A NON-HAUSDORFF COMPLETION

The Abelian Category of C-complete Left Modules over a Topological Ring

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Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Lubkin, Saul, 1939–
A non-Hausdorff completion : the Abelian category of C-complete left modules over a topological ring / Saul Lubkin, University of Rochester, USA.
pages cm
Includes bibliographical references.
ISBN 978-9814667388 (hardcover : alk. paper)
1. Topology. 2. Abelian categories. 3. Algebra, Homological. 4. Commutative algebra. 5. Topological rings. I. Title.
QA611.L83 2015
512'.44--dc23
2015001145

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore

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Dedication: To my wife, Maxine, who encouraged me to complete this book.

Acknowledgement: This manuscript was put into LaTeX by my former graduate student, Chad Gratton, to whom I am very grateful.

Preface/Introduction

Suppose that we have a commutative ring A and an ideal I in A. Then we have the well-known *I-adic completion* $M^{\wedge I}$ of any left A-module M,

$$M^{\wedge I} = \varprojlim_{n \geq 1} M/I^n M \ .$$

The assignment: $M \rightsquigarrow M^{\wedge I}$ is an additive functor, that in general is neither left nor right exact; the usual completion functor fails to have many useful properties, that often make computation difficult.

In this book, we introduce a new functor, $C^I(M)$, the *C-completion of M with respect to the ideal I*. Actually we make this construction in far greater generality—if A is *any* not-necessarily-commutative topological ring with identity such that the topology is given by right ideals and if M is any abstract left A-module, then we define $C(M)$. $C(M)$ can be defined quickly as being the zeroth derived functor of the usual completion functor, $M \rightsquigarrow M^\wedge$. For example, if we choose P_1, P_2 projective left A-modules and an exact sequence

$$P_1 \to P_2 \to M \to 0 \ ,$$

then $C(M) = \texttt{Cok}(P_1^\wedge \to P_2^\wedge)$.

In all cases, the functor $M \rightsquigarrow C(M)$ is right exact. However, unlike $M^\wedge$ $C(M)$ is rarely Hausdorff (not even if the topological ring A is a complete discrete valuation ring that is not a field). Hence $C(M)$ can be thought of as being "a Non-Hausdorff Completion of the abstract left A-module M."

Although $C(M)$ and the traditional $M^\wedge$ are in general different, one can recover $M^\wedge$ from $C(M)$. E.g., under mild conditions,

$$C(M)/(\text{divisible elements}) \approx M^\wedge \ .$$

Thus, $M^\wedge$ can be thought of as being a weaker construction than $C(M)$.

In addition, since the functor C is a right exact functor, it has higher derived functors. These are the higher C-completions, $C_i(M), i \geq 0$. ($C_0(M) = C(M)$.) These are used to construct spectral sequences, that are very useful in computing $C(M)$ and $C_i(M)$, $i \geq 0$.

If A is a topological ring such that the topology is given by right ideals and M is an abstract left A-module, then we define the notion of an *infinite sum* *structure* on the abstract left A-module M. Basically, if $(a_i)_{i\in I}$ are elements in $A^\wedge$ that converge to zero, and if $(m_i)_{i\in I}$ are any elements of the left A-module M, then an infinite sum structure tells us how to define

$$\sum_{i\in I} a_i m_i \in M \ .$$

A left A-module M, together with an infinite sum structure, is called *a C-complete* left A-module. And we define the notion of an *infinitely linear function* between two C-complete left A-modules. For example, if M is any abstract left A-module, then both $M^\wedge$ and $C(M)$ have natural such structures, and, therefore, are naturally C-complete left A-modules, and the natural map: $C(M) \to M^\wedge$ is infinitely linear. The category of all C-complete left A-modules and infinitely linear functions turns out to be a very interesting abelian category, which we shall denote $\mathscr{C}_A$.

