ISI **Lecture Notes** 4

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PROCEEDINGS OF THE SYMPOSIUM ON GRAPH THEORY

ISI Lecture Notes, No. 4

Proceedings of the Symposium on Graph Theory

held at
the Indian Statistical Institute
20-25 December 1976

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Calcutta

First published, 1979, by

THE MACMILLAN COMPANY OF INDIA LIMITED
Delhi Bombay Calcutta Madras

Associated companies throughout the world

SBN : 33390 273 4

Published by S G Wasani for
The Macmillan Company of India Limited, and
printed at Pearl Offset, Kirti Nagar, New Delhi.

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No. 4. A Ramachandra Rao (Ed.): Proceedings of the Symposium on Graph Theory held at the Indian Statistical Institute, 20-25 December 1976

To

Professor S. S. Shrikhande
on his sixtieth birthday

Preface

This publication consists of the *Proceedings of the Symposium on Graph Theory* organized by the Indian Statistical Institute in Calcutta during 20-25 December 1976.

There were three series of expository lectures besides the contributed papers. We have included them also, as the material includes recent results and may not be readily available elsewhere.

It is a pleasure for me to acknowledge the cooperation I have received from various people in connection with the Symposium.

I thank Professor G. Kallianpur, our Director, and Professor B. P. Adhikari of the Division of Theoretical Statistics and Mathematics for their interest in and support for the Symposium and for making facilities available to us.

I thank Dr. S. Bhaskara Rao in particular for his ungrudging and valuable help in organizing the Symposium and the Proceedings.

I also thank my other colleagues who have helped me in various ways, the staff of the Director's office and the office of the Division of Theoretical Statistics and Mathematics for their secretarial assistance, and the staff of our guest house for the arrangements for board and lodging of the participants.

Finally I thank all the participants for their cooperation in making the Symposium a success.

I hope that the papers included in these Proceedings form a significant contribution to Graph Theory and that such symposia will become a regular feature in this country.

Calcutta
August 1978

A RAMACHANDRA RAO

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MULTICOLORINGS OF GRAPHS AND HYPERGRAPHS

C. Berge
M.S.H., 54 Bd. Raspail, Paris-6

For any graph G, Vizing has proved that $q(G) = \Delta(G)$ or $q(G) = \Delta(G) + 1$ where $\Delta(G)$ is the maximum degree of a vertex in G and q(G), the chromatic index of G, is the minimum number of colors needed to color the edges of G so that no two adjacent edges get the same color.

Lucas, in 1902, showed the following : if h is odd, h+1 school girls can go walking in pairs for h days so that every girl walks with every other girl exactly once. This means, of course, that $q(K_{h+1}) = h$ if h is odd.

A multicoloring of the edges of a graph G is an assignment of one or several colors to each edge of G such that at any vertex each color occurs exactly once. For the Petersen graph P_{10}, there is a multicoloring of the edges with 5 colors as shown in figure 1.1. If the vertices of P_{10} represent school girls and edges represent friendship relation then this means that they can go walking for 5 days in pairs of friends such that each girl walks with each of her friends exactly once.

It can be shown that every regular graph has a multicoloring of edges. We conjecture that if the graph is cubic, connected and has no isthmus then there is such a multicoloring with at most 5 colors.

This article is based on the notes taken when Prof. Berge gave an expository talk.

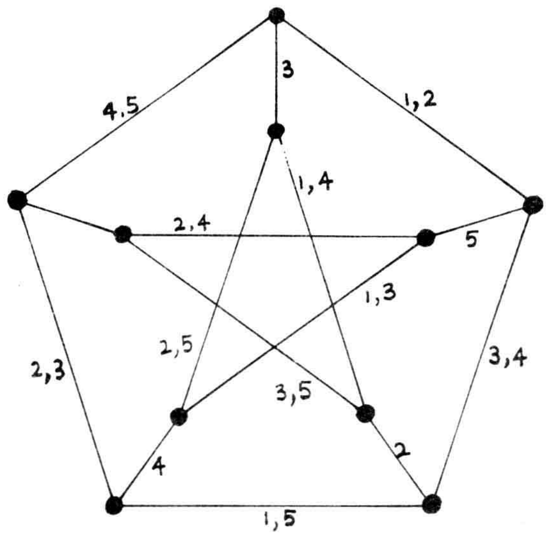

Figure 1.1

Let us define

$$k^*(G) = \begin{cases} \text{the least number of colors in a} \\ \text{multicoloring of the edges of } G \text{ if} \\ \text{one exists, } \infty \text{ otherwise.} \end{cases}$$

It is difficult to determine $k^*(G)$ in general.

It is not very easy, but it can be checked that $k^*(P_{10}) = 5$.

The following theorem gives a necessary and sufficient condition for $k^*(G)$ to be finite.

__Theorem 1__. A simple graph $G = (X, E)$ has a multicoloring of its edges iff G is regularizable and if some regular multigraph H obtained from G by edge-multiplication satisfies

$$m_H(S, X - S) \geq \Delta(H) \quad \text{for every} \quad S \subseteq X \text{ with}$$

$|S|$ odd.

