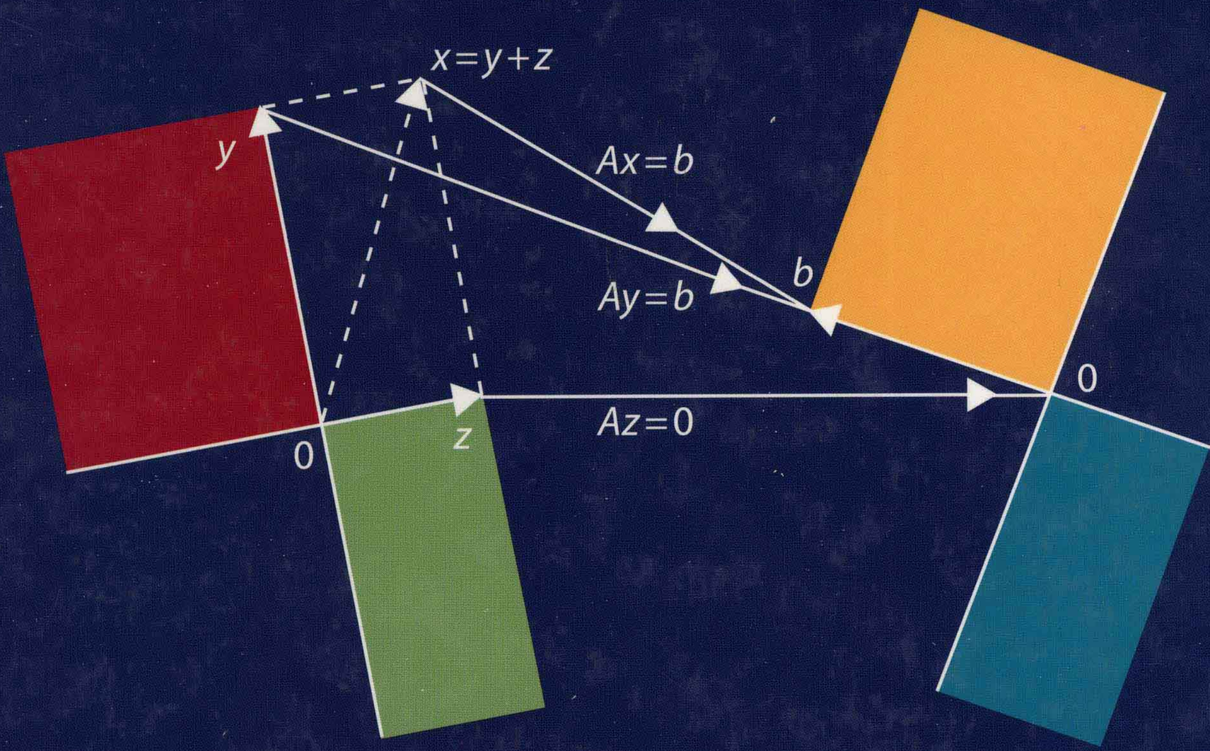
Introduction to LINEAR ALGEBRA

FOURTH EDITION



GILBERT STRANG

INTRODUCTION
TO
LINEAR ALGEBRA

Fourth Edition

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The website for this book is **math.mit.edu/linearalgebra**.
A Solutions Manual is available to instructors by email from the publisher.

Course material including syllabus and Teaching Codes and exams and also videotaped lectures are available on the teaching website: **web.mit.edu/18.06**

Linear Algebra is included in MIT's OpenCourseWare site **ocw.mit.edu**.
This provides video lectures of the full linear algebra course 18.06.
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The front cover captures a central idea of linear algebra.
$A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the (orange) column space of A.
One particular solution $\boldsymbol{y}$ is in the (red) row space: $A\boldsymbol{y} = \boldsymbol{b}$.
Add any vector $\boldsymbol{z}$ from the (green) nullspace of A: $A\boldsymbol{z} = \boldsymbol{0}$.
The complete solution is $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$. Then $A\boldsymbol{x} = A\boldsymbol{y} + A\boldsymbol{z} = \boldsymbol{b}$.
The cover design was the inspiration of a creative collaboration:
Lois Sellers (**birchdesignassociates.com**) and Gail Corbett.

