A Transition to
ABSTRACT MATHEMATICS

Learning Mathematical Thinking and Writing

Second Edition

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Why Read This Book?

One of Euclid's geometry students asked a familiar question more than 2000 years ago. After learning the first theorem, he asked, "What shall I get by learning these things?" Euclid didn't have the kind of answer the student was looking for, so he did what anyone would do—he got annoyed and sarcastic. The story goes that he called his slave and said "Give him threepence since he must make gain out of what he learns."[1]

It's a familiar question: "So how am I ever gonna use this stuff?" I doubt that anyone has ever come up with a good answer, because it's really the wrong question. The first question is not what you're going to do with this stuff, but what this stuff is going to do with you.

This book is not a computer users' manual that will make you into a computer industry millionaire. It's not a collection of tax law secrets that will save you thousands of dollars in taxes. It's not even a compilation of important mathematical results for you to stack on top of the other mathematics you have learned. Instead, it's an entrance into a new kingdom, the world of mathematics, where you learn to think and write as the inhabitants do.

Mathematics is a discipline that requires a certain type of thinking and communicating that many appreciate but few develop to a great degree. Developing these skills involves dissecting the components of mathematical language, analyzing their structure, and seeing how they fit together. Once you have become comfortable with these principles, then your own style of mathematical writing can begin to shine through.

Writing mathematics requires a precision that seems a little stifling at first. It might feel like some pedant is forcing you to use a limited set of words and phrases to express the things you already see clearly with your own mind's eye. Be patient. In time you will see how adapting to the culture of mathematics and adopting its style of communicating will shape all your thinking and writing. You will see your skills of critical analysis become more developed and polished. My hope is

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[1] T. L. Heath, *A History of Greek Mathematics* 1 (Oxford, 1931).

that these skills will influence the way you organize and present your thoughts in everything from English composition papers to late-night bull sessions with friends.

Here is an analogy of what the first principles of this book will do to you. Consider a beginning student of the piano. Music is one of the most creative disciplines, and our piano student has been listening to Chopin for some time. She knows she has a true ear and intuition for music. However, she must begin at the piano by playing scales over and over. These exercises develop her ability to use the piano effectively in order to express the creativity within her. Furthermore, these repetitive tasks familiarize her with the structure of music as an art form, and actually nurture and expand her capacity to express herself in original and creative ways through music. Then, once she has mastered the basic technical skills of hitting the keys, she understands more clearly how enjoyable music can be. She learns this truth: The aesthetic elements of music cannot be fully realized until the technical skills developed by rote exercises have been mastered and can be relegated to the subconscious.

Your first steps to becoming a mathematician are a lot like those for our pianist. You are going to be introduced to the building blocks of mathematical structure, and then you will practice on the precision required to communicate mathematics correctly. The drills you perform in this practice will help you see mathematics as a creative discipline and equip you to appreciate its beauty.

Think of this course as a bicycle trip through a new country. The purposes of the trip are:

- To familiarize you with the territory
- To equip you to explore it on your own
- To give you some panoramic views of the countryside
- To teach you to communicate with the inhabitants
- To help you begin to carve out your own niche

If you're willing to do the work, I promise you will enjoy the trip. Sometimes the hills are steep and the pedaling is tough. Be persistent, knowing that it's worth the effort. You will come back a different person, for this material will have done something with you. Then you'll understand that Euclid really got it right after all, and you will appreciate why his witty response is still fresh and relevant after these 2000 years.

Preface

A Transition to Abstract Mathematics was written under the assumption that students do not yet know how to read upper level mathematics texts. Since the primary purpose of the book is to teach students to write with formal rigor, and since I naturally presume they do not yet appreciate exposition written in that form, two overriding features of style defined the first edition: a loose and informal expository style of writing, and an airtight composition and organization of the logic, so that no student could ever say that any necessary detail had been overlooked or omitted. Consequently, the scope of the first edition was rather narrow and forward focused, where every exercise had an important role in the story and there were no characters too peripheral to the plot.

I believe the second edition maintains the benefits of the first edition's features but is improved in several ways. First, the exposition is still written to the student, but it is tighter and more efficient than before. Second, there are many more exercises than in the first edition. Many of these are essential in that they are the logical basis of later results. The *Instructor's Guide and Solutions Manual* points out which exercises simply must be either assigned or at least discussed because they undergird later results. Others may be assigned, discussed casually, or omitted altogether.