A graph G is called <u>regularizable</u> if it is possible to get a regular graph from G by multiplying each edge of G a suitable number (≥ 1) of times.

The necessity part of Theorem 1 is easy to prove. Given a multicoloring of G with k colors, if each edge is multiplied as many times as the number of colors given to it, we get a regular graph H of degree k which shows that G is regularizable. If $S \subseteq X$ and $|S|$ is odd, then there is at least one arc of H with color i which joins S to X-S (for i = 1,2,..., k). Thus, the number of edge of H between S and X-S is

$$m_H(S, X-S) \geq \triangle(H).$$

We give two applications.

Consider the following arrays A_1 and A_2.

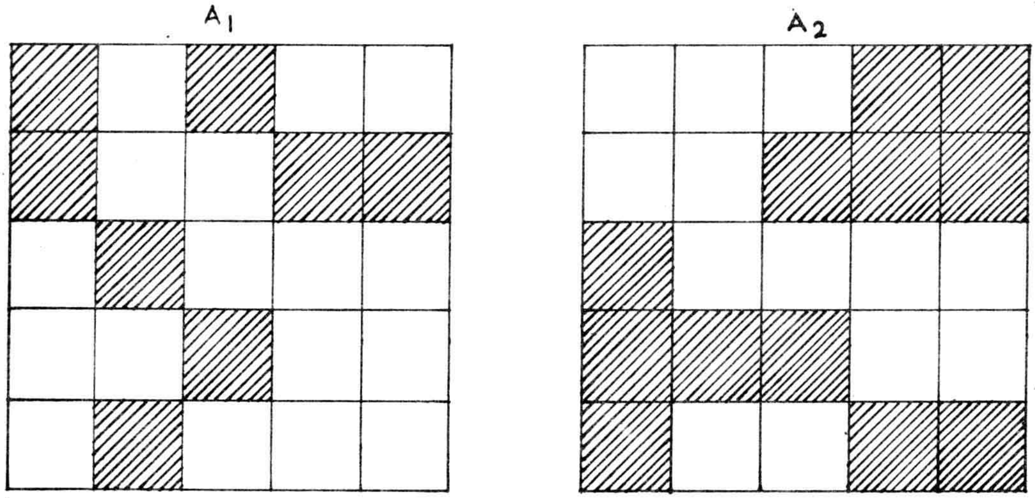

Figure 1.2

Suppose the rows of the array represent workers and columns represent machines and a shaded cell means the corresponding worker cannot operate the corresponding machine. Find the least number of days on which the machines have to be operated so that the five machines are operated simultaneously every day and each machine is operated at least once by every eligible worker. It is easy to see that the answer is $k^*(G)$ where G is the bipartite graph whose vertices correspond to the rows and columns of the array and edges correspond to the unshaded cells in the array.

For the array A_1, the above number is 5 and for the array A_2, the number is ∞ as will be shown later.

As a second application, consider matrices $A = ((a_{ij}))$ where a_{ij} are subsets of a set U. Multiplication of matrices is defined by

$$A.B = ((c_{ij})) \text{ where } c_{ij} = \bigcup_t (a_{it} \cap b_{tj})$$

Note that the matrix $I_U = ((\delta_{ij}))$ where

$$\delta_{ij} = \begin{cases} \emptyset & \text{if } i \neq j, \\ U & \text{if } i = j, \end{cases}$$

acts as an identity matrix. Then it can be proved that a matrix A has an inverse iff the non-empty sets in each row and in each column form a partition of U.

Now we extend the above concepts to hypergraphs.

A <u>hypergraph</u> on a set X is a collection H of non-empty subsets of X whose union is X. The elements of X are the <u>vertices</u> of H and the sets in H are the <u>edges</u> of H.

The *incidence matrix* of a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on the set $X = \{x_1, \ldots, x_n\}$ is the matrix $((a_{ij}))$ where

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in E_j \\ 0 & \text{otherwise.} \end{cases}$$

The *dual* H^* of a hypergraph H is the hypergraph whose incidence matrix is the transpose of the incidence matrix of H.

A hypergraph is called *unimodular* if the determinant of every square submatrix of its incidence matrix is 0, + 1 or -1.

A *multicoloring* of a hypergraph H is an assignment of one or more colors to each vertex of H such that in each edge every color occurs exactly once.

We define

$$k(H) = \begin{cases} \text{the least number of colors in a multicoloring of H if one exists,} \\ \infty \quad \text{otherwise.} \end{cases}$$

It is easy to see that for a graph G, $k^*(G)$ defined earlier is same as $k(G^*)$.

An interesting example of a unimodular hypergraph is an *interval hypergraph* defined thus: its vertices are points on a straight line and edges are strings of points or intervals.

Theorem 2. An interval hypergraph has a multicoloring iff no edge contains another properly.

We give below an example of an interval hypergraph H and a multicoloring with $k(H) = 5$ colors.