Preface

I will be happy with this preface if three important points come through clearly:

1. The beauty and variety of linear algebra, and its extreme usefulness
2. The goals of this book, and the new features in this Fourth Edition
3. The steady support from our linear algebra websites and the video lectures

May I begin with notes about two websites that are constantly used, and the new one.

ocw.mit.edu Messages come from thousands of students and faculty about linear algebra on this OpenCourseWare site. The 18.06 course includes video lectures of a complete semester of classes. Those lectures offer an independent review of the whole subject based on this textbook—the professor's time stays free and the student's time can be 3 a.m. (The reader doesn't have to be in a class at all.) A million viewers around the world have seen these videos (*amazing*). I hope you find them helpful.

web.mit.edu/18.06 This site has homeworks and exams (with solutions) for the current course as it is taught, and as far back as 1996. There are also review questions, Java demos, Teaching Codes, and short essays (*and the video lectures*). My goal is to make this book as useful as possible, with all the course material we can provide.

math.mit.edu/linearalgebra The newest website is devoted specifically to this Fourth Edition. It will be a permanent record of ideas and codes and good problems and solutions. Several sections of the book are directly available online, plus notes on teaching linear algebra. The content is growing quickly and contributions are welcome from everyone.

The Fourth Edition

Thousands of readers know earlier editions of *Introduction to Linear Algebra*. The new cover shows the **Four Fundamental Subspaces**—the row space and nullspace are on the left side, the column space and the nullspace of A^{T} are on the right. It is not usual to put the central ideas of the subject on display like this! You will meet those four spaces in Chapter 3, and you will understand why that picture is so central to linear algebra.

Those were named the Four Fundamental Subspaces in my first book, and they start from a matrix A. Each row of A is a vector in n-dimensional space. When the matrix has m rows, each column is a vector in m-dimensional space. The crucial operation in linear algebra is taking ***linear combinations of vectors***. (That idea starts on page 1 of the book and never stops.) *When we take all linear combinations of the column vectors, we get the column space.* If this space includes the vector b, we can solve the equation $Ax = b$.

I have to stop here or you won't read the book. May I call special attention to the new Section 1.3 in which these ideas come early—with two specific examples. You are not expected to catch every detail of vector spaces in one day! But you will see the first matrices in the book, and a picture of their column spaces, and even an *inverse matrix*. You will be learning the language of linear algebra in the best and most efficient way: by using it.

Every section of the basic course now ends with ***Challenge Problems***. They follow a large collection of review problems, which ask you to use the ideas in that section—-the dimension of the column space, a basis for that space, the rank and inverse and determinant and eigenvalues of A. Many problems look for computations by hand on a small matrix, and they have been highly praised. The new Challenge Problems go a step further, and sometimes they go deeper. Let me give four examples:

Section 2.1: Which row exchanges of a Sudoku matrix produce another Sudoku matrix?

Section 2.4: From the shapes of A, B, C, is it faster to compute AB times C or A times BC?

Background: The great fact about multiplying matrices is that *AB times C gives the same answer as A times BC*. This simple statement is the reason behind the rule for matrix multiplication. If AB is square and C is a vector, it's faster to do BC first. Then multiply by A to produce ABC. The question asks about other shapes of A, B, and C.

Section 3.4: If $Ax = b$ and $Cx = b$ have the same solutions for every b, is $A = C$?

Section 4.1: What conditions on the four vectors r, n, c, ℓ allow them to be bases for the row space, the nullspace, the column space, and the left nullspace of a 2 by 2 matrix?

