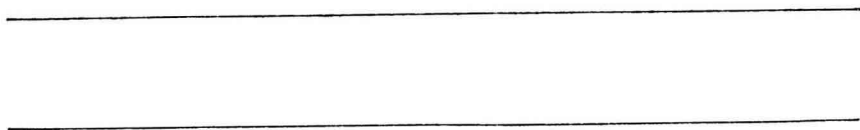




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(problems and solutions)

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PREFACE

This is the translation of the second Russian edition of the book. The experience with the first edition has shown that presentation of the material in the form of problems (there are more than 200 problems in the book) is particularly suitable for independent study of the course of low-temperature plasma physics or some questions of it, so that the book serves as a useful addition to the available textbooks on physics of weakly ionized plasma. The problems in the book either deal with separate theoretical subjects or analyze individual research questions. The emphasis is on presenting the totality of methods and approaches employed for solving such problems without mathematics obscuring the physical nature of the processes involved. At the same time, the solutions of the problems present scientific information of interest for the workers in this field (many problems stem from recent research results and are formulated so as to be of possible use for practical work). The appendices containing reference data serve the same purpose.

Thus, the book is aimed at two types of readers. The first type is graduate students who can study the problems in the book to get acquainted with the field and its theory. The second type is physicists or technologists working with weakly ionized plasmas, who can make use of the information contained in the book.

The bibliography given at the end of the book is a list of textbooks, monographs and reviews where the reader can find more detailed information on the subjects covered by the book. Some of the problems are based on original research results but no references are given in such cases because such problems are formulated primarily for the teaching purposes and no detailed discussion of the results is given.

The appendices serve different purposes. On the one hand, they briefly present the definitions and main relationships for collision cross sections and spectral line broadening. This information is used in the book. On the other hand, the appendices contain some reference data including information on physical constants, simple physical relationships, and selected parameters of atoms, molecules and ions.

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THE BOLTZMANN KINETIC EQUATION

Let us consider an ideal gas. The state of a particle of this gas at a given moment of time is described by its velocity \mathbf{v} , coordinate \mathbf{r} , and the internal degrees of freedom which we shall denote by the generalized parameter I . The ideality of the gas implies that the mean distance $N^{-1/3}$ between the particles (here N is the density of the particles) is much larger than their interaction radius (the characteristic distance between the particles, at which the potential energy of interaction between the particles is of the order of their kinetic energy). This condition of gas ideality can be written as

$$N\sigma^{2/3} \ll 1 \quad (1.1)$$

where σ is the scattering cross section for the particles, at which their states are noticeably changed.

If the ideality condition (1.1) is satisfied, each particle travels as a free particle during most of the time and only for a small fraction ($\sim N\sigma^{2/3}$) of the time interacts with other particles. The state of the colliding particle changes during this short collision time. A simultaneous collision of three particles (that is, when they simultaneously approach each other to a distance of the order of the interaction radius) is a more rare event than collision of two particles. Therefore, when analyzing the motion of a particle in the ideal gas, we can limit ourselves to considering binary collisions of this particle with other gas particles.

The state of the gas can be conveniently described by the function $f(\mathbf{v}_1, \mathbf{r}, I_1, t)$ of distribution of the particles over the states; the number of particles in the unit volume around point \mathbf{r} , having the velocities from \mathbf{v}_1 to $\mathbf{v}_1 + d\mathbf{v}_1$ and the internal state described by the quantum number I_1 , is $f(\mathbf{v}_1, \mathbf{r}, t, I_1) d\mathbf{v}_1$.

The distribution function for the particles is a solution of the equation known as the *Boltzmann kinetic equation*. Let us trace the particles which are at the point \mathbf{r} , travel with the velocity \mathbf{v}_1 and are in the internal state described by the quantum number I_1 . The number of particles in a given state with a given velocity in the unit volume around a given point per unit time is known as the *collision integral* I_c . Then the variation of the

distribution function during time dt is

$$df = f(\mathbf{v}_1 + d\mathbf{v}_1, \mathbf{r} + d\mathbf{r}, t + dt, I_1) - f(\mathbf{v}_1, \mathbf{r}, t, I_1) = I_c dt$$

Since the variations of the coordinate and the velocity of the test particle are

$$d\mathbf{r} = \mathbf{v}_1 dt \quad \text{and} \quad d\mathbf{v}_1 = \frac{\mathbf{F}}{m} dt$$

where \mathbf{F} is the external force acting on the particle and m is the particle's mass, the above equation can be rewritten as

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}_1} = I_c \quad (1.2)$$

§ 1.1. PROPERTIES OF THE KINETIC EQUATION

Problem 1.1. Express the collision integral in terms of the differential cross section of collisions between particles.