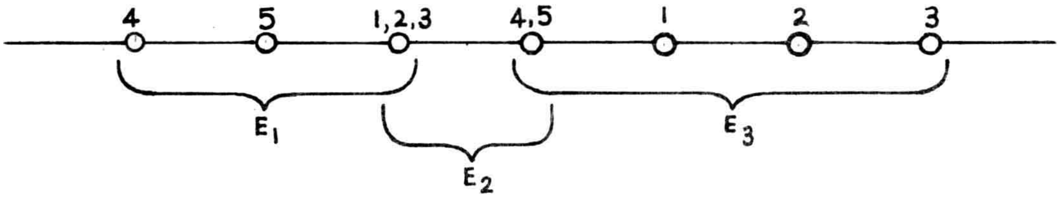

Figure 1.3

Even though the dual of an interval hypergraph is not an interval hypergraph, Theorem 2 remains valid for the dual of an interval hypergraph (which is also a unimodular hypergraph).

Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph on X and $d_H(x)$ denote the number of edges which contain x. Define

$$Q = \{(H', H'') : H' \subseteq H,\ H'' \subseteq H,\ H' \cap H'' = \emptyset,$$
$$d_{H''}(x) \leq d_{H'}(x) \leq d_{H''}(x) + 1 \quad \forall\ x \in X\}.$$

Note that Q is non-empty since $(E_i, \emptyset) \in Q$. Define

$$\mu(H) = \begin{cases} +\infty \text{ if there exists } (H',H'') \in Q \text{ such that } m(H') < m(H'') \\ \max \dfrac{|\{x: d_{H'}(x) = d_{H''}(x) + 1\}|}{m(H') - m(H'')} \text{ otherwise} \end{cases}$$

Then we have

Theorem 3. For any hypergraph H, $k(H) \geq \mu(H)$. If H is unimodular then $k(H) = [\mu(H)]^*$, the smallest integer not less than $\mu(H)$.

If G_1 is the bipartite graph associated with the array A_1 of Figure 1.2, then taking H' = the set of vertices of G_1 corresponding to the last 4 columns of A_1 and H'' = the set of vertices of G_1 corresponding to the first 2 rows of A_1, we see that $(H',H'') \in Q(G_1^*)$ so $k(G_1^*) \geq \mu(G_1^*) \geq \frac{9}{2}$. It is possible to give a multicoloring of G_1^* with 5 colors as given below. Hence it follows that $k(G_1^*) = 5$.

	1,5		3	2,4
	2,3	1,4,5		
1,4		2	5	3
3,5	4		2	1
2		3	1,4	5

Figure 1.4

For the graph G_2 corresponding to the array A_2 of Figure 1.2, we can show that $k(G_2^*) = \infty$ since by taking

$H' = \{\text{rows 3 and 4}\}$, $H'' = \{\text{columns 4 and 5}\}$, we get $\mu(G_2^*) = +\infty$.

<u>Corollary</u> Let H be a unimodular hypergraph. Then there exists no multicoloring of H iff there exist $H', H'' \subseteq H$ such that

$$m(H') = m(H''),$$

$$d_{H'}(x) \geq d_{H''}(x) \quad \text{for all } x \in X,$$

$$d_{H'}(x) > d_{H''}(x) \quad \text{for at least one } x \in X.$$

<u>Proof</u>. If there exist $H', H'' \subseteq H$ with the stated properties and if there exists a multicoloring of H with k colors we get a contradiction by noting that $\Sigma\, n(x)$ over all pairs (x, E) with $x \in X$, $E \in H'$, $x \in E$ is $k.m(H')$ on the one hand, and $\sum_{x \in X} n(x).d_{H'}(x)$ on the other, where $n(x)$ is the number of colors assigned to x in the multicoloring. Conversely if H has no multicoloring then by Theorem 3, $\mu(H) = +\infty$.

<u>Case 1</u>. There exist $H', H'' \subseteq H$ such that

$m(H') < m(H'')$, $H' \cap H'' = \emptyset$ and

$$d_{H''}(x) \leq d_{H'}(x) \leq d_{H''}(x) + 1 \text{ for all } x.$$

Then by augmenting H' with some of the edges of H'', we get two new hypergraphs H' and H'' satisfying the conditions of the corollary.

<u>Case 2</u>. There exists $(H', H'') \in Q$ such that $m(H') = m(H'')$ and $\{x / d_{H'}(x) = d_{H''}(x) + 1\} \neq \emptyset$. In this case, the

conditions of the Corollary are satisfied.

Q.E.D.

We do not know if a statement similar to Theorem 3 is valid for the balanced hypergraphs. However, we can state:

<u>Theorem 4</u>. Let H be a balanced hypergraph. Then H has no multicoloring iff there exist $p_1, p_2, \ldots, p_m \in Z$ satisfying

$$\sum_{i=1}^{m} p_i = 0,$$

$$\sum_{i=1}^{m} p_i \phi_i(x) \geq 0 \quad \text{for all} \quad x \in X$$

$$\sum_{i=1}^{m} p_i \phi_i(x) > 0 \quad \text{for at least one} \quad x \in X,$$

where $\phi_i(x)$ is the characteristic function of the edge E_i.