A third and major change to the second edition is that exercises are now integrated into the flow of the material instead of being placed at the end of each section. I believe this arrangement has several advantages. It better facilitates the students' understanding of how the mathematics is built, one step at a time, because it requires their continual participation in that process at every step. In the second edition, the text speaks clearly to the students and then presents them with exercises right on the heels of every new concept. It also should make daily course organization easier for the instructor, in that it is always clear which exercises may be assigned after a particular day's class meeting.

Other changes to the second edition include a reorganization of the material that comprised Chapter 2 in the first edition. Introductory proof-writing material on set and real number properties has now been divided into two chapters, and the order of the material basically reversed from the first edition. Thus the students'

first theorems involve basic algebraic properties of numbers, which might be a simpler place for them to begin to write proofs than set properties. Chapter 1 now includes a section that enumerates different techniques of proof writing, with plenty of examples but no expectation that a student yet knows how or in what circumstances to employ these techniques. Finally, with exercises integrated into the exposition, certain sections that were quite long in the first edition have now been divided into more sections of more manageable length.

Preface to the First Edition

This text is written for a "transition course" in mathematics, where students learn to write proofs and communicate with a level of rigor necessary for success in their upper level mathematics courses. To achieve the primary goals of such a course, this text includes a study of basic principles of logic, techniques of proof, and fundamental mathematical results and ideas (sets, functions, properties of real numbers), though it goes much further. It is based on two premises: that the most important skill students can learn as they approach the cusp between lower and upper level courses is how to compose clear and accurate mathematical arguments, and that they need more help in developing this skill than they would normally receive by diving into standard upper level courses. By emphasizing how one writes mathematical prose, it is also designed to prepare students for the task of reading upper level mathematics texts. Furthermore, it is my hope that transitioning students in this way gives them a view of the mathematical landscape and its beauty, engaging them to take ownership of their pursuit of mathematics.

Why *This* Text?

I believe students learn best by doing. In many mathematics courses it is difficult to find enough time for students to discover through their own efforts the mathematics we would lead them to find. However, I believe there is no other effective way for students to learn to write proofs. This text is written to them in a format that allows them to do precisely this.

Two principles of this text are fundamental to its design as a tool whereby students learn by doing. First, it does not do too much for them. Proofs are included in this text for only two reasons. Most of them (especially at the beginning) are sample proofs that students can mimic as they write their own proofs to similar theorems. Students must read them because they will need the technique later. Other proofs are included because they are too much to expect of a student at this level. In most of these instances, however, some climactic detail is omitted and relegated to an exercise.

Second, if students are going to learn by doing, they must be presented with doable tasks. This text is designed to be a sequence of stepping stones placed just the right distance apart. Moving from one stone to the next involves writing a proof. Seeing how to step there comes from reading the exposition and calls on the experience that led the student to the current stone. At first, stones are very close together, and there is a lot of guidance. Progressing through the text, stones become increasingly farther apart, and the guidance gets less explicit.

I have written this text with a very deliberate trajectory of style. It is conversational throughout, though sophistication and succinctness of the exposition increase from chapter to chapter.

Organization

This text is organized in the following way. Chapter 0 spells out all assumptions to be used in writing proofs. These are not necessarily standard axioms of mathematics, and they are not presented in the context or language of more abstract mathematical structures. They are designed merely to be a starting point for logical development, so that students appreciate quickly that everything we call on is either stated up front as an assumption or proved from these assumptions. Students can probably read Chapter 0 on their own as the course begins, knowing that it is there primarily as a reference.

Part I begins with logic but does not focus on it. In Chapter 1, truth tables and manipulation of logical symbols are included to give students an understanding of mathematical grammar, of the underlying skeleton of mathematical prose, and of equivalent ways of communicating the same mathematical idea. Chapters 2–4 put these to use right away in proof writing, and allow the students to cut their teeth on the most basic mathematical ideas. These chapters will constitute most, or perhaps all, of the content of the course.

Parts II and III are two completely independent paths, the former into analysis, the latter into algebra. Like Antoni Gaudí's *Sagrada Familia*, the unfinished cathedral in Barcelona, Spain, where narrow spires rise from a foundation to give spectacular views, Parts II and III are purposefully designed to rest on the foundation of Part I and climb quickly into analysis or algebra. Many topics and specific results are omitted along the way, but Parts II and III rest securely on the foundation of Part I and allow students to continue to develop their skills at proof writing by climbing to a height where, I hope, they have a nice view of mathematics.

Flexibility

This text can be used in a variety of ways. It is suitable for use in different class settings, and there is much flexibility in the material one may choose to cover.

