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An Introduction to Stochastic Dynamics

（随机动力系统导论）

Duan Jinqiao

（段金桥）

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Dedicated to the memory of
my grandparents, Duan Chongxiang and Ye Youxiang
and
my parents, Duan Jianhua and Chen Yuying

《纯粹数学与应用数学专著》丛书编委会

Preface

Dynamical systems are often subject to random influences, such as external fluctuations, internal agitation, fluctuating initial conditions, and uncertain parameters. In building mathematical models for these systems, some less-known, less well-understood, or less well-observed processes (e.g., highly fluctuating fast or small scale processes) are ignored because of a lack of knowledge or limitations in our analytical skills and computational capability. This ignorance also contributes to the uncertainty in mathematical models of complex dynamical systems. However, the uncertainty or randomness may have a delicate and profound impact on the overall evolution of complex dynamical systems. Indeed, there is a clear recognition of the importance of taking randomness into account when modeling complex phenomena in biological, chemical, physical, and other systems.

Stochastic differential equations are usually appropriate models for randomly influenced systems. Although the theoretical foundation for stochastic differential equations has been provided by stochastic calculus, better understanding dynamical behaviors of these equations is desirable.

Who is this book for?

There is growing interest in stochastic dynamics in the applied mathematics community. This book is written primarily for applied mathematicians who may not have the necessary background to go directly to advanced reference books or research literature in stochastic dynamics. My goal is to provide an introduction to basic techniques for understanding solutions of stochastic differential equations, from analytical, deterministic, computational and structural perspectives. In deterministic dynamical systems, invariant manifolds and other invariant structures provide global information for dynamical evolution. For stochastic dynamical systems, in addition to these invariant structures, certain computable quantities, such as the mean exit time and escape probability (reminiscent of the quantities like "eigenvalues" and "Poincaré index" in deterministic dynamics, and "entropy" in statistical physics), also offer insights into global dynamics under uncertainty. The mean exit time and escape probability are computed by solving deterministic, local or nonlocal, partial differential equations. Thus, I treat them as deterministic tools for understanding stochastic dynamics. It is my hope that this book will help the reader in accessing advanced monographs and research literature in stochastic dynamics.

What does this book do?

A large part of the materials in this book is based on my lecture notes for the graduate course *Stochastic Dynamics* that I have taught for many times since 1997. Among the students who have taken this course, about two-thirds are from applied mathematics, and the remaining one-third are from departments such as physics, computer science, bioengineering, mechanical engineering, electrical engineering, and chemical engineering. I would like to thank those graduate students for helpful feedback and for solutions to some exercises. For this group of graduate students, selection of topics and choice of presentation style are necessary. Thus, some interesting topics are not included. The choice of topics is personal but is influenced by my teaching to these graduate students, who have basic knowledge in differential equations, dynamical systems, probability, and numerical analysis. Some materials are adopted from my recent research with collaborators, and these include most probable phase portraits in Chapter 5, and random invariant manifolds in Chapter 6, together with mean exit time, escape probability and nonlocal Fokker-Planck equations for systems with non-Gaussian Lévy noise in Chapter 7.

I have tried to strike a balance between mathematical precision and accessibility for the readers of this book. For example, some proofs are presented, whereas some are outlined and others are directed to references. Some definitions are presented in separate paragraphs starting with *Definition*, but many others are introduced less formally as they occur in the body of the text. As far as possible, I have tried to make connections between new concepts in stochastic dynamics and old concepts in deterministic dynamics.

After some motivating examples (Chapter 1), background in analysis and probability (Chapter 2), a mathematical model for white noise (Chapter 3), and a crash course in stochastic differential equations (Chapter 4), I focus on three topics:

• **Quantities that carry stochastic dynamical information (Chapter 5):** This includes moments, probability densities, most probable phase portraits, mean exit time, and escape probability.

• **Structures that build stochastic dynamics (Chapter 6):** This includes the multiplicative ergodic theorem and Hartman-Grobman theorem for linearized stochastic systems, and invariant manifolds for nonlinear stochastic systems.

• **Non-Gaussian stochastic dynamics (Chapter 7):** This is an introduction to systems driven by non-Gaussian, α-stable Lévy motions.

This book is full of examples, together with many figures. There are separate Matlab simulation sections in Chapters 2–4, whereas in Chapters 5 and 7, numerical simulations are included inside various sections. Although Chapter 6 contains no numerical simulations for its nature, it has *Examples* and *Problems* that require detailed derivations or calculations by hand. At the end of each chapter, there are homework problems, including some numerical simulation problems; Matlab is sufficient for this purpose. Most of these problems have been tested in the classroom. Hints or solutions to most problems are provided at the end of the book.

A section with an asterisk may be skipped on a first reading.

Some additional references are provided in the "Further Readings" section, for more advanced readers.

What prerequisites are assumed?

