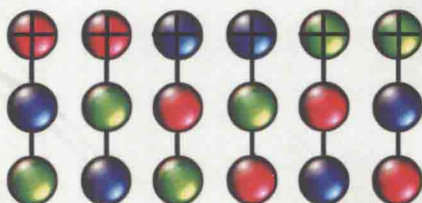
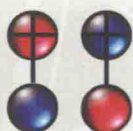
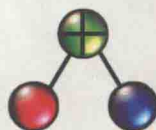
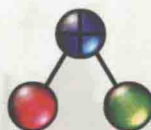
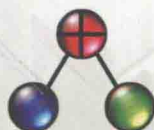
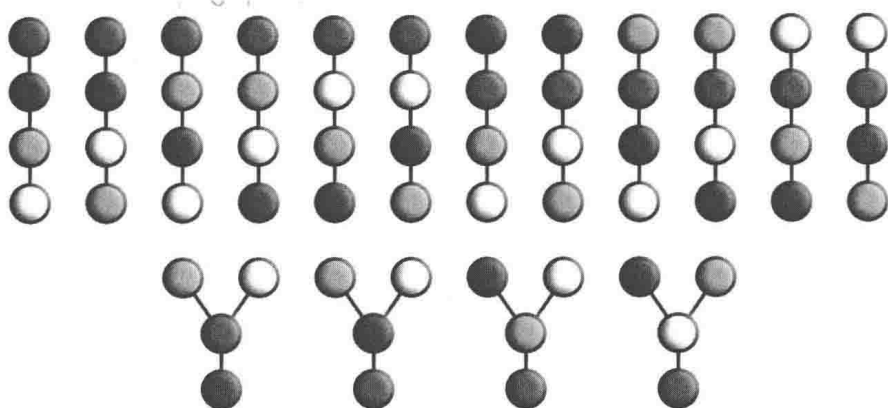


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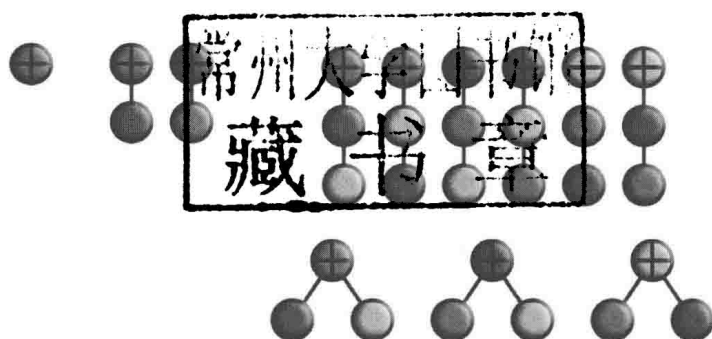


Utkir A Rozikov





GIBBS MEASURES ON CAYLEY TREES



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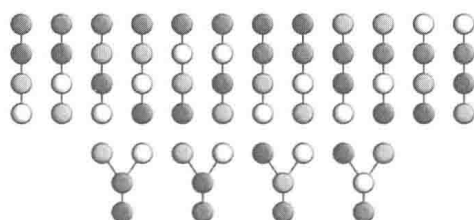
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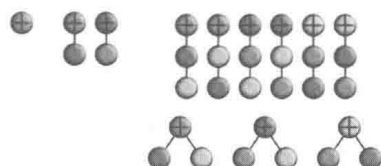
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GIBBS MEASURES ON CAYLEY TREES



To the memory of my brothers
Hamza Rozikov (1965-1992) and Jamshid Rozikov (1972-2004)

Preface

The purpose of this book is to present systematically all known mathematical results on Gibbs measures on Cayley trees (Bethe lattices).

The Gibbs measure is a probability measure, which has been an important object in many problems of probability theory and statistical mechanics. It is the measure associated with the Hamiltonian of a physical system (a model) and generalizes the notion of a canonical ensemble. More importantly, when the Hamiltonian can be written as a sum of parts, the Gibbs measure has the Markov property (a certain kind of statistical independence), thus leading to its widespread appearance in many problems outside of physics, such as biology, Hopfield networks, Markov networks, and Markov logic networks. Moreover, the Gibbs measure is the unique measure that maximizes the entropy for a given expected energy.

Using MathSciNet and a Google search one can see that there are about 600 published works devoted to several models defined on Cayley trees. But most of them are written for physicists audience with little regard for mathematical rigor. There are about 150 papers which contain mathematically rigorous results about Gibbs measures on Cayley trees. This book is mainly based on these mathematical papers.

There are a few books written from a mathematical perspective which are devoted to spin systems and Gibbs measures. Some of these books do not have any result about Gibbs measures on trees; others have a small part devoted to Gibbs measures on trees but were published a long time ago. To the best of my knowledge, there is no mathematical book devoted to (recently) obtained results about Gibbs measures on trees.

The method used for the description of Gibbs measures on Cayley trees is the method of Markov random field theory and recurrent equations of this theory, but the modern theory of Gibbs measures on trees uses new tools

such as group theory, information flows on trees, node-weighted random walks, contour methods on trees, non-linear analysis. This book discusses all the mentioned methods, which were developed recently.