It should be noted that, under reasonably mild conditions—e.g., if the topological ring A is such that the topology is given by denumerably many two-sided ideals $I_i, i \geq 1$, each of which is finitely generated as right ideal, and such that I_i^2 is open, $i \geq 1$, then the category $\mathscr{C}_A$ turns out to be a full exact abelian subcategory of the category $\mathscr{M}_A$ of all abstract left A-modules—that is, every linear map of C-complete left A-modules is then automatically infinitely linear. In all cases, whatever the topological ring A, the "stripping functor": $\mathscr{C}_A \rightsquigarrow \mathscr{M}_A$ that to each C-complete left A-module associates the corresponding abstract left A-module, is always exact and faithful, and preserves direct products. In particular, $\mathscr{C}_A$ is always an exact abelian subcategory of $\mathscr{M}_A$.

We now summarize these constructions, and others, in more detail. For the rest of this Preface, we will refer only to topological rings A such that the topology is given by right ideals.

In Chapter 2, a *C-complete left A-module* is defined to be an abstract left A-module together with an infinite sum structure. For example, if M is an abstract left A-module, then $M^\wedge$ is a C-complete left A-module in an obvious way. In fact, every C-complete left A-module is isomorphic to the cokernel of an infinitely linear map: $F^\wedge \to G^\wedge$, where F and G are free left A-modules. (Note: The map $F^\wedge \to G^\wedge$ need not come from a map in $\mathscr{M}_A : F \to G$.) In Example 3 of Chapter 2, we construct a complete submodule N of $(\mathscr{O}^{(\omega)})^\wedge$, where $A = \mathscr{O}$ is any c.d.v.r. that is not a field, such that $(\mathscr{O}^{(\omega)})^\wedge/N$ is not Hausdorff. However, of course, it is C-complete, for the infinite sum structure inherited from $(\mathscr{O}^{(\omega)})^\wedge$.

In Corollary 2.3.10 of Chapter 2, we show that if A is commutative, then the A-module $Hom_{\mathscr{C}_A}(M, N)$, where $M, N \in \mathscr{C}_A$ also has a natural structure of C-complete A-module (it is given by the infinite sum structure inherited from N^M). In Remark 8 of Section 3 of Chapter 2, if the topological ring A is commutative, if $M, N, L \in \mathscr{C}_A$ and if $f : M \times N \to L$ is a function, then we define what it means for f to be *infinitely bilinear*, and we use this to define $M \bigotimes_A^C N$, the *C-complete tensor product of M and N*. Also, if A is commutative, then we define

$$Hom_{\mathscr{C}_A} : \mathscr{C}_A^0 \times \mathscr{C}_A \rightsquigarrow \mathscr{C}_A \ .$$

The functors $Hom_{\mathscr{C}_A}$ and $\bigotimes_A^C$ are adjoint:

$$Hom_{\mathscr{C}_A}(M \bigotimes_A^C N, L) \approx Hom_{\mathscr{C}_A}(M, Hom_{\mathscr{C}_A}(N, L)) .$$

This is an isomorphism of functors from $\mathscr{C}_A^0 \times \mathscr{C}_A^0 \times \mathscr{C}_A$ into the category of sets (even into the category $\mathscr{C}_A$).

Always, $\mathscr{C}_A$ is abelian and is closed under infinite direct products and inverse limits, and the "stripping functor": $\mathscr{C}_A \rightsquigarrow \mathscr{M}_A$ is exact and preserves arbitrary direct products and inverse limits. Also, $\mathscr{C}_A$ has enough projectives.

Infinite direct sums and arbitrary direct limits also always exist in $\mathscr{C}_A$—but they are very different from the usual construction in $\mathscr{M}_A$: these constructions are pathological in $\mathscr{C}_A$.

Some interesting details: Every finitely presented abstract left $A^\wedge$-module has a natural structure as C-complete left A-module. And, if M is any abstract left A-module, then $C(M)$ can be characterized as being the universal C-complete left A-module into which M maps by a homomorphism of abstract left A-modules. And, the functor $C : \mathscr{M}_A \rightsquigarrow \mathscr{C}_A$ preserves arbitrary direct limits.