The Start of the Course

The equation $Ax = b$ uses the language of linear combinations right away. The vector Ax is *a combination of the columns of A*. The equation is asking for ***a combination that produces b***. The solution vector x comes at three levels and all are important:

1. ***Direct solution*** to find x by forward elimination and back substitution.

2. ***Matrix solution*** using the inverse of A: $x = A^{-1}b$ (if A has an inverse).

3. ***Vector space solution*** $x = y + z$ as shown on the cover of the book:

   Particular solution (to $Ay = b$) plus ***nullspace solution*** (to $Az = 0$)

Direct elimination is the most frequently used algorithm in scientific computing, and the idea is not hard. Simplify the matrix A so it becomes triangular—then all solutions come quickly. I don't spend forever on practicing elimination, it will get learned.

The speed of every new supercomputer is tested on $Ax = b$: it's pure linear algebra. IBM and Los Alamos announced a new world record of 10^{15} operations per second in 2008.

That *petaflop speed* was reached by solving many equations in parallel. High performance computers avoid operating on single numbers, they feed on whole submatrices.

The processors in the Roadrunner are based on the Cell Engine in PlayStation 3. What can I say, video games are now the largest market for the fastest computations.

Even a supercomputer doesn't want the inverse matrix: too slow. Inverses give the simplest formula $x = A^{-1}b$ but not the top speed. And everyone must know that determinants are even slower—there is no way a linear algebra course should begin with formulas for the determinant of an n by n matrix. Those formulas have a place, but not first place.

Structure of the Textbook

Already in this preface, you can see the style of the book and its goal. That goal is serious, to explain this beautiful and useful part of mathematics. You will see how the applications of linear algebra reinforce the key ideas. I hope every teacher will learn something new; familiar ideas can be seen in a new way. The book moves gradually and steadily from *numbers* to *vectors* to *subspaces*—each level comes naturally and everyone can get it.

Here are ten points about the organization of this book:

1. Chapter 1 starts with vectors and dot products. If the class has met them before, focus quickly on linear combinations. The new Section 1.3 provides three independent vectors whose combinations fill all of 3-dimensional space, and three dependent vectors in a plane. ***Those two examples are the beginning of linear algebra.***

2. Chapter 2 shows the row picture and the column picture of $Ax = b$. The heart of linear algebra is in that connection between the rows of A and the columns: the same numbers but very different pictures. Then begins the algebra of matrices: an elimination matrix E multiplies A to produce a zero. The goal here is to capture the whole process—start with A and end with an ***upper triangular*** U.

   Elimination is seen in the beautiful form $A = LU$. The ***lower triangular*** L holds all the forward elimination steps, and U is the matrix for back substitution.

3. Chapter 3 is linear algebra at the best level: ***subspaces***. The column space contains all linear combinations of the columns. The crucial question is: ***How many of those columns are needed***? The answer tells us the dimension of the column space, and the key information about A. We reach the Fundamental Theorem of Linear Algebra.

4. Chapter 4 has m equations and only n unknowns. It is almost sure that $Ax = b$ has no solution. We cannot throw out equations that are close but not perfectly exact. When we solve by ***least squares***, the key will be the matrix $A^{\mathrm{T}}A$. This wonderful matrix $A^{\mathrm{T}}A$ appears everywhere in applied mathematics, when A is rectangular.

5. ***Determinants*** in Chapter 5 give formulas for all that has come before—inverses, pivots, volumes in n-dimensional space, and more. We don't need those formulas to compute! They slow us down. But $\det A = 0$ tells when a matrix is singular, and that test is the key to eigenvalues.

6. ***Section* 6.1 *introduces eigenvalues for* 2 *by* 2 *matrices*.** Many courses want to see eigenvalues early. It is completely reasonable to come here directly from Chapter 3, because the determinant is easy for a 2 by 2 matrix. *The key equation is* $Ax = \lambda x$.