In the bibliography of this book I have collected the possibly maximal list of references which are related to Gibbs measures on trees. Moreover, the last section of Chapter 11 contains a brief review and references on quantum models of statistical mechanics on trees, a review about p -adic Gibbs measures on Cayley trees, and a comparison between real-valued Gibbs measures and p -adic measures.

Thus, the book informs the reader about what has been (mathematically) done in the theory of Gibbs measures on trees and about where the corresponding results were published. The book is divided into 11 chapters; at the end of each chapter I give commentaries and references related to the chapter.

Chapter 1 is devoted to algebraic properties of the Cayley tree. We give a (non-commutative) group representation of a Cayley tree of order k , which is a free product of $k + 1$ cyclic groups of the second order. We construct several subgroups of the group. Moreover, the partition structures of the Cayley tree with respect to normal subgroups of the group are studied. These results are applied in the following chapters to describe periodic and weakly periodic Gibbs measures on the Cayley tree.

In the first section of Chapter 2 we give general definitions of configuration space, Hamiltonian and Gibbs measures. This chapter contains all known results about Gibbs measures of the Ising model on Cayley trees. Some of the results were obtained very recently. We show that the Ising model may have up to three translation-invariant Gibbs measures. It may have *only* periodic measures with period two, which are a ‘chess-board’ periodic. To describe more ‘rich’ set of Gibbs measures we introduce the notion of weakly periodic Gibbs measures and show that the Ising model has at least seven such measures (under some conditions on parameters). The extremality criterion of the disordered Gibbs measure is proved. Moreover we give two constructions of uncountable sets of non-periodic Gibbs measures. We show that under some conditions on the temperature, one can construct for each known Gibbs measure on a Cayley tree of order k_0 a new Gibbs measure on the Cayley tree of order k , $k > k_0$. Some explicit formulae of the free energies (and entropies) according to these Gibbs measures are presented.

Chapter 3 is devoted to two Ising type models with competing interactions on Cayley trees. One of them is known as Vannimenus’s model; the

second is a model with four competing interactions (external field, nearest neighbor, second neighbors and triples of neighbors) on the Cayley tree of order two. We show non-uniqueness of the Gibbs measure for some parameter values of the models. Our second result gives a complete description of periodic Gibbs measures for the models. We also construct uncountably many non-periodic extreme Gibbs measures.

In Chapter 4 we consider a process on a tree T in which information is transmitted from the root of the tree to all the nodes of the tree. Each node inherits information from its parent with some probability of error. The transmission process is assumed to have identical distribution on all the edges, and different edges of the tree are assumed to act independently. The basic question of this chapter is: Does the configuration obtained at level n of T typically contain significant information on the root variable? This problem arose independently in biology, information theory and statistical physics. For models of statistical physics on trees, the problem is related to extremality of the disordered Gibbs measure. In this chapter, we give results and challenges related to this problem. In the following chapters we shall apply the results to extremality conditions of Gibbs measures.

Chapter 5 contains results related to Gibbs measures of the q -state Potts model on Cayley trees. The description of such measures is reduced to solution of a vector-valued functional equation. We show that under some conditions on the parameters there exist $q + 1$ distinct translation-invariant Gibbs measures. We apply the results of Chapter 4 to find the extremality conditions of the disordered Gibbs measure. Moreover using the Bleher-Ganikhodjaev construction we show the existence of an uncountable set of non-translation-invariant Gibbs measures. Compared to the Ising model, one can see that many problems related to the Potts model are open. For example, periodic and weakly periodic Gibbs measures have not been studied yet.

In Chapter 6 we consider a nearest-neighbor SOS (solid-on-solid) model, with several spin values $0, 1, \dots, m$, $m \geq 2$, and zero external field, on a Cayley tree of order k . We mainly assume that $m = 2$ or $m = 3$ and study Gibbs measures.

For $m = 2$, in the anti-ferromagnetic case, we show that the translation-invariant Gibbs measure is unique for all temperatures. In the ferromagnetic case, for $m = 2$, the number of such measures varies with the temperature: this gives an interesting example of phase transition. Here we identify a critical inverse temperature, $\beta_{\text{cr}}^1 \in (0, \infty)$ such that for all $0 \leq \beta \leq \beta_{\text{cr}}^1$, there exists a unique translation-invariant measure and for all $\beta > \beta_{\text{cr}}^1$ there

are exactly three such measures. For $\beta > \beta_{\text{cr}}^1$ we also construct a continuum of distinct, non-translation-invariant Gibbs measures. A complete description of the set of periodic Gibbs measures for the SOS model on a Cayley tree is given. A complete description of periodic Gibbs measures means a characterization of such measures with respect to any given normal subgroup of finite index in the representation group of the tree. We show that (i) for an ferromagnetic SOS model, for any normal subgroup of finite index, each periodic Gibbs measure is in fact translation-invariant. Further, (ii) for an anti-ferromagnetic SOS model, for any normal subgroup of finite index, each periodic Gibbs measure is either translation-invariant or has period two (i.e., is a chess-board Gibbs measure). For $m = 3$ similar results are obtained. But the case $m \geq 4$ is not studied yet.