First, because this text speaks directly to the student, it can naturally be used in a setting where students are given responsibility for the momentum of the class. It is written so that students can read the material on their own first, then bring to class the fruits of their work on the exercises, and present these to the instructor and

each other for discussion and critique. If class time and size limit the practicality of such a student-driven approach, then certainly other approaches are possible. To illustrate, we may consider three components of a course's activity and arrange them in several ways. The components are (1) the students' reading of the material, (2) the instructor's elaboration on the material, and (3) the students' work on the exercises, either to be presented in class or turned in. When I teach from this text, (1) is first, (3) follows on its heels, and (2) and (3) work in conjunction until a section is finished. Others might want to arrange these components in another order, for example, beginning with (2), then following with (1) and (3).

Which material an instructor would choose to cover will depend on the purpose of the course, personal taste, and how much time there is. Here are two broad options.

1. To proceed quickly into either analysis or algebra, first cover the material from Part I that lays the foundation. Almost all sections and exercises of Part I are necessary for Parts II and III. However, the *Instructor's Guide and Solutions Manual* notes precisely which sections, theorems, and exercises are necessary for each path, and which may be safely omitted without leaving any holes in the logical progression. Of course, even if a particular result is necessary later, one might decide that to omit its proof details does not deprive the students of a valuable learning experience. The instructor might choose simply to elaborate on how one would go about proving a certain theorem, then allow the students to use it as if they had proved it themselves.

2. Cover Part I in its entirety, saving specific analysis and algebra topics for later courses. This option might be most realistic for courses of two or three units where all the Part I topics are required. Even with this approach, there would likely be time to cover the beginnings of Parts II and/or III. This might be the preferred choice for those who do not want to study analysis or algebra with the degree of depth and breadth characteristic of this text.

Acknowledgments

It takes an entire team to write a book. It strikes me how a well-coordinated team can pull this feat off without ever meeting each other face to face, or in some cases, without even knowing who the other team members are. So it is with the second edition of *Transition to Abstract Mathematics*.

First, enthusiastic thanks go to the staff at Elsevier who coordinated and drove this project to its completion: editors Lauren Schultz and Gavin Becker, project manager Julie Ochs, Leah Ackerson in marketing, and cover designer Eric DeCicco. Next are the professors and students who provided valuable input as the revised manuscript took shape: Michael Coco at Lynchburg College, Jessica Knapp at Pima Community College, and Will Cousins at Pepperdine University.

Finally, my deepest appreciation goes to the many students who have made the adventurous transition to abstract mathematics under my direction over the past few years. I take endless delight in seeing them gain their mathematical legs as they learn to stand, walk, and then run on their own. Without exception, I can read in their demeanor and hear in their conversation that they have developed a new way of thinking and communicating, as well as a new level of confidence in their ability to play the mathematical game.

Contents

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Notation and Assumptions

Suppose you have just opened a new jigsaw puzzle. What are the first things you do? First, you pour all the pieces out of the box. Then you sort through and turn them all face up, taking a quick look at each one to determine whether it is an inside or outside piece, and you arrange them somehow so that you will have an idea of where certain types of pieces can be found later. You don't study each piece in depth, nor do you start trying to fit any of them together. You merely lay all the pieces out on the table and briefly familiarize yourself with them. This is the point of the game where you set the stage, knowing that everything you will need later has been put in a place where you can find it when you need it.

In this introductory chapter, we lay out all the pieces we will use for our work in this course. It is essential that you read it now, in part because you need some preliminary exposure to the ideas, but mostly because you need to have spelled out precisely what you can use without proof in Part I, where this chapter will serve you as a reference. Give this chapter a casual but complete reading for now. You have probably encountered most of the ideas before. But don't try to remember it all, and certainly don't expect to understand everything either. That is not the point. Right now, we are just organizing the pieces. The two issues we address in this chapter are: (1) Set terminology and notation, and (2) Assumptions about the real numbers.

0.1 Set Terminology and Notation

Sets are perhaps the most fundamental mathematical entity. Intuitively, we think of a set as a collection of things, where the collection itself is regarded as a single entity. Sets may contain numbers, points in the xy-plane, functions, ice cream cones, steak knives, worms, even other sets. We will denote many of our sets with uppercase letters (A, B, C), or sometimes with scripted letters ($\mathcal{F}$, $\mathcal{S}$, $\mathcal{T}$). First, we need a way of stating whether a certain thing is or is not in a set.

Definition 0.1.1 If A is a set and x is an entity in A, we write $x \in A$, and say that x is an *element* of A. To write $x \notin A$ means that x is not an element of A.

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