NONLINEAR
PHYSICAL
SCIENCE

CNC

Nail H. Ibragimov
Ranis N. Ibragimov

Applications of Lie Group Analysis in Geophysical Fluid Dynamics

李群分析在地球物理流体动力学中的应用

高等教育出版社
HIGHER EDUCATION PRESS

Nail H. Ibragimov
Ranis N. Ibragimov

Applications of Lie Group Analysis in Geophysical Fluid Dynamics

李群分析在地球物理
流体动力学中的应用

Liqun Fenxi zai Diqiu Wuli
Liuti Donglixue zhong de Yingyong

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

World Scientific

NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

Nail H. Ibragimov
Department of Mathematics and Science
Blekinge Institute of Technology
S-371 79 Karlskrona, Sweden

Ranis N. Ibragimov
Department of Mathematics
College of Science, Mathematics and Technology
University of Texas at Brownsville
Brownsville TX 78520, USA

图书在版编目（CIP）数据

李群分析在地球物理流体动力学中的应用=Applications of Lie Group Analysis in Geophysical Fluid Dynamics: 英文 / (瑞典) 伊布拉基莫夫(Ibragimov, N.H.), (加) 伊布拉基莫夫(Ibragimov, R.N.)著. —北京: 高等教育出版社, 2011.4
（非线性物理科学 / 罗朝俊, (瑞典) 伊布拉基莫夫主编）
ISBN 978-7-04-031954-5

Ⅰ. ①李… Ⅱ. ①伊… ②伊… Ⅲ. ①李群-应用-海洋地球物理学: 流体动力学-英文 Ⅳ. ①P738.1

中国版本图书馆 CIP 数据核字(2011)第 038208 号

策划编辑 王丽萍 **责任编辑** 李华英 **封面设计** 杨立新
责任印制 朱学忠

出版发行	高等教育出版社	**购书热线**	010-58581118
社　　址	北京市西城区德外大街 4 号	**咨询电话**	400-810-0598
邮政编码	100120	**网　　址**	http://www.hep.edu.cn
			http://www.hep.com.cn
		网上订购	http://www.landraco.com
经　　销	蓝色畅想图书发行有限公司		http://www.landraco.com.cn
印　　刷	涿州市星河印刷有限公司	**畅想教育**	http://www.widedu.com
开　　本	787 × 1092 1/16	**版　　次**	2011 年 4 月第 1 版
印　　张	14.25	**印　　次**	2011 年 4 月第 1 次印刷
字　　数	240 000	**定　　价**	59.00 元

物料号 31954-00

Preface

This is the first monograph dealing with applications of Lie group analysis and theory of resonant interactions to the modeling equations governing internal wave propagation in the deep ocean and in costal areas. Some specific applications to engineering problems related to reflection of internal waves from sloping bottom are subject of this book.

The book provides new conservation laws and exact solutions to the nonlinear theory of internal gravity waves in the ocean which allows to investigate the anisotropic properties of internal waves through the prism of *Lie group analysis*. Anisotropy is ubiquitous property of internal waves. But the importance to find exact solutions has been recognized only in recent years. Although it was known already in early 1970s that linear sinusoidal wavetrains satisfy the nonlinear equations for internal waves, the fact that plane-wave beams are nonlinear solutions as well had passed unnoticed until it was recognized in recent studies [76]. Since then, all analytical modeling of internal wave beams has been based on the very well-known physical property of a stratified fluid that the frequency of internal waves is independent of the magnitude of wavenumber vector so that any sinusoidal plane-wave disturbance in a uniformly stratified Boussinesq fluid obeying the linear dispersion relation satisfies the nonlinear equations of motion. However, the fact that the exact solutions representing the wave beams can be found on a regular basis, i.e., by solving the nonlinear equations in question, still remains unnoticed. It is shown in the book for the first time that internal wave beams representing exact solutions to the equation of motion of stratified fluid can be found by solving the given model as invariant solutions of nonlinear equations of motion. On the illustrative basis, it will be also shown here that the presence of the invariant solutions makes it possible to construct more general class of disturbances which represents wave beams propagating in certain direction coinciding with the beam energy radiation and is in the transverse direction to the general uniform plane wave profile.