Let us define the constant $W(\mathbf{v}_1, I_1; \mathbf{v}_2, I_2 \rightarrow \mathbf{v}'_1, I'_1; \mathbf{v}'_2, I'_2)$ for collisions between particles in the states \mathbf{v}_1, I_1 and \mathbf{v}_2, I_2 in the course of which they go over to the states \mathbf{v}'_1, I'_1 and \mathbf{v}'_2, I'_2 , respectively. The *principle of detailed balancing* (see A2.6) yields

$$W(\mathbf{v}_1, I_1; \mathbf{v}_2, I_2 \rightarrow \mathbf{v}'_1, I'_1; \mathbf{v}'_2, I'_2) = W(\mathbf{v}'_1, I'_1; \mathbf{v}'_2, I'_2 \rightarrow \mathbf{v}_1, I_1; \mathbf{v}_2, I_2) \quad (1.3)$$

If the gas consists of one species, the collision integral, by definition, is

$$I_c = \int (f'_1 f'_2 - f_1 f_2) W d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2 \quad (1.4)$$

where $f_1 \equiv f(\mathbf{v}_1, \mathbf{r}, I_1, t)$ and the distribution functions f_2, f'_1 and f'_2 are defined in a similar way. Equation (1.4) implies summation over the internal quantum numbers I_2, I'_1 and I'_2 .

The number of integrations in eq. (1.4) can be reduced if the laws of conservation of momentum and energy are taken into account, which use the properties of the transition probability W . This can be done by considering the differential scattering cross section defined as the ratio of the number of particles scattered into the solid angle element per unit time per one scattering particle to the flux density of the scattered particles. The number of scattering events per unit time in the unit volume, in which the first particle changes its velocity from the range $\mathbf{v}_1, \mathbf{v}_1 + d\mathbf{v}_1$ to $\mathbf{v}'_1, \mathbf{v}'_1 + d\mathbf{v}'_1$ and the second particle, from the range $\mathbf{v}_2, \mathbf{v}_2 + d\mathbf{v}_2$ to $\mathbf{v}'_2, \mathbf{v}'_2 + d\mathbf{v}'_2$ with a given variation of the internal quantum numbers, is $f_1 f_2 W d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2$.

The number of the scattering particles in the unit volume is $f_1 d\mathbf{v}_1$ and the flux density of the scattered particles is $|\mathbf{v}_2 - \mathbf{v}_1| f_2 d\mathbf{v}_2$, so that the differential scattering cross section is

$$d\sigma = \frac{f_1 f_2 W d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2}{f_1 d\mathbf{v}_1 f_2 |\mathbf{v}_2 - \mathbf{v}_1| d\mathbf{v}_2} = \frac{W d\mathbf{v}'_1 d\mathbf{v}'_2}{|\mathbf{v}_2 - \mathbf{v}_1|}$$

Substitution of this equation into eq. (1.4) yields the following modification of the Boltzmann kinetic equation (1.2):

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \frac{\partial f_1}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \frac{\partial f_1}{\partial \mathbf{v}_1} = \int (f'_1 f'_2 - f_1 f_2) |\mathbf{v}_1 - \mathbf{v}_2| d\sigma d\mathbf{v}_2 \quad (1.5)$$

As before, summation over the internal degrees of freedom I'_1, I'_2 and I_2 is implied. Equation (1.5) describes a one-component system. The collision integral for a multicomponent system must take into account the collisions of the test particle with the particles of each species.

Problem 1.2. Demonstrate that the functional $H(t) = \int f(\mathbf{v}_1, t) \ln f(\mathbf{v}_1, t) d\mathbf{v}_1$, defined for a gas on which no external fields are acting, satisfies the condition $dH/dt \leq 0$.