In Chapter 3, Section 2, we study the divisible part of C-complete left A-modules. For example, if the topology of A is given by denumerably many open right ideals, and if M is a C-complete left A-module, then M always has no non-zero infinitely divisible elements (i.e., there is no non-zero submodule of M that is divisible). And then also for every abstract left A-module M, $C(M) = 0$ *iff* $M^\wedge = 0$ *iff* M is A-divisible. And then also for every $M \in \mathscr{C}_A$ we have the short exact sequence:

$$0 \to (\text{div } M) \to M \to M^\wedge \to 0$$

in $\mathscr{C}_A$, where (div M) denotes the divisible part of M; and if $N \in \mathscr{M}_A$, then we have the short exact sequence

$$0 \to \text{div } (C(N)) \to C(N) \to \widehat{N} \to 0$$

in $\mathscr{C}_A$. Of course, these hypotheses are very mild, and hold in all serious current applications to algebraic geometry and commutative algebra. And then,

$$N^\wedge = C(N)/(\text{divisible elements}) ,$$

so that $N^\wedge$ is "$C(N)$ made Hausdorff", for all abstract left A-modules N.

Note: If the topology of A is the right t-adic for some element $t \in A$, such that, e.g., *either* t is not a left divisor of zero, *or* A is right Noetherian, then

$$Ker(M \to C(M)) = \{\text{infinitely } t - \text{divisible elements of } M\} .$$

And

$$M/(\text{div } M) \hookrightarrow M^\wedge,$$
$$M/(\text{infinitely divisible part of } M) \hookrightarrow C(M),$$
$$M^\wedge = \frac{C(M)}{\left[\frac{\text{div }(M)}{(\text{inf.div.}M)}\right]}.$$

In Chapter 4, we study the higher C-completions, $C_i(M), i \geq 0$ —these are the left derived functors of the functor C from $\mathscr{M}_A$ into $\mathscr{C}_A$—or, equivalently, of the usual A-adic completion functor, $M \rightsquigarrow M^\wedge$, from $\mathscr{M}_A$ into $\mathscr{C}_A$.

If B_* is any non-negatively indexed chain complex of abstract left A-modules, then we have the two spectral sequences in the category $\mathscr{C}_A$ starting with

$$^I E^1_{p,q} = C_q(B_p)$$

and

$$^{II} E^2_{p,q} = C_p(H_q(B_*)),$$

both abutting at the same sequence $K_n, n \geq 0$, in $\mathscr{C}_A$ (but with different filtrations). From these, we deduce the *spectral sequence of the* C-completion: B_* as above, if also

$$C_i(B_q) = 0, i \geq 1, q \geq 0, \qquad (*)$$

then we have a first quadrant homological spectral sequence:

$$E^2_{p,q} = C_p(H_q(B_*)) \Rightarrow H_n(C(B_*)),\ n \geq 0.$$

Note: Condition $(*)$ holds if the topology of A is given by denumerably many right ideals, and if B_i is left flat as A-module, all $i \geq 0$.

The above spectral sequence is very important in many computations involving cohomology of completions and p-adic cohomology of algebraic varieties and schemes. For example, the short exact sequence (I.8) of [PPWC], and of [COC], Chapter 2, is a very special case of this spectral sequence.

A corollary of the spectral sequence: If the topology of A is given by denumerably many right ideals $I_1 \supset I_2 \supset I_3 \supset \cdots$, then we have the short exact sequence

$$0 \to \left(\varprojlim_{j\geq 1}{}^1 \mathtt{Tor}^A_1(A/I_j, M)\right) \to C(M) \to \widehat{M} \to 0,$$

for every abstract right A-module M. And, if the I_j are two-sided ideals that are finitely generated as right ideals, and if I_j^2 is open for all $j \geq 1$ (i.e., if for all $j \geq 1$, $I_j^2 \supset I_k$ for some $k \geq j$), then for every C-complete left A-module M, we have that

$$\text{div}(M) \approx \varprojlim_{j\geq 0}{}^1 \mathtt{Tor}^A_1(A/I_j, M)$$

as C-complete left A-modules.