In Chapter 7 we consider models which have “hard constraints”. Such models are defined by the space $\text{Hom}(\Gamma^k, H)$ of homomorphisms from a Cayley tree Γ^k to a fixed finite constraint graph H . For any assignment λ of positive real activities to the nodes of H , there is at least one Gibbs measure on $\text{Hom}(\Gamma^k, H)$, but there may be more than one (phase transition). We mainly consider the case where graph H contains two vertices or three vertices. In such simple cases, a hard core model with two spin values and several hard core models with three spin values are discussed. In the last section of this chapter we give a model with two spin values (without hard constraints), but with interaction radius equal to two. We show that this model can be “transformed” to a nearest-neighbor interaction model with 8 spin values and with hard constraints on the Cayley tree. In each case we construct several kinds of Gibbs measures of these models.

Chapter 8 is devoted to a nearest-neighbor Potts model, with countable spin values $0, 1, \dots$, and non-zero external field, on a Cayley tree of order k . We study translation-invariant ‘splitting’ Gibbs measures, which depend on k and a probability measure ν (with $\nu(i) > 0$ on the set of all non-negative integer numbers $\Phi = \{0, 1, \dots\}$). This problem is reduced to the description of the solutions of an infinite system of equations. For any $k \geq 1$ and any fixed probability measure ν we show that the set of translation-invariant splitting Gibbs measures contains at most one point, independently on parameters of the Potts model with countable set of spin values on a Cayley tree. Also we give a description of the class of measures ν on Φ such that with respect to each element of this class the infinite system of equations has unique solution $\{a^i, i = 1, 2, \dots\}$, where $a \in (0, 1)$.

In Chapter 9 we present very recently obtained results for models with nearest-neighbor interactions and with the set $[0, 1]$ of spin values, on a

Cayley tree of order $k \geq 1$. We reduce the problem of describing the “splitting Gibbs measures” of the model to the description of the solutions of some non-linear integral equation. For $k = 1$ we show that the integral equation has a unique solution. In case $k \geq 2$ some models (with the set $[0, 1]$ of spin values) which have a unique splitting Gibbs measure are constructed. Also for the Potts model with uncountable set of spin values it is proven that there is unique splitting Gibbs measure. For arbitrary $k \geq 2$ we find a sufficient condition under which the integral equation has unique solution; hence under this condition the corresponding model has unique splitting Gibbs measure. Finally, we construct several models with the set $[0, 1]$ of spin values and show that each of the constructed model has at least two translational-invariant Gibbs measures.

Chapter 10 is devoted to recently developed contour methods on Cayley trees. In the first section of this chapter we consider a one-dimensional model with nearest-neighbor interactions $I_n, n \in \mathbb{Z}$, and spin values ± 1 . We show that under some conditions on parameters I_n the phase transition occurs for the model. We define a notion of “phase separation point” between two phases. We prove that the expectation value of the point is zero and its mean square fluctuation is bounded by a constant $C(\beta)$ which tends to $\frac{1}{4}$ if $\beta \rightarrow \infty$. Here $\beta = \frac{1}{T}$, $T > 0$ -temperature. In other sections of this chapter we consider a q -component model and the Ising model with competing two-step interactions on a Cayley tree of order $k \geq 1$. We constructively describe (periodic and weakly periodic) ground states and verify the Peierls condition for these models. We define the notion of a contour for the models on the Cayley tree. Using a contour argument we show the existence of several different Gibbs measures. This chapter also contains a general contour argument for a finite range lattice models on Cayley tree with two basic properties: the existence of only a finite number of ground states and with Peierls type condition. We define a general contour for such models on the Cayley tree. By a contour argument we show the existence of s different (where s is the number of ground states) Gibbs measures.

Chapter 11 contains several models not discussed in previous chapters. In Sections 1-9 we give the definitions of these models and describe some known results for each model. The last section contains a review of other models. This section is divided into three subsections: the first is devoted to classical (real-valued) models, the second subsection to quantum models, and the third one to models with p -adic values. Moreover, we give a brief description of the differences of behavior between classical (real) models and p -adic models on Cayley trees.

The bibliography is possibly a maximal list of mathematical references related to Gibbs measures on trees.

The book is based on materials collected by the author during several visits to Statistical Laboratory, DPMMS, University of Cambridge and Newton Institute (Cambridge, UK); International Center for Theoretical Physics (Trieste, Italy); Physical department of “La Sapienza” University (Rome, Italy); Institut des Hautes Etudes Scientifiques (IHES, Bures-sur-Yvette, France); Universite du Sud Toulon Var and Centre de Physique Theorique (Marseille, France); University Santiago de Compostela (Santiago de Compostela, Spain); Institute for Applied Mathematics of University of Bonn, (Bonn, Germany).

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Utkir A. Rozikov
Tashkent, Uzbekistan
March 2013

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