Differentiating $H(t)$ with respect to time and using the Boltzmann equation with the collision integral in the form (1.4), we obtain

$$\frac{dH}{dt} = \int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2 W (f'_1 f'_2 - f_1 f_2) (1 + \ln f_1)$$

The integrand can be symmetrized by making the substitutions $\mathbf{v}_1 \rightarrow \mathbf{v}_2$, $\mathbf{v}_2 \rightarrow \mathbf{v}_1$:

$$\frac{dH}{dt} = \int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2 W (f'_1 f'_2 - f_1 f_2) \left(1 + \frac{1}{2} \ln f_1 f_2 \right)$$

Making the substitution $\mathbf{v}_{1,2} \rightarrow \mathbf{v}'_{1,2}$, $\mathbf{v}'_{1,2} \rightarrow \mathbf{v}_{1,2}$ in the integrand and using the principle of detailed balancing (1.3), we obtain

$$\frac{dH}{dt} = \int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}'_1 d\mathbf{v}'_2 W (f'_1 f'_2 - f_1 f_2) \ln \frac{f_1 f_2}{f'_1 f'_2}$$

The function of the type $(x - y) \ln (y/x)$ is non-positive for any positive x and y . Hence, the functional $H(t)$ satisfies the condition

$$\frac{dH}{dt} \leq 0 \quad (1.6)$$

This relationship is known as the *Boltzmann H-theorem*. The functional H with the opposite sign is identical to the entropy of the system, so that

eq. (1.6) expresses the entropy increase law for the system under consideration.

Problem 1.3. Find the velocity distribution for the gas particles in thermodynamic equilibrium.

In this case $dH/dt = 0$, which is true if $\ln(ff'/f'f) = 0$, that is, if $\ln f$ is an additive function of the integrals of motion. Using the laws of conservation of momentum and energy in collisions, we can write for the equilibrium distribution function

$$\ln f = C + C_1 \mathbf{p} + C_2 \varepsilon$$

whence

$$f = A \exp[-\alpha(\mathbf{v} - \mathbf{v}_0)^2]$$

where $\varepsilon = mv^2/2$ is the particle's energy, $\mathbf{p} = m\mathbf{v}$ is its momentum, and m is the particle's mass. The above equation contains three indefinite constants.

The constant A can be found from the normalization condition $\int f(\mathbf{v}) d\mathbf{v} = N$ for the distribution function, where N is the number density of the gas particles. Another constant is expressed through the velocity \mathbf{v}_0 of the directed motion of the gas particles. The third constant, α , can be found from the condition that in the reference system in which there is no directed motion of the particles the mean energy of the particles should be $(3/2)T$, where T is the gas temperature. We obtain

$$\begin{aligned} \frac{3}{2} T &= \frac{\int \frac{m}{2} (\mathbf{v} - \mathbf{v}_0)^2 \exp[-\alpha(\mathbf{v} - \mathbf{v}_0)^2] d\mathbf{v}}{\int \exp[-\alpha(\mathbf{v} - \mathbf{v}_0)^2] d\mathbf{v}} \\ &= -\frac{m}{2} \frac{d}{d\alpha} \ln \int \exp[-\alpha(\mathbf{v} - \mathbf{v}_0)^2] d\mathbf{v} = -\frac{m}{2} \frac{d}{d\alpha} \ln \alpha^{-3/2} = \frac{3m}{4\alpha} \end{aligned}$$

and, hence, $\alpha = m/2T$. Thus, the equilibrium velocity distribution function normalized for the gas density, which is known as the *Maxwellian distribution*, has the following form:

$$f(\mathbf{v}) = N \left(\frac{m}{2\pi T} \right)^{3/2} \exp \left[-\frac{m(\mathbf{v} - \mathbf{v}_0)^2}{2T} \right] \equiv N\varphi(v_x)\varphi(v_y)\varphi(v_z) \quad (1.7)$$

where

$$\varphi(v_x) = \left(\frac{m}{2\pi T} \right)^{1/2} \exp \left[-\frac{m(v_x - v_{0x})^2}{2T} \right], \quad \int_{-\infty}^{+\infty} \varphi(v_x) dv_x = 1 \quad (1.8)$$

The H -theorem yields the following important property of the distribution function $f(\mathbf{v}, t)$: in the absence of external fields the system tends to the equilibrium corresponding to the equilibrium distribution (1.7). Hence, the distribution function which is a solution of the Boltzmann equation (1.2) does not satisfy the principle of time reversibility. Indeed, let at $t = -T$ ($T \rightarrow \infty$) the system be in a certain non-equilibrium state described by the distribution function $f_0(\mathbf{v})$. According to the H -theorem, at $t = T$ the system goes over to the equilibrium state described by the Maxwellian distribution (1.7). If, on the contrary, at $t = -T$ the system is in the equilibrium state, then at $t = T$, according to the H -theorem, the system remains in this state rather than pass over to the state described by the distribution function $f_0(\mathbf{v})$, as should be expected from reversibility of time (that is, the requirement that the system develops in the reverse direction when t is replaced by $-t$). Hence, the Boltzmann kinetic equation does not satisfy the principle of time reversibility, while the laws of mechanics from which this equation has been derived satisfy this principle.