   Eigenvalues and eigenvectors are an astonishing way to understand a square matrix. They are not for $Ax = b$, they are for dynamic equations like $du/dt = Au$. The idea is always the same: *follow the eigenvectors*. In those special directions, A acts like a single number (the eigenvalue λ) and the problem is one-dimensional.

   Chapter 6 is full of applications. One highlight is ***diagonalizing a symmetric matrix***. Another highlight—not so well known but more important every day—is the diagonalization of *any matrix*. This needs two sets of eigenvectors, not one, and they come (of course!) from $A^{\mathrm{T}}A$ and AA^{T}. This Singular Value Decomposition often marks the end of the basic course and the start of a second course.

7. Chapter 7 explains the ***linear transformation*** approach—it is linear algebra without coordinates, the ideas without computations. Chapter 9 is the opposite—all about how $Ax = b$ and $Ax = \lambda x$ are really solved. Then Chapter 10 moves from real numbers and vectors to complex vectors and matrices. The Fourier matrix F is the most important complex matrix we will ever see. And the ***Fast Fourier Transform*** (multiplying quickly by F and F^{-1}) is a revolutionary algorithm.

8. Chapter 8 is full of applications, more than any single course could need:

   8.1 *Matrices in Engineering*—differential equations replaced by matrix equations

   8.2 *Graphs and Networks*—leading to the edge-node matrix for Kirchhoff's Laws

   8.3 *Markov Matrices*—as in Google's *PageRank* algorithm

   8.4 *Linear Programming*—a new requirement $x \geq 0$ and minimization of the cost

   8.5 *Fourier Series*—linear algebra for functions and digital signal processing

   8.6 *Matrices in Statistics and Probability*—$Ax = b$ is weighted by average errors

   8.7 *Computer Graphics*—matrices move and rotate and compress images.

9. Every section in the basic course ends with a ***Review of the Key Ideas***.

10. How should computing be included in a linear algebra course? It can open a new understanding of matrices—every class will find a balance. I chose the language of MATLAB as a direct way to describe linear algebra: eig(ones(4)) will produce the eigenvalues $4, 0, 0, 0$ of the 4 by 4 all-ones matrix. *Go to* ***netlib.org*** *for codes.*

    You can freely choose a different system. More and more software is open source.

The new website ***math.mit.edu/linearalgebra*** provides further ideas about teaching and learning. Please contribute! Good problems are welcome by email: ***gs@math.mit.edu***. Send new applications too, linear algebra is an incredibly useful subject.

The Variety of Linear Algebra

Calculus is mostly about one special operation (the derivative) and its inverse (the integral). Of course I admit that calculus could be important .... But so many applications of mathematics are discrete rather than continuous, digital rather than analog. The century of data has begun! You will find a light-hearted essay called "Too Much Calculus" on my website. ***The truth is that vectors and matrices have become the language to know.***

Part of that language is the wonderful variety of matrices. Let me give three examples:

Symmetric matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Orthogonal matrix $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

Triangular matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A key goal is learning to "read" a matrix. You need to see the meaning in the numbers. This is really the essence of mathematics—patterns and their meaning.

May I end with this thought for professors. You might feel that the direction is right, and wonder if your students are ready. ***Just give them a chance***! Literally thousands of students have written to me, frequently with suggestions and surprisingly often with thanks. They know this course has a purpose, because the professor and the book are on their side. Linear algebra is a fantastic subject, enjoy it.

Help With This Book

I can't even name all the friends who helped me, beyond thanking Brett Coonley at MIT and Valutone in Mumbai and SIAM in Philadelphia for years of constant and dedicated support. The greatest encouragement of all is the feeling that you are doing something worthwhile with your life. Hundreds of generous readers have sent ideas and examples and corrections (and favorite matrices!) that appear in this book. *Thank you all.*

Background of the Author

This is my eighth textbook on linear algebra, and I have not written about myself before. I hesitate to do it now. It is the mathematics that is important, and the reader. The next paragraphs add something personal as a way to say that textbooks are written by people.

