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Transcendental Dynamics and Complex Analysis

Edited by
Philip J. Rippon and
Gwyneth M. Stallard

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A Tribute to Noel Baker

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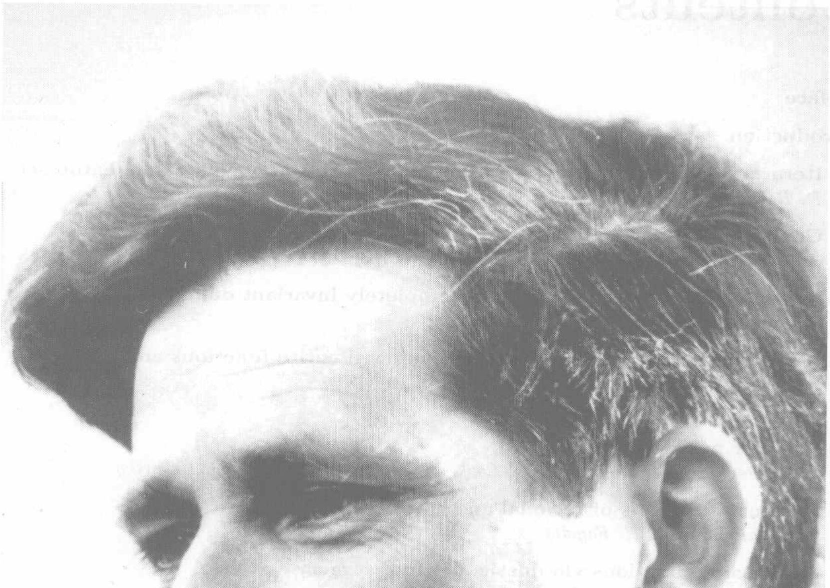

Professor Noel Baker (1932–2001)

PREFACE

This book was written in honour of Noel Baker following his sudden death in 2001. It comprises a collection of articles written by friends, colleagues and former students of Noel. In particular, we are delighted that Noel's MSc supervisor and long-time friend, George Szekeres, was able to contribute a paper to this volume — he sadly died before the book was published.

All of these articles deal with topics that interested Noel and, in most cases, they are in areas where Noel's own work has been very influential. Several of the papers are survey articles that we hope will be a valuable addition to the literature. There are also new results that Noel would have been delighted to have seen. Most of the papers deal with the iteration of transcendental meromorphic functions — the field in which Noel was pre-eminent and in which he carried out much of the pioneering work — and there are also some papers in closely related topics that he would have enjoyed. As this volume shows, much of the recent work in complex dynamics (as the subject of iteration theory is now called) builds on ideas and techniques that Noel introduced and that will continue to be used by all those who work in this field. We hope that this book will be a fitting memorial to a man who inspired so many of us.

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INTRODUCTION

In this introduction, we summarise the mathematical career of Noel Baker and indicate how the papers in this volume relate to his work. Much of the material is taken from the obituary of Noel Baker that appeared in the *Bulletin of the London Mathematical Society* [17].

Noel Baker was born on 10 August 1932 and died, of a heart attack, on 20 May 2001. He grew up in Australia and was first introduced to the theory of iteration by his MSc supervisor, George Szekeres, who suggested that he work on the functional equation

$$f(f(z)) = F(z),$$

where f and F are analytic functions. In his first mathematical paper (1), Noel used the theory of iteration of analytic functions, which had been developed principally by Fatou and Julia and which was not well known at that time. He used this theory to show, amongst other things, that if F belongs to a certain class of entire functions, which includes the exponential function, then the above equation has no entire solution. This first paper also contains examples that were constructed using Wiman-Valiron theory. Throughout his career Noel was to find ever more techniques from classical complex analysis that can usefully be applied to iteration theory.

In 1955 Noel won a German government scholarship to the University of Tübingen, where he worked under Hellmuth Kneser. Noel's doctoral thesis, published in (2), continued his study of functional equations. From 1957 to 1959, Noel taught mathematics at the University of Alberta in Edmonton, Canada. In 1959 he moved to Imperial College London, where he remained until retirement in 1997.