Thus, we have found that the distribution function which is a solution of the Boltzmann equation cannot describe the system exactly. The distribution function is seen to contain a much smaller amount of information than is needed for the exact description of the system. Indeed, to describe the system exactly we have to specify the initial states for all the particles, that is, the number of parameters describing the system is proportional to the number of particles in it. But when we introduce a distribution function describing the probability that particles with given parameters are in a given space element, we thus perform averaging over the initial conditions. Therefore, the distribution function which is a solution of the Boltzmann equation (1.2) describes the most probable state of the system for given macroscopic parameters.

If the system consists of a large number of particles, the most probable state is practically the exact state, since in such a system fluctuations (deviations from the most probable state) are small. Moreover, it appears practically unfeasible to find the exact state of a system consisting of a large number of particles, since the number of parameters describing the system is very large. The number of parameters increases with the number of particles in the system. The distribution function satisfying eq. (1.2) has a smaller number of variables and this number does not depend on the number of particles of a given species in the system. Thus, description of a system with the Boltzmann equation (1.2) gives the most probable state of the system and is, therefore, accurate if the number of particles in the system is large. The advantage of this approach is that it yields the most important information on the development of a system by selecting a limited number of parameters describing it.

Problem 1.4. Find the number of particles in a given state for a system of a large number of particles in a thermodynamic equilibrium (the Boltzmann distribution).

Let a system consist of n particles in various states. The energy of a particle in the state k is ε_k . Then, if n_k is the number of particles in the state k and E is the total energy of the given closed system, we obtain the following relationships for the total number of particles and the total energy:

$$n = \sum_k n_k \quad \text{and} \quad E = \sum_k \varepsilon_k n_k \quad (1.9)$$

We have to find the distribution of particles over states. We shall assume that the number of particles in a given state is large ($n_k \gg 1$). Furthermore, we shall make use of the Boltzmann statistics in which the probability that a particle is in a given state is not affected by other particles present in this state. The probability of a given distribution of particles over the states $P(n_1, n_2, \dots, n_k, \dots)$ is proportional to the number of ways by which this distribution can be realized.

To find the number of such ways, first, from the total number of particles we shall select n_1 particles in state 1. This can be done by $C_{n_1}^n$ ways ($C_{n_1}^n$ is the number of combinations). Then we shall select n_2 particles in state 2 from the remaining $n - n_1$ particles, which can be done by $C_{n_2}^{n-n_1}$ ways. This procedure is continued until we find that the probability of a given distribution of particles over the states, which is proportional to the number of ways for realizing this distribution, is

$$P(n_1, n_2, \dots, n_k, \dots) = A C_{n_1}^n C_{n_2}^{n-n_1} \dots C_{n_k}^{n-n_1-n_2-\dots-n_{k-1}} = A \sum_{i=1}^k \frac{n!}{\prod_i n_i!}$$

where A is the normalization constant. The probability $P(n_1, \dots, n_k, \dots)$ has a maximum at some values $n_k = \bar{n}_k$ of occupation numbers which are the most probable numbers of particles in given states. Let us use the condition for a maximum of the distribution function. If the numbers of particles in three states n_i, n_k and n_l are varied by amounts small in comparison with \bar{n}_i, \bar{n}_k and \bar{n}_l ($n_i = \bar{n}_i + \delta_i, n_k = \bar{n}_k - \delta_k, n_l = \bar{n}_l + \delta_l$), then in the first approximation in the power series in δ_i/n_i the probability P is not changed: $P(\dots, \bar{n}_i, \bar{n}_k, \bar{n}_l, \dots) = P(\dots, \bar{n}_i + \delta_i, \bar{n}_k - \delta_k, \bar{n}_l + \delta_l, \dots)$. Hence we obtain $[(n_i + \delta_i)! \approx n_i(n_i)^{\delta_i}]$

$$(\bar{n}_i)^{\delta_i} (\bar{n}_l)^{\delta_l} = (\bar{n}_k)^{\delta_k}$$

Furthermore, the conditions for balance of particles and balance of energy yield the following relationships:

$$\delta_k = \delta_i + \delta_l, \quad \varepsilon_k \delta_k = \varepsilon_i \delta_i + \varepsilon_l \delta_l$$

Using these relationships to eliminate δ_l from the above equation, we obtain

$$\left(\frac{\bar{n}_k}{\bar{n}_i} \right)^{\frac{1}{\varepsilon_k - \varepsilon_i}} = \left(\frac{\bar{n}_l}{\bar{n}_k} \right)^{\frac{1}{\varepsilon_l - \varepsilon_k}}$$