If the topology of A is given by denumerably right ideals, and if M is a flat left A-module, then

$$C(M) = M^\wedge \text{ and } C_i(M) = 0, i \geq 1 .$$

As a special case:

If A is a commutative ring, and t is an element that is not a divisor of zero, and if the topology of A is the t-adic topology, then

$$\operatorname{div}(C(M)) = \varprojlim_{i \geq I} \left(\text{precise } t^i\text{-torsion in } \frac{M^\wedge}{M} \right)$$

where "{precise t^i-torsion in an A-module N}" means "Ker $(t^i : N \to N)$", and where, if $M \in \mathscr{M}_A$, then "$M^\wedge/M$" is shorthand for "$(\mathtt{Cok}(M \to M^\wedge))$". And, in this case $\mathtt{Ker}(M \to C(M)) = \{$infinitely t-divisible elements in $M\}$, for all abstract A-modules M.

In Chapter 5, we study direct sums and direct limits in $\mathscr{C}_A$. As we have noted above, direct sum is usually very different from the direct sum of abstract left A-modules and is not exact. Because of its unusual behavior, we use the symbol

$$\int_{i \in I} M_i$$

to denote the direct sum of objects M_i in $\mathscr{C}_A, i \in I$. The notation

$$\bigoplus_{i \in I} M_i$$

will mean the direct sum in $\mathscr{M}_A$—i.e., as abstract left A-modules, ignoring the infinite sum structures. Under mild conditions, we have that

$$\int_{i \in I} M_i = C\left(\bigoplus_{i \in I} M_i \right) \text{ in } \mathscr{C}_A ,$$

whenever $M_i \in \mathscr{C}_A$, all $i \in I$. And $C\left(\varinjlim_{i \in D} M_i\right)$ is the direct limit in $\mathscr{C}_A$ of any direct system $(M_i, \alpha_{ij})_{i,j \in D}$ of objects and maps in $\mathscr{C}_A$—where, as usual $\varinjlim_{i \in D} M_i$ denotes the direct limit in $\mathscr{M}_A$, ignoring the C-complete left A-module structures of the M_i, $i \in i$.

The functor C from $\mathscr{M}_A$ into $\mathscr{C}_A$ always preserves arbitrary sums and direct limits; in particular, we have that

$$C\left(\bigoplus_{i \in I} M_i \right) = \int_{i \in I} C(M_i) ,$$

for all $M \in \mathscr{M}_A$ and all sets I.

The natural map from the direct sum into the direct product in $\mathscr{C}_A$:

$$\int_{i \in I} M_i \longrightarrow \prod_{i \in I} M_i$$

is almost never injective. In fact, under very mild conditions,

$$\operatorname{div}\left(\int_{i\in I} M_i\right) = \texttt{Ker}\left(\int_{i\in I} M_i \longrightarrow \prod_{i\in I} M_i\right) ,$$

and this is often non-zero. For example, if $A = \mathscr{O}$ is a c.d.v.r. not a field with uniformizing parameter t, then

$$\operatorname{div}\left(\int_{i\geq 1} \mathscr{O}/t^i\mathscr{O}\right) = \operatorname{Ker}\left(\int_{i\geq 1} \mathscr{O}/t^i\mathscr{O} \longrightarrow \prod_{i\geq 1} \mathscr{O}/(t^i\mathscr{O})\right)$$

and is non-zero. And, therefore, the C-complete $\mathscr{O}$-module

$$\int_{i\geq 1} \mathscr{O}/t^i\mathscr{O} = C\left(\bigoplus_{i\geq 1}(\mathscr{O}/t^i\mathscr{O})\right)$$

is a C-complete $\mathscr{O}$-module that is not complete.

In §5.6, since the direct sum $\int_{i\in I} M_i$ in the category $\mathscr{C}_A$ is usually not exact, but is always right exact, and since $\mathscr{C}_A$ always has enough projectives, we define and study the *higher direct sums*

$$(M_i)_{i\in I} \rightsquigarrow \int_{i\in I}^{n} M_i, \quad n \geq 0 ,$$

which are by definition the higher left derived functors of $\int_{i\in I}$. For example, we always have the first quadrant homological spectral sequence in the category $\mathscr{C}_A$,

$$E^2_{p,q} = \int_{i\in I}^{p} C_q(M_i) \Rightarrow C_n\left(\bigoplus_{i\in I} M_i\right) \tag{$*$}$$

where $M_i \in \mathscr{C}_A$, all $i \in I$.