In his research, Noel worked on many problems in complex analysis and had a wide range of collaborators, but iteration theory, his great love, was for many years a lone interest. However, when the subject was reborn around 1980, partly as a result of the advent of accessible computer graphics, it became clear to the new adherents that Noel had for many years been quietly and carefully completing the foundations begun earlier in the century by the French mathematicians Pierre Fatou and Gaston Julia. He had also pointed the way towards many future developments, both by proving new results and by posing challenging problems. In the explosion of research on iteration theory that took place in the subsequent years, many of the papers published on iteration made reference to Noel's work and he received many invitations to speak at international conferences on iteration. At these he would often appear reserved, much preferring to let others speak about the latest work, even though he was the acknowledged authority on many matters, and the person whose judgement about the validity of a new proof was always sought. Noel continued his research after his retirement and one of his last papers was dedicated to George Szekeres on the occasion of the latter's 90th birthday.

Noel's early work on functional equations led him to consider problems about periodic points, which play a very important role in complex dynamics. It was already known that for an entire function there must be infinitely many periodic points of period p, for all $p \geq 2$, but Noel considered the unsolved problem of the existence of periodic points of a given *exact* period. He showed in (6) that for all non-linear entire functions there exist periodic points of exact period p, for all p with at most one exception; for example, $f(z) = z + e^z$ has no fixed points. In a later paper (13) Noel showed that for a polynomial the only possible exceptional value in this result is $p = 2$, the corresponding exceptional functions being $f(z) = z^2 - z$ and other quadratics 'similar' to this one. He also conjectured that for a transcendental entire function the only possible exceptional value is $p = 1$, and this was proved by Bergweiler [2].

We now describe the origins of complex dynamics. Let f be a rational function of degree at least 2 or a transcendental entire function. The set of points near which the sequence of iterates f^n forms a normal family is called the *Fatou set* $F(f)$ and its complement is called the *Julia set* $J(f)$. Roughly speaking, the dynamics are stable on the Fatou set and chaotic on the Julia set. Also, the Julia set often exhibits great topological complexity as well as 'self-similarity'; for example, the paper in this volume by Devaney *et al* discusses a family of rational functions whose Julia sets in some cases contain Cantor sets of curves and in other cases contain Sierpinski curves.

The fundamental properties of the sets $F(f)$ and $J(f)$ were first established for rational functions in [13] and [8], and for transcendental entire functions in [9]. In the last paper, Fatou studied the iteration of transcendental entire functions in some detail, giving examples that pointed to significant differences to the theory that had been developed for rational functions. He asked the following fundamental questions about a transcendental entire function f:

1. Are the repelling periodic points of f dense in $J(f)$?
2. Are there examples where $J(f) = \mathbb{C}$? In particular, is this true for $f(z) = e^z$?
3. Can $J(f)$ be totally disconnected?
4. Must $J(f)$ contain infinitely many unbounded analytic curves, at each point of which $f^n \to \infty$?

Question 1 is of great theoretical importance, and it had been answered 'yes' for rational functions by both Fatou and Julia. Fatou had also given an example of a rational function f for which $J(f)$ is totally disconnected, and Lattès [14] an example for which $J(f) = \mathbb{C}$. Most of Fatou's questions were solved by Noel during the decade 1965–1975, as we now indicate.

The first question was answered in the affirmative in the paper (22), which is of fundamental importance in complex dynamics and appropriately dedicated to Hellmuth Kneser. Here, Noel called on a deep covering theorem due to Ahlfors (see [11, page 148]) to show that arbitrarily close to each point of $J(f)$ there is a repelling periodic point of f. From this, he deduced the general result that if f is any non-linear entire function, then the set of entire functions that commute with f is countable. Many authors have tried to simplify the proof in (22) that the repelling periodic points are dense in $J(f)$, in order to avoid the deep theorem of Ahlfors. Eventually, more elementary proofs based on a renormalisation technique were given by Schwick [18], Bargmann [1], and Berteloot and Duval [4].

Two years later, in (25), Noel answered the first part of Fatou's second question by showing that there is a function of the form $f(z) = kze^z$, where $k > 0$, such that $J(f) = \mathbb{C}$. A proof that if $f(z) = e^z$, then we have $J(f) = \mathbb{C}$ was given ten years later by Misiurewicz [15].