I was born in Chicago and went to school in Washington and Cincinnati and St. Louis. My college was MIT (and my linear algebra course was *extremely abstract*). After that came Oxford and UCLA, then back to MIT for a very long time. I don't know how many thousands of students have taken 18.06 (more than a million when you include the videos on ***ocw.mit.edu***). The time for a fresh approach was right, because this fantastic subject was only revealed to math majors—**we needed to open linear algebra to the world.**

Those years of teaching led to the Haimo Prize from the Mathematical Association of America. For encouraging education worldwide, the International Congress of Industrial and Applied Mathematics awarded me the first Su Buchin Prize. I am extremely grateful, more than I could possibly say. What I hope most is that you will like linear algebra.

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Chapter 1

Introduction to Vectors

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $\boldsymbol{v}+\boldsymbol{w}$. We multiply them by numbers c and d to get $c\boldsymbol{v}$ and $d\boldsymbol{w}$. Combining those two operations (adding $c\boldsymbol{v}$ to $d\boldsymbol{w}$) gives the ***linear combination*** $c\boldsymbol{v}+d\boldsymbol{w}$.

Linear combination $$c\boldsymbol{v}+d\boldsymbol{w}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

Example $\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$ is the combination with $c=d=1$

Linear combinations are all-important in this subject! Sometimes we want one particular combination, the specific choice $c=2$ and $d=1$ that produces $c\boldsymbol{v}+d\boldsymbol{w}=(4,5)$. Other times we want *all the combinations* of $\boldsymbol{v}$ and $\boldsymbol{w}$ (coming from all c and d).

The vectors $c\boldsymbol{v}$ lie along a line. When $\boldsymbol{w}$ is not on that line, **the combinations** $c\boldsymbol{v}+d\boldsymbol{w}$ **fill the whole two-dimensional plane**. (I have to say "two-dimensional" because linear algebra allows higher-dimensional planes.) Starting from four vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}$ in four-dimensional space, their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}+f\boldsymbol{z}$ are likely to fill the space—but not always. The vectors and their combinations could even lie on one line.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space). The first steps are the operations in Sections 1.1 and 1.2. Then Section 1.3 outlines three fundamental ideas.

1.1 ***Vector addition*** $\boldsymbol{v}+\boldsymbol{w}$ ***and linear combinations*** $c\boldsymbol{v}+d\boldsymbol{w}$.

1.2 ***The dot product*** $\boldsymbol{v}\cdot\boldsymbol{w}$ ***of two vectors and the length*** $\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$.

1.3 ***Matrices*** A, ***linear equations*** $A\boldsymbol{x}=\boldsymbol{b}$, ***solutions*** $\boldsymbol{x}=A^{-1}\boldsymbol{b}$.

1.1 Vectors and Linear Combinations

"You can't add apples and oranges." In a strange way, this is the reason for vectors. We have two separate numbers v_1 and v_2. That pair produces a ***two-dimensional vector*** $\boldsymbol{v}$:

$$\textbf{Column vector} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{aligned} v_1 &= \text{first component} \\ v_2 &= \text{second component} \end{aligned}$$

We write $\boldsymbol{v}$ as a ***column***, not as a row. The main point so far is to have a single letter $\boldsymbol{v}$ (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in *lightface italic*).

Even if we don't add v_1 to v_2, we do *add vectors*. The first components of $\boldsymbol{v}$ and $\boldsymbol{w}$ stay separate from the second components:

$$\textbf{VECTOR ADDITION} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

You see the reason. We want to add apples to apples. Subtraction of vectors follows the same idea: *The components of* $\boldsymbol{v} - \boldsymbol{w}$ *are* $v_1 - w_1$ *and* $v_2 - w_2$.

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c. There are two ways to double a vector. One way is to add $\boldsymbol{v} + \boldsymbol{v}$. The other way (the usual way) is to multiply each component by 2:

$$\textbf{SCALAR MULTIPLICATION} \qquad 2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad \text{and} \quad -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar".

