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Fourier Analysis and Hausdorff Dimension

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FOURIER ANALYSIS AND HAUSDORFF DIMENSION

During the past two decades there has been active interplay between geometric measure theory and Fourier analysis. This book describes part of that development, concentrating on the relationship between the Fourier transform and Hausdorff dimension.

The main topics concern applications of the Fourier transform to geometric problems involving Hausdorff dimension, such as Marstrand type projection theorems and Falconer's distance set problem, and the role of Hausdorff dimension in modern Fourier analysis, especially in Kakeya methods and Fourier restriction phenomena. The discussion includes both classical results and recent developments in the area. The author emphasizes partial results of important open problems, for example, Falconer's distance set conjecture, the Kakeya conjecture and the Fourier restriction conjecture. Essentially self-contained, this book is suitable for graduate students and researchers in mathematics.

Pertti Mattila is Professor of mathematics at the University of Helsinki and an expert in geometric measure theory. He has authored the book *Geometry of Sets and Measures in Euclidean Spaces* as well as more than 80 other scientific publications.

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To John Marstrand

Preface

This is a book on geometric measure theory and Fourier analysis. The main purpose is to present several topics where these areas meet including some of the very active recent interplay between them. We shall essentially restrict ourselves to questions involving the Fourier transform and Hausdorff dimension leaving many other aspects aside.

The book is intended for graduate students and researchers in mathematics. The prerequisites for reading it are basic real analysis and measure theory. Familiarity with Hausdorff measures and dimension and with Fourier analysis is certainly useful, but all that is needed will be presented in Chapters 2 and 3. Although most of the material has not appeared in book form, there is overlap with several earlier books. In particular, Mattila [1995] covers part of Chapters 4–7, Wolff [2003] of Chapters 14, 19, 20 and 22, and Stein [1993] of 14 and 19–21. Several other overlaps are mentioned in the text. The surveys Iosevich [2001], Łaba [2008], [2014], Mattila [2004], Mitsis [2003a] and Tao [2001], [2004] are closely related to the themes of the book.

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This book grew out of several graduate courses I have taught at the Department of Mathematics and Statistics of the University of Helsinki. I am grateful to the department for excellent facilities and for the students and post docs attending the courses for inspiration and for their comments and questions. For financial support I am also indebted to the Academy of Finland. Many mathematicians have been of great help in the preparation of this book. In particular, my special thanks are due to Vasilis Chousionis, who helped me in many aspects, and to Kenneth Falconer, who read large parts of the manuscript and made a great number of valuable comments on mathematics, style and language. Terence Tao's blogs and lecture notes have been very useful as well as Ana Vargas's master's thesis, which she kindly sent to me. For many comments and corrections I would like to thank Anthony Carbery, Marton Elekes, Burak Erdoğan, Risto Hovila, Tuomas Hytönen, Alex Iosevich, Tamás Keleti, Sangyuk Lee, José María Martell, Tuomas Orponen, Keith Rogers, Tuomas Sahlsten, Andreas Seeger, Pablo Shmerkin, Ville Suomala, Terence Tao, Ana Vargas and Laura Venieri. Finally I am much obliged to Jon Billam, Clare Dennison, Samuel Harrison and others at the Cambridge University Press for their help and for accepting the book for publication.

Contents

1

Introduction

The main object of this book is the interplay between geometric measure theory and Fourier analysis on $\mathbb{R}^n$. The emphasis will be more on the first in the sense that on several occasions we look for the best known results in geometric measure theory while our goals in Fourier analysis will usually be much more modest. We shall concentrate on those parts of Fourier analysis where Hausdorff dimension plays a role. Much more between geometric measure theory and Fourier analysis has been and is going on. Relations between singular integrals and rectifiability have been intensively studied for more than two decades; see the books David and Semmes [1993], Mattila [1995] and Tolsa [2014], the survey Volberg and Eiderman [2013], and Nazarov, Tolsa and Volberg [2014] for recent break-through results. Relations between harmonic measure, partial differential equations (involving a considerable amount of Fourier analysis) and rectifiability have recently been very actively investigated by many researchers; see, for example, Kenig and Toro [2003], Hofmann, Mitrea and Taylor [2010], Hofmann, Martell and Uriarte-Tuero [2014], and the references given therein.

In this book there are two main themes. Firstly, the Fourier transform is a powerful tool on geometric problems concerning Hausdorff dimension, and we shall give many applications. Secondly, some basic problems of modern Fourier analysis, in particular those concerning restriction, are related to geometric measure theoretic Kakeya (or Besicovitch) type problems. We shall discuss these in the last part of the book. We shall also consider various particular constructions of measures and the behaviour of their Fourier transforms.

The contents of this book can be divided into four parts.

PART I Preliminaries and some simpler applications of the Fourier transform.

PART II Specific constructions.

PART III Deeper applications of the Fourier transform.

PART IV Fourier restriction and Kakeya type problems.

Parts I and III are closely linked together. They are separated by Part II only because much of the material in Part III is rather demanding and Part II might be more easily digestible. In any case, the reader may jump over Part II without any problems. On the other hand, the sections of Part II are essentially independent of each other and only rely on Chapters 2 and 3. Part IV is nearly independent of the others. In addition to the basics of the Fourier transform, given in Chapter 3, the reader is advised to consult Chapter 11 on Besicovitch sets and Chapter 14 on oscillatory integrals before reading Part IV.

The applicability of the Fourier transform on Hausdorff dimension stems from the following three facts. First, the Hausdorff dimension of a Borel set $A \subset \mathbb{R}^n$, $\dim A$, can be determined by looking at the behaviour of Borel measures μ with compact support $\operatorname{spt}\mu \subset A$. We denote by $\mathcal{M}(A)$ the family of such measures μ with $0 < \mu(A) < \infty$. More precisely, by Frostman's lemma $\dim A$ is the supremum of the numbers s such that there exists $\mu \in \mathcal{M}(A)$ for which

$$\mu(B(x,r)) \leq r^s \quad \text{for } x \in \mathbb{R}^n, \quad r > 0. \tag{1.1}$$

This is easily transformed into an integral condition. Let

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y$$

be the s-energy of μ. Then $\dim A$ is the supremum of the numbers s such that there exists $\mu \in \mathcal{M}(A)$ for which

$$I_s(\mu) < \infty. \tag{1.2}$$

For a given μ the conditions (1.1) and (1.2) may not be equivalent, but they are closely related: (1.2) implies that the restriction of μ to a suitable set with positive μ measure satisfies (1.1), and (1.1) implies that μ satisfies (1.2) for any $s' < s$. Defining the Riesz kernel k_s, $k_s(x) = |x|^{-s}$, the s-energy of μ can be written as

$$I_s(\mu) = \int k_s * \mu\, d\mu.$$

For $0 < s < n$ the Fourier transform of k_s (in the sense of distributions) is $\widehat{k_s} = \gamma(n,s)k_{n-s}$ where $\gamma(n,s)$ is a positive constant. Thus we have by Parseval's theorem

$$I_s(\mu) = \int \widehat{k_s}|\widehat{\mu}|^2 = \gamma(n,s)\int |x|^{s-n}|\widehat{\mu}(x)|^2\, dx.$$