Noel answered Fatou's third question in the negative in (32). If $J(f)$ is totally disconnected, then $F(f)$ must have a single unbounded multiply connected component. Noel had already constructed in (9) an example of a transcendental entire function for which $F(f)$ has at least one multiply connected component. This function was of the form

$$f(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right),$$

in which the positive constants $r_1 < r_2 < \ldots$ have the property that

$$f(A_n) \subset A_{n+1}, \quad \text{where } A_n = \{z : r_n^2 < |z| < r_{n+1}^{1/2}\}.$$

However, Noel did not determine in (9) whether $F(f)$ has a single unbounded multiply connected component or a sequence of bounded multiply connected components. In (33) he used Schottky's theorem [11, page 169], yet another result from classical complex analysis, to show that the latter must be the case. This solved another important problem in complex dynamics, open since the work of Fatou and Julia, by showing that the above function has a sequence of *wandering domains*, that is, distinct components U_n of $F(f)$ such that $f(U_n) \subset U_{n+1}$, for $n = 1, 2, \ldots$. In contrast, Sullivan [16] showed that rational functions do not have wandering domains. The paper (32), written later than but published earlier than (33), used Schottky's theorem once again to show that a transcendental entire function cannot have an unbounded multiply connected component of $F(f)$, thus proving that $J(f)$ can never be totally disconnected.

The results in (32) and (33) led to much further work. In (53), Noel showed that wandering domains for transcendental entire functions may be infinitely connected. For many years it was not known whether such wandering domains could be finitely connected. In this volume, Kisaka and Shishikura show that they can in fact have any given finite connectivity.

The result in (32) shows that if f is a transcendental entire function, then $J(f)$ must contain a continuum, so its Hausdorff dimension $\dim_H J(f)$ is at least 1. It remains an open question whether $\dim_H J(f) = 1$ is possible. In this volume, there is a survey article on dimensions of Julia sets by Stallard, complemented by a survey article on fractal measures and ergodic theory by Kotus and Urbański.

Noel's wandering domains example mentioned earlier shows that the answer to Fatou's fourth question (as stated here) is 'no'. However, the structure of the 'escaping set', where $f^n \to \infty$, continues to stimulate much work, including the paper by Rottenfusser and Schleicher that appears in this volume.

Sullivan's remarkable result [16] that rational functions do not have wandering domains was proved using new techniques based on quasiconformal conjugacy. Noel quickly saw that these new techniques would also apply to various families of transcendental entire functions, and a proof that exponential functions have no wandering domains appeared in (49). This was one of a number of papers at that time that established many of the basic dynamical properties of the exponential family and began the description of the corresponding parameter space, the 'exponential Mandelbrot set', which has since been the subject of much study — see, for example, the paper by Rempe and Schleicher in this volume.

In (41), Noel initiated another major development by showing that if a transcendental entire function f has order of growth at most $1/2$, minimal type, then $F(f)$ has no unbounded invariant components, and he also gave a more restrictive condition on the maximum modulus of f that forces every component of $F(f)$ to be bounded. The question of whether the latter conclusion follows from order at most $1/2$, minimal type, remains open, though many authors have obtained partial results in this direction; this volume contains a survey article on this problem by Hinkkanen.

A key step in Noel's proof in (41) is to exclude unbounded invariant components of $F(f)$ in which $f^n \to \infty$. He did this by establishing estimates for the growth of iterates in such components, which he later refined in (57). In recognition of his work on Fatou components of this type, Eremenko and Lyubich introduced the name *Baker domain* for such components in [7]. In this fundamental paper, Eremenko and Lyubich showed that if the set $S(f)$ of inverse function singularities of a transcendental entire function f is bounded, then f has no Baker domains and if $S(f)$ is finite, then f has no wandering domains; see also [10]. A survey article on Baker domains by Rippon appears in this volume.

Yet another fundamental contribution to the iteration of transcendental entire functions came in the papers (65), (73) and (74). Once again an unbounded invariant component U of $F(f)$ was considered, but now the aim was to describe the nature of the boundary of U. Some special cases had been investigated by other authors, following the appearance of computer pictures of Julia sets, but Noel and his students Weinreich and Domínguez attacked the general case. In (65), it was shown that

- if U is not a Baker domain (that is, U is an attracting basin, a parabolic basin, or a Siegel disc), then ∂U is sufficiently complicated that ∞ belongs to the impression of every prime end of U;
- if ∂U is a Jordan curve in the extended complex plane $\hat{\mathbb{C}}$ (and such U do exist), then not only must U be a Baker domain, but f must be univalent in U.