The book also provides a concise but systematic introduction to the resonant theory of oceanic internal waves. Resonant wave interactions have been an active area of research from the very beginning. Resonant interactions play an important role in producing mixing in the interior of the ocean providing an important link in the presumed energy cascade from large to small scales. The Resonant Triad Model (RTM) developed in our recent studies in [33] is introduced in this book and it is used to provide an alternative approach to the Thorpe's problem [79] in the modeling of self-resonant internal waves, i.e., the waves for which a resonant interaction occurs at second order between the incident and reflected internal waves off slopes to study the special and temporal structure of mixing process in the costal areas of the ocean. The RTM represents the extension of the McComas' & Bretherton's three-wave hydrostatic model [55] developed previously for a single resonant triad process which ignores the effects of the earth's rotation and under hydrostatic approximation to the case of the non-hydrostatic analytical model involving arbitrarily large number of rotating internal waves with frequencies spanning the range of possible frequencies, i.e., between the maximum of the buoyancy frequency (vertical motion) and a minimum of the inertial frequency (horizontal motion). As an illustrative example, it is shown in this book that the classification of resonant interactions into the sum, middle and difference interaction classes identifies the value of latitude, which is classified as the singular latitude, at which the coalescence of the middle and difference interaction classes occurs. At this value of latitude, the value of the bottom slope at which the second-order frequency and wavenumber components of the incident and reflected waves satisfy the internal wave dispersion relation can be approximated by two latitude-dependent parameters in the limiting case when latitude approaches its singular value. Such coalescence phenomenon had passed unnoticed in previous studies. Although other analytical, numerical and statistical models designed to study the resonant interactions of internal waves in the ocean have been developed over the last decade, the crucial difference between the RTM model presented here and the previous studies is that the RTM model is based on understanding the dynamics of each individual wave in the model. Up to the present, the models based in that specification do not exist.

Overall, the main purpose of the book is to provide the link between modern achievements in Lie group analysis of differential equations and nonlinear internal wave modeling which is expected to result in a better observational knowledge of the spatial distribution of mixing than achieved to date, and to contribute to the improvement of diapycnal mixing parameterizations intended for ocean general circulation models and climate models.

The present book is meant for specialists and graduate students in applied mathematics interested in physics of the ocean.

Contents

Part I
Internal Waves in Stratified Fluid

Chapter 1
Introduction

Internal waves occur in density-stratified fluid in the presence of gravitational field. They arise as a result of perturbation which force the stratified fluid to move vertically (i.e., against gravity). *Interfacial waves* occurring between two superimposed layers of different densities are a familiar phenomenon, particularly at the upper free surface of the ocean in the form of surface waves. In the continuously stratified interior of the ocean the restoring force of gravity is much weaker, and as a consequence, the periods of internal waves are much larger than those of surface waves.

Internal waves are imperceptible part of the world's oceans. They represent a random superposition of many waves with different amplitudes, wave number, and frequencies spanning the possible range of frequencies between the inertial frequency and the buoyancy frequency. Typical velocities of internal waves are about 5 cm $\cdot\ s^{-1}$ and typical vertical displacements range from a few meters to a few tens of kilometers, and the vertical wavelengths from about one meter to about one kilometer. Because of the complexity of the internal wave field, comparatively little is known about the dynamical processes which govern the internal wave field in the ocean. From the practical standpoint, understanding of such processes is important for many reasons. One of them is that internal waves play an important climatic role, permitting information about changes in one region of the ocean to be transmitted to another. They also advect and disperse pollutants, chemical and biological tracers and affect the transmission of sound.

However, one of the main reasons for studying internal waves is the fact that they are suspected to play an important role in the dynamics of the ocean, especially in affecting the large-scale general circulation model (see e.g., [60]; [83] and [80]). Particularly, recent interest in internal waves study is due to internal wave's important role in the processes forming horizontal and vertical exchange in the ocean; mutual interactions between internal waves produce mixing in the interior of the ocean providing an important link in the presumed energy cascade from large to small scales (see e.g., [40]; [18]; [32] and [81]). One of the practical needs to better understand mixing processes in the ocean resides in the fact that mixing plays a role in maintaining a gradual transition between the sun-warmed surface layer of the ocean and the upwelling cold, dense water formed at high latitudes ([11]). To understand how internal waves affect the general circulation and how they cascade energy from large to small scales, one has to study their dynamics. It involves the study of generation and dissipation mechanisms of nonlinear interactions which are thought to be responsible for the universal shape and level of the observed internal wave spectrum (see e.g., [75]; [81]).

The inquiry of understanding of internal wave interactions in the vicinity of oceanic bottom irregularities is motivated by the consensus of the previous related studies that the understanding of the reflection of internal waves over sloping topography plays a crucial role in determining exchange between the costal ocean and the adjacent deep waters. Particularly, current estimates suggest that 40%—50% of the required energy available for mixing due to internal waves in the interior of the ocean is injected by tide-topography interactions with the remainder coming from wind forcing ([82]). The observations by [69]; [48] and [47] based on measurements of microstructure show that mixing is considerably increased over bottom irregularities. Tidal flows result in mixing directly above the boundaries, in which case an issue of particular importance is the rate at which oceanic fluid from interior is exchanged with fluid at the sloping boundary. This exchange may improve the efficiency of boundary mixing, allowing it to contribute significantly to the global overturning circulation.