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$—***adding vectors and multiplying by scalars***.

The order of addition makes no difference: $\boldsymbol{v} + \boldsymbol{w}$ equals $\boldsymbol{w} + \boldsymbol{v}$. Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. Check also by an example:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix} \qquad \boldsymbol{w} + \boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix}.$$

Linear Combinations

Combining addition with scalar multiplication, we now form "*linear combinations*" of $\boldsymbol{v}$ and $\boldsymbol{w}$. Multiply $\boldsymbol{v}$ by c and multiply $\boldsymbol{w}$ by d; then add $c\boldsymbol{v} + d\boldsymbol{w}$.

DEFINITION ***The sum of $c\boldsymbol{v}$ and $d\boldsymbol{w}$ is a*** linear combination ***of $\boldsymbol{v}$ and $\boldsymbol{w}$.***

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\boldsymbol{v}$:

$$\begin{aligned} 1\boldsymbol{v} + 1\boldsymbol{w} &= \text{sum of vectors in Figure 1.1a} \\ 1\boldsymbol{v} - 1\boldsymbol{w} &= \text{difference of vectors in Figure 1.1b} \\ 0\boldsymbol{v} + 0\boldsymbol{w} &= \textbf{\textit{zero vector}} \\ c\boldsymbol{v} + 0\boldsymbol{w} &= \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v} \end{aligned}$$

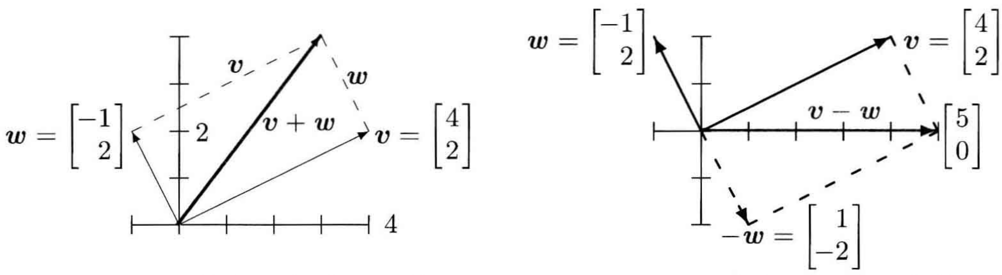
The zero vector is always a possible combination (its coefficients are zero). Every time we see a "space" of vectors, that zero vector will be included. This big view, taking *all* the combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$, is linear algebra at work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). That vector $\boldsymbol{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are $4, 2$. This point is another representation of the vector—so we have three ways to describe $\boldsymbol{v}$:

Represent vector $\boldsymbol{v}$	Two numbers	Arrow from $(0, 0)$	Point in the plane

We add using the numbers. We visualize $\boldsymbol{v} + \boldsymbol{w}$ using arrows:

Vector addition (head to tail) ***At the end of $\boldsymbol{v}$, place the start of $\boldsymbol{w}$.***



$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1.1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the diagonal shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ gives the same answer as $\boldsymbol{v} + \boldsymbol{w}$.

These are different ways along the parallelogram (in this example it is a rectangle). The sum is the diagonal vector $\boldsymbol{v} + \boldsymbol{w}$.

The zero vector $\mathbf{0} = (0, 0)$ is too short to draw a decent arrow, but you know that $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$. For $2\boldsymbol{v}$ we double the length of the arrow. We reverse $\boldsymbol{w}$ to get $-\boldsymbol{w}$. This reversing gives the subtraction on the right side of Figure 1.1.

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of $\boldsymbol{v}$ are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$. Now we allow vectors to have three components (v_1, v_2, v_3).

The xy plane is replaced by three-dimensional space. Here are typical vectors (still column vectors but with three components):

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The vector $\boldsymbol{v}$ corresponds to an arrow in 3-space. Usually the arrow starts at the "origin", where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***.