And, under mild conditions on A, the natural infinitely linear function:

$$C_p\left(\bigoplus_{i\in I} M_i\right) \longrightarrow \int_{i\in I}^{p} M_i$$

is an isomorphism, all $p \geq 0$, whenever $M_i \in \mathscr{C}_A$, all $i \in I$. And, under the same mild conditions, the spectral sequence $(*)$ simplifies to

$$E^2_{p,q} = C_p\left(\bigoplus_{i\in I} C_q(M_i)\right) \Rightarrow C_n\left(\bigoplus_{i\in I} M_i\right) .$$

Sometimes, however, the infinite direct sum $\int_{i\in I} M_i$ in $\mathscr{C}_A$ *is* exact: For example, if A is a right t-adic ring (meaning that there is an element $t \in A$ such that the topology of A has an open neighborhood base at zero consisting of the right ideals $t^iA, i \geq 0$), and such that the t-torsion is bounded below (meaning that there is an integer $n \geq 1$ such that $t^m a = 0$ in A implies that $t^n a = 0$, all $a \in A$, all $m \geq 1$), then, for every set I, the I-fold direct sum

$$\int_{i \in I} : \quad \mathscr{C}_A^I \rightsquigarrow \mathscr{C}_A$$

is exact.

In §5.7, we study some of the consequences of the fact that $\int_{i \in I}$ is usually not exact.

An abelian category $\mathscr{A}$ obeys the *Eilenberg Moore Axiom* $(P1)$ *iff* denumerable direct products exist and the functor "denumerable direct product": $\mathscr{A}^\omega \rightsquigarrow \mathscr{A}$ is exact. $\mathscr{A}$ obeys the *Eilenberg–Moore Axiom* $(P2)$ *iff* denumerable direct products exist, and if also whenever

$$\cdots \to A_{i+1} \to A_i \to \cdots \to A_1$$

is an inverse system in which all the maps are epimorphisms, then the induced map:

$$\left[\varprojlim_{i \geq 1} A_i\right] \longrightarrow A_1$$

is an epimorphism. $\mathscr{A}$ *obeys the E-M Axiom* $(S1)$ (resp. $(S2)$) *iff* the dual category $\mathscr{A}^0$ obeys the $(P1)$ (resp. $(P2)$). We show the well-known facts that $(P2) \Rightarrow (P1)$, and that $(S2) \Rightarrow (S1)$; and that if $\mathscr{A}$ has enough injectives, and if denumerable direct sums exist, and if denumerable direct limit is exact, then $\mathscr{A}$ obeys $(S2)$, and therefore also $(S1)$.

Now, if A is any ring, and t is an element in the center of A that is not nilpotent, and if we give A the t-adic topology, then the abelian category $\mathscr{C}_A$ does not obey the EM Axiom $(S2)$.

In fact, if A is a commutative ring and $t \in A$, then t is not nilpotent *iff* $\mathscr{C}_A$ does not obey EM $(S2)$.

And, we also show that, given a ring A, and an element t in the center of A that is not nilpotent, and such that the t-torsion is bounded below, then if we give A the t-adic topology, we have that the abelian category $\mathscr{C}_A$ obeys the Eilenberg–Moore Axiom $(S1)$ but not $(S2)$. Since $\mathscr{C}_A$ does not obey $(S2)$, it follows that it also does not have enough injectives. And, in this case, $\mathscr{C}_A$ is an exact full abelian subcategory of the category of left A-modules $\mathscr{M}_A$.

As a special case, if $\mathscr{O}$ is a complete discrete valuationg ring not a field, then the full subcategory $\mathscr{C}_\mathscr{O}$ of the category of $\mathscr{O}$-modules obeys $(S1)$ but not $(S2)$, and also does not have enough injectives (but does have enough projectives). Such examples are hard to come by, and $\mathscr{C}_\mathscr{O}$ is a pretty natural such example.

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