The key tool introduced in this work arises from the fact that if Ψ is a conformal map from the unit disc D onto U, then $\Psi^{-1} \circ f \circ \Psi$ is an *inner function*, that is, an analytic self-map of D whose angular limits have modulus 1 almost everywhere on ∂D. The paper (65) initiated a version of Fatou-Julia theory for inner functions, a topic now of interest in its own right, and this theory was taken further in (73). Further results on this theory are given in the paper by Bargmann in this volume.

Many of Noel's final papers are joint papers with his last student, Domínguez, and concern the connectedness properties of the Julia set. Many of these results are described and extended in the paper by Domínguez and Fagella in this volume.

Fatou-Julia theory of the iteration of *general* transcendental meromorphic functions was established in the fundamental papers (62), (63), (64) and (66) by Baker, Kotus and Lü. The Fatou set $F(f)$ is here taken to be the set of points near which the iterates f^n are defined and form a normal family, and then $J(f) = \hat{\mathbb{C}} \setminus F(f)$. Many of the basic results turn out to be similar to those for rational and entire functions, but there are some striking differences. For example, in (62) the authors showed that $J(f)$ is once again the closure of the repelling periodic points of f, and this fact is used to give a complete classification of those transcendental meromorphic functions, such as $f(z) = \tan z$, for which $J(f)$ is a subset of the real line; there are no transcendental entire functions for which the Julia set is contained in the real line. Then, in (63), they used techniques from approximation theory, pioneered by Eremenko and Lyubich [6], to construct transcendental meromorphic functions with wandering domains of all possible connectivities.

The question of periodic components was taken up in (64), where the authors showed that precisely five possible types can arise for a transcendental meromorphic function, namely, attracting basins, parabolic basins, Siegel discs, Herman rings and Baker domains. Moreover, any invariant components of $F(f)$ must be simply connected, doubly connected, or infinitely connected.

But perhaps the most striking result here was the construction of a transcendental meromorphic function f with a preperiodic component of $F(f)$ of any given finite connectivity. This construction used the powerful technique of quasiconformal surgery, introduced by Shishikura [19], which also appears in many of the papers in this volume — namely, those by Drasin and Langley, Domínguez and Fagella, and Kisaka and Shishikura. Finally, in (66), Sullivan's method of quasiconformal conjugacy was adapted to show that a transcendental meromorphic function of finite type has no wandering domains. These four papers opened a new and fruitful area of research, made even more accessible by the excellent survey article [3], which appeared soon after.

One of the differences between the iteration of entire functions and meromorphic functions is the number of completely invariant components of the Fatou set that can occur. In (24) Noel proved that a transcendental entire function can have at most one completely invariant component of the Fatou set. In (64) Baker, Kotus and Lü proved that a transcendental meromorphic function of finite type can have at most two completely invariant Fatou components, and in this volume it is shown by Bergweiler and Eremenko that, in these circumstances, the Julia set must be a Jordan curve. (An example of a function with these properties is $f(z) = \tan z$.) It is an open question whether a general transcendental meromorphic function can have at most two completely invariant Fatou components.

Fatou-Julia theory can be developed in many further directions. For a transcendental meromorphic function f, the iterates f^n need not be meromorphic. It is desirable, however, to have a closed system of iterates, so that we can consider, for example, the Fatou set of f^n, for $n \geq 2$. To obtain such a system, Noel's student Herring [12], and independently Bolsch [5], developed Fatou-Julia theory for functions, such as $f(z) = e^{\tan z}$, which are meromorphic outside certain compact totally disconnected subsets of $\hat{\mathbb{C}}$. Much of this theory, and its subsequent developments, is expounded in Noel's last papers (75), (77), (78) and (79).

This volume also contains papers that, while not explicitly about complex dynamics, are on closely related topics. The paper by Hayman and Hinkkanen is concerned with the growth of meromorphic functions that belong to certain normal families, the paper by Beardon and Minda classifies conformal automorphisms of finitely connected regions of the plane, and the paper by Szekeres is on possible connections between 'regular growth' and Abel's functional equation, a topic in which Noel had a great interest. Finally, the paper by Bullett and Freiburger is on the theory of holomorphic correspondences, a generalisation of complex dynamics. Here they investigate, for the first time, holomorphic correspondences that involve transcendental entire functions.

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