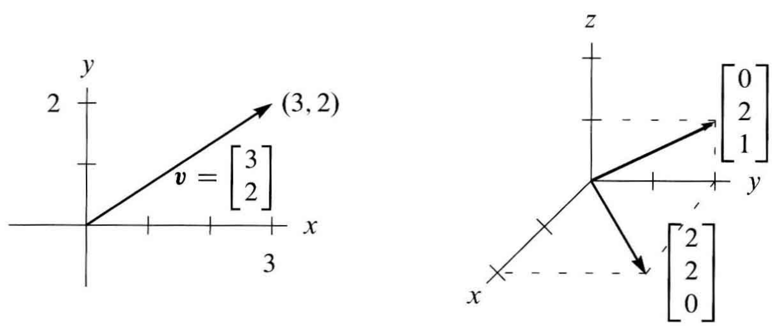


Figure 1.2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

From now on $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ***is also written as*** $\boldsymbol{v} = (1, 1, -1)$.

The reason for the row form (in parentheses) is to save space. But $\boldsymbol{v} = (1, 1, -1)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[\,1 \quad 1 \quad -1\,]$ is absolutely different, even though it has the same three components. That row vector is the "transpose" of the column $\boldsymbol{v}$.

In three dimensions, $\boldsymbol{v} + \boldsymbol{w}$ is still found a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When $\boldsymbol{w}$ starts at the end of $\boldsymbol{v}$, the third side is $\boldsymbol{v} + \boldsymbol{w}$. The other way around the parallelogram is $\boldsymbol{w} + \boldsymbol{v}$. Question: Do the four sides all lie in the same plane? *Yes*. And the sum $\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{w}$ goes completely around to produce the ______ vector.

A typical linear combination of three vectors in three dimensions is $\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w}$:

Linear combination
Multiply by 1, 4, −2
Then add

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$$

The Important Questions

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$. Will you take the big step from *one* combination to *all* combinations? Every c and d and e are allowed. Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space:

1. What is the picture of *all* combinations $c\boldsymbol{u}$?

2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$?

3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

1. The combinations $c\boldsymbol{u}$ fill a ***line***.

2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane***.

3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

The zero vector $(0, 0, 0)$ is on the line because c can be zero. It is on the plane because c and d can be zero. The line of vectors $c\boldsymbol{u}$ is infinitely long (forward and backward). It is the plane of all $c\boldsymbol{u} + d\boldsymbol{v}$ (combining two vectors in three-dimensional space) that I especially ask you to think about.

Adding all $c\boldsymbol{u}$ on one line to all $d\boldsymbol{v}$ on the other line fills in the plane in Figure 1.3.

When we include a third vector $\boldsymbol{w}$, the multiples $e\boldsymbol{w}$ give a third line. Suppose that third line is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$. Then combining all $e\boldsymbol{w}$ with all $c\boldsymbol{u} + d\boldsymbol{v}$ fills up the whole three-dimensional space.

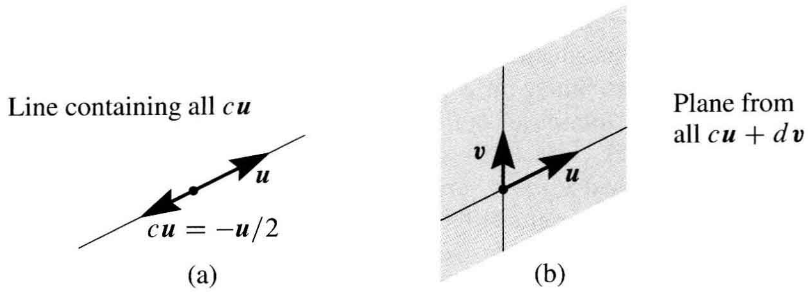


Figure 1.3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

This is the typical situation! **Line**, then **plane**, then **space**. But other possibilities exist. When $\boldsymbol{w}$ happens to be $c\boldsymbol{u} + d\boldsymbol{v}$, the third vector is in the plane of the first two. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ will not go outside that $\boldsymbol{uv}$ plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

