so-called Imperial or British units (still common in the USA) as well as in the SI units. Each problem in which Imperial units are used is followed by a version of the same problem in SI units, so that the reader may choose the system of units he or she prefers.

Data frequently used are presented in the appendices, with Appendix A offering a review of matrix operations usually needed in structural analysis. Matrix notation is extensively used in this book because this makes it possible to present equations in a compact form, helping the reader to concentrate on the overall operations without being distracted by algebraic or arithmetical details.

Several computer programs are briefly described in Appendix L. These include the computer programs available from the web site mentioned above and four nonlinear analysis programs. An order form at the end of the book can be used to obtain the nonlinear analysis programs. The computer programs can be employed in structural engineering practice and also to aid the study of structural analysis. However, understanding the book does not depend upon the availability of these programs.

The text has been developed by the first and principal author in teaching, over a number of years, undergraduate and graduate courses at the University of Calgary, Canada. The third author, who joined in the fifth edition, has also been teaching the subject at Calgary. Teaching and understanding the needs of the students have helped in preparing a better edition, believed to be easier to study.

Chapters 1 to 13, 18, and 19 contain basic material which should be covered in the first courses. From the remainder of the book, a suitable choice can be made to form a more advanced course. The contents have been selected to make the book suitable not only for the student but also for the practicing engineer who wishes to obtain guidance on the most convenient methods of analysis for a variety of types of structures.

Dr. Ramez Gayed, research associate at the University of Calgary, has checked the new material, the solution of the examples, and some of the answers of the problems, and provided the figures for the sixth edition.

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April 2008

Notation

The following is a list of symbols which are common in the various chapters of the text; other symbols are used in individual chapters. All symbols are defined in the text when they first appear.

A	Any action, which may be a reaction or a stress resultant. A stress resultant at a section of a framed structure is an internal force: bending moment, shearing force or axial force.
a	Cross-sectional area.
D_i or D_{ij}	Displacement (rotational or translational) at coordinate i . When a second subscript j is provided it indicates the coordinate at which the force causing the displacement acts.
E	Modulus of elasticity.
EI	Flexural rigidity.
F	A generalized force: a couple or a concentrated load.
FEM	Fixed-end moment.
f_{ij}	Element of flexibility matrix.
G	Modulus of elasticity in shear.
I	Moment of inertia or second moment of area.
i, j, k, m, n, p, r	Integers.
J	Torsion constant (length ⁴), equal to the polar moment of inertia for a circular cross section.
l	Length.
M	Bending moment at a section, e.g. M_n = bending moment at sections. In beams and grids, a bending moment is positive when it causes tension in bottom fibers.
M_{AB}	Moment at end A of member AB . In plane structures, an end-moment is positive when clockwise. In general, an end-moment is positive when it can be represented by a vector in the positive direction of the axes x , y , or z .
N	Axial force at a section or in a member of a truss.
P, Q	Concentrated loads.
q	Load intensity.
R	Reaction.
S_{ij}	Element of stiffness matrix.
s	Used as a subscript, indicates a statically determinate action.
T	Twisting moment at a section.
u	Used as a subscript, indicates the effect of unit forces or unit displacements.

V	Shearing force at a section.
W	Work of the external applied forces.
ϵ	Strain.
η	Influence ordinate.
ν	Poisson's ratio.
σ	Stress.
τ	Shearing stress.
$\{ \}$	Braces indicate a vector, i.e. a matrix of one column. To save space, the elements of a vector are sometimes listed in a row between two braces.
$[\]$	Brackets indicate a rectangular or square matrix.
$[T]_{n \times m}^T$	Superscript T indicates matrix transpose. $n \times m$ indicates the order of the matrix which is to be transposed resulting in an $m \times n$ matrix.
\longleftrightarrow	Double-headed arrow indicates a couple or a rotation: its direction is that of the rotation of a right-hand screw progressing in the direction of the arrow.
\longrightarrow	Single-headed arrow indicates a load or a translational displacement.
$z \begin{array}{c} \searrow \\ \downarrow \\ \nearrow \end{array} \begin{array}{c} x \\ y \end{array}$	Axes: the positive direction of the z axis points away from the reader.