For the reader, it is desirable to have basic knowledge of dynamical systems, such as the material contained in

- Chapters 1–2 of *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* by J. Guckenheimer and P. Holmes; or
- Chapters 1–2 of *Introduction to Applied Nonlinear Dynamical Systems and Chaos* by S. Wiggins; or
- Chapters 1–2 of *Differential Equations and Dynamical Systems* by L. Perko; or
- Chapters 1–3 of *Nonlinear Dynamics and Chaos* by S. H. Strogatz.

Ideally, it is also desirable to have elementary knowledge of stochastic differential equations, such as

- Chapters 1–6 of *Stochastic Differential Equations* by L. Arnold; or
- Chapters 1–5 of *An Introduction to Stochastic Differential Equations* by L. C. Evans; or
- Chapters 1–5 of *Stochastic Differential Equations* by B. Oksendal; or
- Chapters 1–3 of *Stochastic Methods* by C. Gardiner.

Realizing that some readers may not be familiar with stochastic differential equations, I review this topic in Chapter 4.

Acknowledgements

I would like to thank Philip Holmes for suggesting that I write this book back in 2004, when we were taking an academic tour in China. Steve Wiggins has also encouraged me to publish this book. I am especially grateful to Ludwig Arnold, who has always inspired and encouraged my learning and research in stochastic dynamics. I appreciate Bernt Oksendal's encouragement and comments. I have benefited from many years of productive collaboration and interaction with many colleagues, especially Peter Bates, Peter Baxendale, Dirk Blomker, Tomas Caraballo, Michael Cranston, Hans Crauel, Manfred Denker, David Elworthy, Franco Flandoli, Hongjun Gao, Martin Hairer, Peter Imkeller, Peter E. Kloeden, Kening

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My wife, Xiong Yan, and my children, Victor and Jessica, are constant sources of inspiration and happiness. Their love and understanding made this book possible.

Duan Jinqiao
Wuhan, April 2014

Notations

$\triangleq$: Is defined to be

$|x|$: Absolute value of $x \in \mathbb{R}^2$

$\|x\|$: Euclidean norm of $x \in \mathbb{R}^n$

$a \wedge b \triangleq \min\{a, b\}$

$a \vee b \triangleq \max\{a, b\}$

$a^+ \triangleq \max\{a, 0\}$

$a^- \triangleq \max\{-a, 0\}$

B_t: Brownian motion

$\mathcal{B}(\mathbb{R}^n)$: Borel σ-field of $\mathbb{R}^n$

$\mathcal{B}(S)$: Borel σ-field of state space S

$\text{Supp}(f) \triangleq$ Closure of $\{x \in \mathbb{R}^n : f(x) \neq 0\}$: The support of function f

$C(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$

$C_0(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$ which have compact support

$C^k(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$ which have up to k-th order continuous derivatives

$C_0^k(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$ which (i) have up to k-th order continuous derivatives, and (ii) have compact support

$C^\infty(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$ which have derivatives of all orders

$C_0^\infty(\mathbb{R}^n)$: Space of continuous functions on $\mathbb{R}^n$ which (i) have derivatives of all orders, and (ii) have compact support

$C^\alpha(D)$: Space of functions which are locally Hölder continuous in D with exponent α

$C^\alpha(\bar{D})$: Space of functions which are uniformly Hölder continuous in D with exponent α

$C^{k,\alpha}(D)$: Space of continuous functions in D whose k-th order derivatives are locally Hölder continuous in D with exponent α

$\delta(\xi)$: Dirac delta function

$\mathbb{E}$: Expectation

$\mathbb{E}^x$: Expectation with respect to the probability measure $\mathbb{P}^x$ induced by a solution process starting at x

$F_X(x)$: Distribution function of the random variable X

$\mathcal{F}^X$ or $\sigma(X)$: σ-field generated by the random variable X

$\mathcal{F}^{X_t} = \sigma(X_s, s \in \mathbb{R})$: σ-field generated by a stochastic process X_t. It is the smallest σ-field with which X_t is measurable for every t.

$\mathcal{F}^{\xi}$: σ-field generated by the stochastic process ξ_t

$\mathcal{F}_t^B \triangleq \sigma(B_s : s \leqslant t)$: Filtration generated by Brownian motion B_t

$\mathcal{F}_\infty \triangleq \sigma(\bigcup_{t \geqslant 0} \mathcal{F}_t)$

$\mathcal{F}_{t^+} \triangleq \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$

$\mathcal{F}_{t^-} \triangleq \sigma(\bigcup_{s<t} \mathcal{F}_s)$

$\mathcal{F}_t^X \triangleq \sigma(X_s : 0 \leqslant s \leqslant t)$: Filtration generated by a stochastic process X_t

$\mathcal{F}_{-\infty}^t \triangleq \sigma(\cup_{s \leqslant t} \mathcal{F}_s^t)$: Also denoted as $\bigvee_{s \leqslant t} \mathcal{F}_s^t$