■ REVIEW OF THE KEY IDEAS ■

1. A vector $\boldsymbol{v}$ in two-dimensional space has two components v_1 and v_2.
2. $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1,\ v_2 + w_2)$ and $c\boldsymbol{v} = (cv_1, cv_2)$ are found a component at a time.
3. A linear combination of three vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.
4. Take *all* linear combinations of $\boldsymbol{u}$, or $\boldsymbol{u}$ and $\boldsymbol{v}$, or $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. In three dimensions, those combinations typically fill a line, then a plane, and the whole space $\mathbf{R}^3$.

■ WORKED EXAMPLES ■

1.1 A The linear combinations of $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (0, 1, 1)$ fill a plane. *Describe that plane.* Find a vector that is *not* a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution **The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ fill a plane in $\mathbf{R}^3$.** The vectors in that plane allow any c and d. The plane of Figure 1.3 fills in between the "$\boldsymbol{u}$-line" and the "$\boldsymbol{v}$-line".

$$\text{Combinations} \quad c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix} \text{ fill a plane.}$$

Four particular vectors in that plane are $(0, 0, 0)$ and $(2, 3, 1)$ and $(5, 7, 2)$ and $(\pi, 2\pi, \pi)$. The second component $c + d$ is always the sum of the first and third components. *The vector* $(1, 2, 3)$ *is not in the plane, because* $2 \neq 1 + 3$.

Another description of this plane through $(0,0,0)$ is to know that $\boldsymbol{n} = (1,-1,1)$ is **perpendicular** to the plane. Section 1.2 will confirm that 90° angle by testing dot products: $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{w}\cdot\boldsymbol{n} = 0$.

1.1 B For $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (0,1)$, describe all points $c\boldsymbol{v}$ with **(1)** *whole numbers c* **(2)** *nonnegative* $c \geq 0$. Then add all vectors $d\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\boldsymbol{w}$.

Solution

(1) The vectors $c\boldsymbol{v} = (c,0)$ with whole numbers c are **equally spaced points** along the x axis (the direction of $\boldsymbol{v}$). They include $(-2,0)$, $(-1,0)$, $(0,0)$, $(1,0)$, $(2,0)$.

(2) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a ***half-line***. It is the *positive* x axis. This half-line starts at $(0,0)$ where $c = 0$. It includes $(\pi,0)$ but not $(-\pi,0)$.

(1′) Adding all vectors $d\boldsymbol{w} = (0,d)$ puts a vertical line through those points $c\boldsymbol{v}$. We have infinitely many ***parallel lines*** from (*whole number c, any number d*).

(2′) Adding all vectors $d\boldsymbol{w}$ puts a vertical line through every $c\boldsymbol{v}$ on the half-line. Now we have a ***half-plane***. It is the right half of the xy plane (any $x \geq 0$, any height y).

1.1 C Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution In applying mathematics, many problems have two parts:

1 *Modeling part* Express the problem by a set of equations.

2 *Computational part* Solve those equations by a fast and accurate algorithm.

Here we are only asked for the first part (the equations). Chapter 2 is devoted to the second part (the algorithm). Our example fits into a fundamental model for linear algebra:

$$\text{Find} \quad c_1,\ldots,c_n \quad \text{so that} \quad c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b}.$$

For $n = 2$ we could find a formula for the $\boldsymbol{c}$'s. The "elimination method" in Chapter 2 succeeds far beyond $n = 100$. For n greater than 1 million, see Chapter 9. Here $n = 2$:

Vector equation $$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The required equations for c and d just come from the two components separately:

Two scalar equations $$\begin{aligned} 2c - d &= 1 \\ -c + 2d &= 0 \end{aligned}$$

You could think of those as two lines that cross at the solution $c = \dfrac{2}{3}, d = \dfrac{1}{3}$.