The SI system of units of measurement

Length	meter	m
	millimeter = 10^{-3} m	mm
Area	square meter	m ²
	square millimeter = 10^{-6} m ²	mm ²
Volume	cubic meter	m ³
Frequency	hertz = 1 cycle per second	Hz
Mass	kilogram	kg
Density	kilogram per cubic meter	kg/m ³
Force	newton	N
	= a force which applied to a mass of one kilogram gives it an acceleration of one meter per second, i.e. $1\text{N} = 1\text{kgm/s}^2$	
Stress	newton per square meter	N/m ²
	newton per square millimeter	N/mm ²
Temperature interval	degree Celsius	deg C; °C
Nomenclature for multiplication factors		
	10^9 giga G	
	10^6 mega M	
	10^3 kilo k	
	10^{-3} milli m	
	10^{-6} micro μ	
	10^{-9} nano n	

Contents

<i>Preface to the sixth edition</i>	xxii
<i>Notation</i>	xxv
<i>The SI system of units of measurement</i>	xxvii
1 Structural analysis modeling	1
1.1 Introduction	1
1.2 Types of structures	2
1.2.1 Cables and arches	8
1.3 Load path	11
1.4 Deflected shape	14
1.5 Structural idealization	15
1.6 Framed structures	16
1.6.1 Computer programs	18
1.7 Non-framed or continuous structures	19
1.8 Connections and support conditions	19
1.9 Loads and load idealization	21
1.9.1 Thermal effects	22
1.10 Stresses and deformations	23
1.11 Normal stress	25
1.11.1 Normal stresses in plane frames and beams	26
1.11.2 Examples of deflected shapes and bending moment diagrams	28
1.11.3 Deflected shapes and bending moment diagrams due to temperature variation	30
1.12 Comparisons: beams, arches and trusses	30
Example 1.1 Load path comparisons: beam, arch and truss	30
Example 1.2 Three-hinged, two-hinged, and totally fixed arches	34
1.13 Strut-and-tie models in reinforced concrete design	37
1.13.1 B- and D-regions	38
Example 1.3 Strut-and-tie model for a wall supporting an eccentric load	40
1.13.2 Statically indeterminate strut-and-tie models	40

1.14	<i>Structural design</i>	41	
1.15	<i>General</i>	42	
	<i>Problems</i>	42	
2	Statically determinate structures		46
2.1	<i>Introduction</i>	46	
2.2	<i>Equilibrium of a body</i>	48	
	Example 2.1 <i>Reactions for a spatial body: a cantilever</i>	49	
	Example 2.2 <i>Equilibrium of a node of a space truss</i>	51	
	Example 2.3 <i>Reactions for a plane frame</i>	52	
	Example 2.4 <i>Equilibrium of a joint of a plane frame</i>	52	
	Example 2.5 <i>Forces in members of a plane truss</i>	53	
2.3	<i>Internal forces: sign convention and diagrams</i>	53	
2.4	<i>Verification of internal forces</i>	56	
	Example 2.6 <i>Member of a plane frame: V and M-diagrams</i>	58	
	Example 2.7 <i>Simple beams: verification of V and M-diagrams</i>	59	
	Example 2.8 <i>A cantilever plane frame</i>	59	
	Example 2.9 <i>A simply-supported plane frame</i>	60	
	Example 2.10 <i>M-diagrams determined without calculation of reactions</i>	61	
	Example 2.11 <i>Three-hinged arches</i>	62	
2.5	<i>Effect of moving loads</i>	63	
	2.5.1 <i>Single load</i>	63	
	2.5.2 <i>Uniform load</i>	63	
	2.5.3 <i>Two concentrated loads</i>	64	
	Example 2.12 <i>Maximum bending moment diagram</i>	66	
	2.5.4 <i>Group of concentrated loads</i>	67	
	2.5.5 <i>Absolute maximum effect</i>	67	
	Example 2.13 <i>Simple beam with two moving loads</i>	69	
2.6	<i>Influence lines for simple beams and trusses</i>	69	
	Example 2.14 <i>Maximum values of M and V using influence lines</i>	71	
2.7	<i>General</i>	73	
	<i>Problems</i>	73	
3	Introduction to the analysis of statically indeterminate structures		80
3.1	<i>Introduction</i>	80	
3.2	<i>Statical indeterminacy</i>	80	
3.3	<i>Expressions for degree of indeterminacy</i>	84	
3.4	<i>General methods of analysis of statically indeterminate structures</i>	88	
3.5	<i>Kinematic indeterminacy</i>	88	
3.6	<i>Principle of superposition</i>	92	
3.7	<i>General</i>	94	
	<i>Problems</i>	95	