Chapter 2
Governing Equations

In this section we offer a concise introduction to the geophysical fluid dynamics equations used in this book. For the sake of simplicity, we ignore viscosity, diffusion and make the incompressibility approximation.

Let us consider an ocean basin consisting of two horizontal boundaries at $z = h(x,y,t)$ on the free surface and $z = f(x,y,t)$ at the bottom. The function f is assumed to be given while the function h is considered to be unknown. First, we argue out the question: "What is a stratification and how it appears?" Our starting point here is the equations of a three-dimensional fluid motion away from frictional boundary layers in the simplest case of incompressible, inviscid flow.

A motionless background state is perturbed by velocity components u, v and w, pressure p, and the density ρ. In the domain of the continuous motion, i.e., in the domain where the velocity vector $\vec{u} = (u,v,w)$, pressure p, density ρ and their first derivatives are continuous, the equations of motion are the Euler's equation in the rotating reference frame (see e.g., Section 7.2 in [15])

$$\vec{u}_t + (\vec{u} \cdot \nabla)\,\vec{u} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega} \times \vec{u} - g\,\vec{k}, \tag{2.1}$$

$$\rho_t + \vec{u}\nabla\rho = 0, \tag{2.2}$$

$$\operatorname{div}(\vec{u}) = 0, \tag{2.3}$$

in which the subscripts mean the partial differentiation, ∇ is the gradient operator, u, v, w correspond to zonal, meridional and vertical components of the velocity vector, g is the gravitational acceleration and $\vec{k}$ denotes a unit vector in the vertical direction. In this notation, Eq. (2.1) represents three momentum equations and Eqs. (2.2) and (2.3) stand for the incompressibility condition of the mass conservation law. By convention, the horizontal x coordinate is increasing eastward and the transverse horizontal coordinate y northward while z is in the vertical direction. The term $2\vec{\Omega} \times \vec{u}$ is referred to as the *Coriolis acceleration*. Note that the momentum balance equation (2.1) is a vector equation of motion valid for any coordinate system rotating with the Earth.

The components of angular velocity of the Earth in the local Cartesian coordinates are given by [44]

$$\Omega_{(x)} = 0,\ \Omega_{(y)} = \Omega\cos\theta,\ \Omega_{(z)} = \Omega\sin\theta,$$

where θ is latitude and $\Omega = 2\pi$ rad/day $\approx 0.73\times 10^{-4}\ \mathrm{s}^{-1}$ is the rate of the earth rotation. Thus

$$2\vec{\Omega}\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \cos\theta & \sin\theta \\ u & v & w \end{vmatrix} = (f_h w - fv,\ fu - f_h u), \tag{2.4}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors in the xy- directions respectively, and for convenience, the following notation is introduced:

$$f_h = 2\Omega\cos\theta,\ \ f = 2\Omega\sin\theta. \tag{2.5}$$

The coefficient $f = 2\Omega\sin\theta$ (which is defined as being twice the vertical component of the Coriolis force) is called the *Coriolis frequency*. Terms involving f_h are usually neglected, and a scaling analysis shows them to have a small effect ([15]). Although the Coriolis frequency varies with latitude, this variation is important only for phenomena having very long length scales (thousands of kilometers). Therefore the Coriolis forces are unimportant for the vertical motions. Correspondingly, the only component of the Earth's angular velocity taken into account in our model for Coriolis forces is $\Omega_{(z)}$ which couples horizontal flows to horizontal flows.

2.1 Stratification

To better understand exclusively the phenomenon of stratification, let us consider first the particular case of fluid motion ignoring the effects of the earth's rotation. Then, in the stationary and irrotational flow regime, i.e., when $f = 0$ and $\vec{u} = 0$, Equations (2.2) and (2.3) are satisfied identically while the first equation becomes $\nabla p = \vec{g}\rho$. Since $\vec{g} = (0,0,-g)$, the latter equations are written as

$$p_x = p_y = 0,\ \ p_z = -g\rho. \tag{2.6}$$

Equation (2.6) implies that the pressure is an arbitrary function of z- direction only, i.e., $p = p_0(z)$ which also means that the density is also a function of z, i.e., $\rho = \rho_0(z)$ so that the following *hydrostatic equation* holds:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0(z)\,g. \tag{2.7}$$

If $\rho_0(z) \neq$const., then in the stationary case the fluid flakes (splits) into horizontal layers by planes z =const. along which ρ_0 =const. Such splitting phenomenon is known as *stratification* and the layers of constant density are called *stratus* .