This equation shows that each of the relationships does not depend on the numbers of the states chosen, that is, it is constant for a given total number of particles and the energy of the system. To meet this condition, the dependence of \bar{n}_k on the energy of this state must be expressed as

$$\bar{n}_k = B \exp(-\varepsilon_k/T) \quad (1.10)$$

where the normalization constant B and the temperature of the system T are found from conditions (1.9). Equation (1.10) is the equilibrium distribution of particles over states and is known as the *Boltzmann distribution*. A special case of this distribution is the Maxwellian distribution found in Problem 1.3. The Boltzmann distribution yields the following relationship between the densities of atoms N_i and N_k in given states in thermodynamic equilibrium:

$$\frac{N_i}{N_k} = \frac{g_i}{g_k} \exp\left(\frac{\varepsilon_k - \varepsilon_i}{T}\right) \quad (1.11)$$

(here g_i and g_k are the statistical weights of these states).

Problem 1.5. Derive the distribution function for the particles with half-integral spin if not more than one particle can be in each state (the *Fermi-Dirac distribution*).

If the probability that a particle is in a given state is not small, the Boltzmann distribution (1.10) is not applicable. The distribution of identical particles over states depends then on the statistics of these particles. We shall find such distribution if not more than one particle can be in each state.

Let the particles in the g_k states have the energy ε_k and $p(n_k)$ be the number of ways to place n_k particles in the states with the energy ε_k . The probability of a given distribution of particles over states is

$$P(n_1, n_2, \dots, n_k, \dots) = C \prod_m p(n_k)$$

where the normalization constant C does not depend on the form of distribution.

As follows from the solution of Problem 1.4, the function P is at a maximum for the most probable values of occupation numbers $n_k = \bar{n}_k$. In this case $\ln P$ also is at a maximum, and we obtain

$$\sum_k \frac{p'(\bar{n}_k)}{p(\bar{n}_k)} \delta n_k = 0$$

where $\delta n_k = n_k - \bar{n}_k$ is the deviation of the occupation number from the mean value ($|\delta n_k| \ll \bar{n}_k$). Furthermore, eq. (1.9) yields

$$\sum_k \delta n_k = 0, \quad \sum_k \varepsilon_k \delta n_k = 0$$

Multiply the second of the above equations by $-\mu/T$ and the third by $1/T$ (where μ and T are constants) and add up the three equations. The result is

$$\sum_k \delta n_k \left[\frac{p'(\bar{n}_k)}{p(\bar{n}_k)} - \frac{\mu}{T} + \frac{\varepsilon_k}{T} \right] = 0$$

Since δn_k can have arbitrary values, the above equation is satisfied if the term in the square brackets is zero. Hence we obtain the relationship for the most probable occupation numbers:

$$\frac{p'(\bar{n}_k)}{p(\bar{n}_k)} - \frac{\mu}{T} + \frac{\varepsilon_k}{T} = 0$$

In particular, we find that for the Fermi-Dirac distribution, when n_k particles are in g_k -states with the energy ε_k and not more than one particle is in each state ($n_k \leq g_k$), $p(n_k)$ is equal to the number of combinations of n_k things g_k at a time. Hence, we obtain

$$\ln p(n_k) = \ln(g_k!) - [\ln(n_k!) - \ln(g_k - n_k)!]$$

and, since $\ln(n!) \approx \int_0^n \ln n \, dn$ for $n \gg 1$, we have

$$\frac{d \ln p(n_k)}{dn_k} = \ln \frac{g_k}{g_k - n_k}$$

and

$$\bar{n}_k = \frac{g_k}{\exp \left[\frac{\varepsilon_k - \mu}{T} \right] + 1}$$

On the average, $\left[\exp \left(\frac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1}$ particles are in one state with the energy ε_k . The constants T (the temperature of the particles) and μ (the chemical potential of the distribution) can be found from the normalization conditions (1.9):

$$\begin{aligned} \sum_k g_k \left[\exp \left(\frac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1} &= n \\ \sum_k \varepsilon_k g_k \left[\exp \left(\frac{\varepsilon_k - \mu}{T} \right) + 1 \right]^{-1} &= E \end{aligned} \quad (1.12)$$

For example, let us calculate the chemical potential of the free Fermi particles with the mass m , whose number density is N . The energy of the particles in the range from ε to $\varepsilon + d\varepsilon$ (here $\varepsilon = p^2/2m$) corresponds to