4	Force method of analysis	99
4.1	<i>Introduction</i>	99
4.2	<i>Description of method</i>	99
	<i>Example 4.1 Structure with degree of indeterminacy = 2</i>	100
4.3	<i>Released structure and coordinate system</i>	103
	4.3.1 <i>Use of coordinate represented by a single arrow or a pair of arrows</i>	104
4.4	<i>Analysis for environmental effects</i>	104
	4.4.1 <i>Deflected shapes due to environmental effects</i>	106
	<i>Example 4.2 Deflection of a continuous beam due to temperature variation</i>	106
4.5	<i>Analysis for different loadings</i>	107
4.6	<i>Five steps of force method</i>	107
	<i>Example 4.3 A stayed cantilever</i>	108
	<i>Example 4.4 A beam with a spring support</i>	109
	<i>Example 4.5 Simply-supported arch with a tie</i>	110
	<i>Example 4.6 Continuous beam: support settlement and temperature change</i>	112
	<i>Example 4.7 Release of a continuous beam as a series of simple beams</i>	114
4.7	<i>Equation of three moments</i>	118
	<i>Example 4.8 The beam of Example 4.7 analyzed by equation of three moments</i>	120
	<i>Example 4.9 Continuous beam with overhanging end</i>	121
	<i>Example 4.10 Deflection of a continuous beam due to support settlements</i>	123
4.8	<i>Moving loads on continuous beams and frames</i>	124
	<i>Example 4.11 Two-span continuous beam</i>	126
4.9	<i>General</i>	128
	<i>Problems</i>	129
5	Displacement method of analysis	135
5.1	<i>Introduction</i>	135
5.2	<i>Description of method</i>	135
	<i>Example 5.1 Plane truss</i>	136
5.3	<i>Degrees of freedom and coordinate system</i>	139
	<i>Example 5.2 Plane frame</i>	140
5.4	<i>Analysis for different loadings</i>	143
5.5	<i>Analysis for environmental effects</i>	144
5.6	<i>Five steps of displacement method</i>	144
	<i>Example 5.3 Plane frame with inclined member</i>	145
	<i>Example 5.4 A grid</i>	148