Let us consider now some special class of exact solutions of system (2.1) — (2.3) given by

$$u = u_0(z), \quad v = v_0(z), \quad w = 0, \quad \rho = \rho_0(z), \quad p = p_0(z), \tag{2.8}$$

where $u_0(z), v_0(z), \rho_0(z)$ are arbitrary functions of z and $p_0(z)$ is determined by the hydrostatic equation (2.7). The solution (2.8) represents the fluid motion in which there is no vertical motion, the pressure is hydrostatic and the horizontal layers are preserved. For this reason, the fluid motion determined by (2.8) is called a *sliding motion* (in some literature, e.g., [44], such motion is also called shear flow).

2.2 Linear model for small disturbances

Let us look for solution of system (2.1) — (2.3) in the form

$$\vec{u} = \vec{u}_0 + \vec{u}', \quad \rho = \rho_0 + \rho', \quad p = p_0 + p', \tag{2.9}$$

where $\vec{u}', \rho'$ and p' are spacial and time dependent small disturbances of an arbitrary sliding flow.

It is convenient to introduce the *material derivative* operator D by

$$D = \partial_t + \vec{u} \cdot \nabla. \tag{2.10}$$

Then, in the linear approximation, we can represent D as the sum: $D = D^0 + D'$, where

$$D^0 = \partial_t + u_0(z)\,\partial_x + v_0(z)\,\partial_y \tag{2.11}$$

and

$$D' = u'(x,y,z,t)\,\partial_x + v(x,y,z,t)\,\partial_y + w'(x,y,z,t)\,\partial_z. \tag{2.12}$$

The substitution of the expansion (2.9) into the mass conservation law (2.3) gives

$$u'_x + v'_y + w'_z = 0 \tag{2.13}$$

while the incompressibility condition (2.2) yields

$$D_0\rho' + w'\rho_{0_z} = 0. \tag{2.14}$$

To write the expression $1/\rho$ in the corresponding linearized momentum equation (2.1), we first note that

$$\frac{1}{\rho_0 + \rho'} - \frac{1}{\rho_0} = -\frac{\rho'}{\rho_0^2 + \rho_0\rho'}. \tag{2.15}$$

Thus using the Taylor series expansion, we can replace the term $1/\rho$ by its linear approximation as follows:

$$\frac{1}{\rho} \approx \frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2} + o\left(\rho'^2\right). \tag{2.16}$$

Respectively, in the linear approximation, the $x-$momentum equation (2.1) takes the form

$$D_0 u' + w' u_{0_z} + \frac{p'_x}{\rho_0} = 0. \tag{2.17}$$

While the second $y-$momentum equation can be obtained likewise by observing that the first and the second equations are symmetric with respect to the index permutation $x \to y$, $u \to v$, the third $z-$momentum equation (2.1) takes the form

$$D_0 w' + \frac{p'_z}{\rho_0} + \frac{g\rho'}{\rho_0} = 0. \tag{2.18}$$

Finally, combining the results of (2.13), (2.14), (2.17) and (2.18) we arrive at the following linearized equations of motion of stratified fluid:

$$D_0 u' + w' u_{0_z} + \frac{p'_x}{\rho_0} = 0, \tag{2.19}$$

$$D_0 v' + w' v_{0_z} + \frac{p'_y}{\rho_0} = 0, \tag{2.20}$$

$$D_0 w' + \frac{p'_z}{\rho_0} + \frac{g\rho'}{\rho_0} = 0, \tag{2.21}$$

$$D_0 \rho' + w' \rho_{0_z} = 0, \tag{2.22}$$

$$u'_x + v'_y + w'_z = 0. \tag{2.23}$$

Equations (2.19) — (2.23) represent five linear equations for five unknowns u', v', w', p' and ρ'.

The symmetry of the model in x and y allows one to look for solution in the form

$$(u, v, w, \rho, p) = (U(z), V(z), iW(z), R(z), P(z)) \exp(i\phi), \tag{2.24}$$

where we denote $\phi = kx + ly - \omega t$. The procedure is similar to a Fourier transform, in which case the disturbance amplitude functions U, V, iW, R and P would represent the z-dependent envelope of a disturbance with horizontal wavenumbers k, l and frequency ω. The substitution (2.24) is more general, however, because ω can be a complex number. Thus, for example, the real part of ω denotes the disturbance frequency and the imaginary part denotes the growth rate.

Since the equations (2.19) — (2.23) are linear, we can exclude the amplitude functions $U, V, R(z)$ and $P(z)$ to write Eqs. (2.19) — (2.23) as the single equation in terms of $W(z)$ only. To this end, let us denote