$\mathcal{F}_s^\infty \triangleq \sigma(\cup_{t \geqslant s} \mathcal{F}_s^t)$: Also denoted as $\bigvee_{t \geqslant s} \mathcal{F}_s^t$

$H(f)$: Hessian matrix of a scalar function $f : \mathbb{R}^n \to \mathbb{R}^1$

$H(\xi)$: Heaviside function

$H^k(D)$: Sobolev space

$H_0^k(D)$: Sobolev space of functions with compact support

$\|\cdot\|_k$: Sobolev norm in $H^k(D)$ or $H_0^k(D)$

lim in m.s.: Convergence in mean square, i.e., convergence in $L^2(\Omega)$

l^p: Space of infinite sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty |x_i|^p < \infty$

$L^2(\mathbb{R}^n)$: Space of square-integrable functions defined on $\mathbb{R}^n$

$L^p(\mathbb{R}^n)$: Space of p-integrable functions defined on $\mathbb{R}^n$, with $p \geqslant 1$

$L^p(D)$: Space of p-integrable functions defined on a domain $D \subset \mathbb{R}^n$, with $p \geqslant 1$

$L^2(\Omega)$ or $L^2(\Omega, \mathbb{R}^n)$: Space of random variables, taking values in Euclidean space $\mathbb{R}^n$, with finite variance

$L^2(\Omega)$ or $L^2(\Omega, H)$: Space of random variables, taking values in Hilbert space H, with finite variance

$L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^n)$: $\{X : \mathbb{E}|X|^p < \infty\}$ for $p \geqslant 1$

$L_t(\omega)$: Lévy motion

$L_t^\alpha(\omega)$: α-stable Lévy motion

$\mathbb{N}$: Set of the natural numbers

$\mathcal{N}(\mu, \sigma^2)$: Normal (or Gaussian) distribution with mean μ and variance σ^2

$\nu(dy)$: Lévy jump measure

$\mathbb{P}$: Probability measure

$\mathbb{P}(A)$ or $\mathbb{P}\{A\}$: Probability of an event A

P^X: Distribution measure induced by the random variable X

$\mathbb{P}^x$: Probability measure induced by a solution process starting at x

$P(\lambda)$: Poisson distribution with parameter $\lambda > 0$

$\mathbb{R}$: Two-sided time set

$\mathbb{R}^+$: One-sided time set $\{t : t \geqslant 0\}$

$\mathbb{R}^1$: One dimensional Euclidean space

$\mathbb{R}^n$: n-dimensional Euclidean space

$\sigma(X)$ or $\mathcal{F}^X$: σ-field generated by the random variable X. It is the smallest σ-field with which X is measurable.

$\mathrm{Tr}(A)$: Trace of A

$U(a,b)$: Uniform distribution on the interval $[a,b]$

$\bigvee_{s\leqslant t}\mathcal{F}_s^t \triangleq \sigma(\cup_{s\leqslant t}\mathcal{F}_s^t)$: Also denoted as $\mathcal{F}_{-\infty}^t$

$\bigvee_{t\geqslant s}\mathcal{F}_s^t \triangleq \sigma(\cup_{t\geqslant s}\mathcal{F}_s^t)$: Also denoted as $\mathcal{F}_s^{\infty}$

Contents

Chapter 1

Introduction

Noisy fluctuations are abundant in complex systems. In some cases, noise is not negligible, whereas in some other situations, noise could even be beneficial. It is desirable to have a better understanding of the impact of noise on dynamical evolution of complex systems. In other words, it becomes crucial to take randomness into account in mathematical modeling of complex phenomena under uncertainty.

In 1908, Langevin devised a stochastic differential equation for the motion of Brownian particles in a fluid, under random impacts of surrounding fluid molecules. This stochastic differential equation, although important for understanding Brownian motion, went largely unnoticed in the mathematical community until after stochastic calculus emerged in the late 1940s. Introductory books on stochastic differential equations (SDEs) include [8, 88, 213].

The goal for this book is to examine and present select dynamical systems concepts, tools, and methods for understanding solutions of SDEs. To this end, we also need basic information about deterministic dynamical systems modeled by ordinary differential equations (ODEs), as presented in the first couple of chapters in one of the references [110, 290].

In this introductory chapter, we present a few examples of deterministic and stochastic dynamical systems, then briefly outline the contents of this book.

1.1 Examples of deterministic dynamical systems

We recall a few examples of deterministic dynamical systems, where short time-scale forcing and nonlinearity can affect dynamics in a profound way.

Example 1.1 A double-well system.

Consider a one-dimensional dynamical system $\dot{x} = x - x^3$. It has three equilibrium states, $-1, 0$ and 1, at which the vector field $x - x^3$ is zero. Observe that

$$\dot{x} = x - x^3 = x(1-x^2)\begin{cases} < 0, & -1 < x < 0 \text{ or } 1 < x < \infty, \\ = 0, & x = -1, 0, 1, \\ > 0, & -\infty < x < -1 \text{ or } 0 < x < 1. \end{cases}$$