5.7	<i>Analysis of effects of displacements at the coordinates</i>	151
	<i>Example 5.5 A plane frame: condensation of stiffness matrix</i>	152
5.8	<i>General</i>	153
	<i>Problems</i>	153
6	Use of force and displacement methods	159
6.1	<i>Introduction</i>	159
6.2	<i>Relation between flexibility and stiffness matrices</i>	159
	<i>Example 6.1 Generation of stiffness matrix of a prismatic member</i>	161
6.3	<i>Choice of force or displacement method</i>	161
	<i>Example 6.2 Reactions due to unit settlement of a support of a continuous beam</i>	162
	<i>Example 6.3 Analysis of a grid ignoring torsion</i>	163
6.4	<i>Stiffness matrix for a prismatic member of space and plane frames</i>	164
6.5	<i>Condensation of stiffness matrices</i>	167
	<i>Example 6.4 End-rotational stiffness of a simple beam</i>	168
6.6	<i>Properties of flexibility and stiffness matrices</i>	169
6.7	<i>Analysis of symmetrical structures by force method</i>	171
6.8	<i>Analysis of symmetrical structures by displacement method</i>	174
	<i>Example 6.5 Single-bay symmetrical plane frame</i>	176
	<i>Example 6.6 A horizontal grid subjected to gravity load</i>	177
6.9	<i>Effect of nonlinear temperature variation</i>	179
	<i>Example 6.7 Thermal stresses in a continuous beam</i>	182
	<i>Example 6.8 Thermal stresses in a portal frame</i>	185
6.10	<i>Effect of shrinkage and creep</i>	187
6.11	<i>Effect of prestressing</i>	188
	<i>Example 6.9 Post-tensioning of a continuous beam</i>	190
6.12	<i>General</i>	192
	<i>Problems</i>	192
7	Strain energy and virtual work	198
7.1	<i>Introduction</i>	198
7.2	<i>Geometry of displacements</i>	199
7.3	<i>Strain energy</i>	201
	<i>7.3.1 Strain energy due to axial force</i>	205
	<i>7.3.2 Strain energy due to bending moment</i>	206
	<i>7.3.3 Strain energy due to shear</i>	207
	<i>7.3.4 Strain energy due to torsion</i>	208
	<i>7.3.5 Total strain energy</i>	208
7.4	<i>Complementary energy and complementary work</i>	208
7.5	<i>Principle of virtual work</i>	211

7.6	<i>Unit-load and unit-displacement theorems</i>	212
7.7	<i>Virtual-work transformations</i>	214
	<i>Example 7.1 Transformation of a geometry problem</i>	216
7.8	<i>Castigliano's theorems</i>	217
	7.8.1 <i>Castigliano's first theorem</i>	217
	7.8.2 <i>Castigliano's second theorem</i>	219
7.9	<i>General</i>	221
8	Determination of displacements by virtual work	223
8.1	<i>Introduction</i>	223
8.2	<i>Calculation of displacement by virtual work</i>	223
8.3	<i>Displacements required in the force method</i>	226
8.4	<i>Displacement of statically indeterminate structures</i>	226
8.5	<i>Evaluation of integrals for calculation of displacement by method of virtual work</i>	229
	8.5.1 <i>Definite integral of product of two functions</i>	231
	8.5.2 <i>Displacements in plane frames in terms of member end moments</i>	232
8.6	<i>Truss deflection</i>	232
	<i>Example 8.1 Plane truss</i>	233
	<i>Example 8.2 Deflection due to temperature: statically determinate truss</i>	234
8.7	<i>Equivalent joint loading</i>	235
8.8	<i>Deflection of beams and frames</i>	237
	<i>Example 8.3 Simply-supported beam with overhanging end</i>	237
	<i>Example 8.4 Deflection due to shear in deep and shallow beams</i>	239
	<i>Example 8.5 Deflection calculation using equivalent joint loading</i>	241
	<i>Example 8.6 Deflection due to temperature gradient</i>	242
	<i>Example 8.7 Effect of twisting combined with bending</i>	243
	<i>Example 8.8 Plane frame: displacements due to bending, axial and shear deformations</i>	244
	<i>Example 8.9 Plane frame: flexibility matrix by unit-load theorem</i>	247
	<i>Example 8.10 Plane truss: analysis by the force method</i>	248
	<i>Example 8.11 Arch with a tie: calculation of displacements needed in force method</i>	250
8.9	<i>General</i>	251
	<i>Problems</i>	251
9	Further energy theorems	260
9.1	<i>Introduction</i>	260
9.2	<i>Betti's and Maxwell's theorems</i>	260

9.3	<i>Application of Betti's theorem to transformation of forces and displacements</i>	262
	Example 9.1 <i>Plane frame in which axial deformation is ignored</i>	266
9.4	<i>Transformation of stiffness and flexibility matrices</i>	266
9.5	<i>Stiffness matrix of assembled structure</i>	269
	Example 9.2 <i>Plane frame with inclined member</i>	270
9.6	<i>Potential energy</i>	271
9.7	<i>General</i>	273
	<i>Problems</i>	273
10	Displacement of elastic structures by special methods	278
10.1	<i>Introduction</i>	278
10.2	<i>Differential equation for deflection of a beam in bending</i>	278
10.3	<i>Moment–area theorems</i>	280
	Example 10.1 <i>Plane frame: displacements at a joint</i>	282
10.4	<i>Method of elastic weights</i>	283
	Example 10.2 <i>Parity of use of moment–area theorems and method of elastic weights</i>	284
	Example 10.3 <i>Beam with intermediate hinge</i>	286
	Example 10.4 <i>Beam with ends encastré</i>	287
	10.4.1 <i>Equivalent concentrated loading</i>	287
	Example 10.5 <i>Simple beam with variable I</i>	291
	Example 10.6 <i>End rotations and transverse deflection of a member in terms of curvature at equally-spaced sections</i>	292
	Example 10.7 <i>Bridge girder with variable cross section</i>	294
10.5	<i>Method of finite differences</i>	295
10.6	<i>Representation of deflections by Fourier series</i>	297
	Example 10.8 <i>Triangular load on a simple beam</i>	298
10.7	<i>Representation of deflections by series with indeterminate parameters</i>	298
	Example 10.9 <i>Simple beam with a concentrated transverse load</i>	302
	Example 10.10 <i>Simple beam with an axial compressive force and a transverse concentrated load</i>	303
	Example 10.11 <i>Simple beam on elastic foundation with a transverse force</i>	304
10.8	<i>General</i>	304
	<i>Problems</i>	305
11	Applications of force and displacement methods: column analogy and moment distribution	309
11.1	<i>Introduction</i>	309
11.2	<i>Analogous column: definition</i>	309

11.3	<i>Stiffness matrix of nonprismatic member</i>	310
11.3.1	<i>End rotational stiffness and carryover moment</i>	311
	<i>Example 11.1 Nonprismatic member: end-rotational stiffness, carryover factors and fixed-end moments</i>	312
11.4	<i>Process of moment distribution</i>	314
	<i>Example 11.2 Plane frame: joint rotations without translations</i>	316
11.5	<i>Moment-distribution procedure for plane frames without joint translation</i>	318
	<i>Example 11.3 Continuous beam</i>	319
11.6	<i>Adjusted end-rotational stiffnesses</i>	321
11.7	<i>Adjusted fixed-end moments</i>	323
	<i>Example 11.4 Plane frame symmetry and antisymmetry</i>	324
	<i>Example 11.5 Continuous beam with variable section</i>	326
11.8	<i>General moment distribution procedure: plane frames with joint translations</i>	328
	<i>Example 11.6 Continuous beam on spring supports</i>	329
11.9	<i>General</i>	332
	<i>Problems</i>	332
12	<i>Influence lines</i>	341
12.1	<i>Introduction</i>	341
12.2	<i>Concept and application of influence lines</i>	341
12.3	<i>Müller-Breslau's principle</i>	342
12.3.1	<i>Procedure for obtaining influence lines</i>	346
12.4	<i>Correction for indirect loading</i>	348
12.5	<i>Influence lines for a beam with fixed ends</i>	348
12.6	<i>Influence lines for plane frames</i>	351
	<i>Example 12.1 Bridge frame: influence line for member end moment</i>	353
12.7	<i>Influence lines for grids</i>	355
	<i>Example 12.2 Grid: influence line for bending moment in a section</i>	357
	<i>Example 12.3 Torsionless grid</i>	359
12.8	<i>General superposition equation</i>	363
12.9	<i>Influence lines for arches</i>	364
12.10	<i>Influence lines for trusses</i>	366
	<i>Example 12.4 Continuous truss</i>	366
12.11	<i>General</i>	369
	<i>Problems</i>	370
13	<i>Effects of axial forces on flexural stiffness</i>	373
13.1	<i>Introduction</i>	373
13.2	<i>Stiffness of a prismatic member subjected to an axial force</i>	373