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Abstract

We study the complex geometry and coherent cohomology of nonclassical Mumford-Tate domains and their quotients by discrete groups. Our focus throughout is on the domains D which occur as open $G(\mathbb{R})$-orbits in the flag varieties for $G = SU(2,1)$ and $Sp(4)$, regarded as classifying spaces for Hodge structures of weight three. In the context provided by these basic examples, we formulate and illustrate the general method by which correspondence spaces $\mathcal{W}$ give rise to Penrose transforms between the cohomologies $H^q(D, L)$ of distinct such orbits with coefficients in homogeneous line bundles.

Turning to the quotients, representation theory allows us to define subspaces of $H^q(\Gamma\backslash D, L)$ called cuspidal automorphic cohomology, which via the Penrose transform are endowed in some cases with an arithmetic structure. We demonstrate that the arithmetic classes assume arithmetic values at CM points in $\mathcal{W}$, up to a transcendental factor that depends only on the CM type.

The representations related to this result are certain holomorphic discrete series representations of $G(\mathbb{R})$. We conclude with a discussion of how our framework may also be used to study the K-types and $\mathfrak{n}$-cohomology of (non-holomorphic) totally degenerate limits of discrete series, and to give an alternative treatment of the main result of Carayol (1998). These especially interesting connections will be further developed in future works.

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Contents

Introduction

The objective of this work is to study aspects of the *automorphic cohomology groups* $H^q(X, L_\mu)$ on quotients $X = \Gamma\backslash D$ by an arithmetic group Γ acting on a class of homogeneous complex manifolds $D = G_\mathbb{R}/T$. Here $G_\mathbb{R}$ is the connected real Lie group associated to a reductive $\mathbb{Q}$-algebraic group G, $T \subset G_\mathbb{R}$ is a compact maximal torus, and μ the weight associated to a character of T that gives a homogeneous holomorphic line bundle $L_\mu \to D$. These D's may be realized as Mumford-Tate domains that arise in Hodge theory, and in general we shall follow the terminology and notations from the monograph [**GGK1**].[1] We shall say that D is *classical* if it equivariantly fibres holomorphically or anti-holomorphically over an Hermitian symmetric domain; otherwise it is *non-classical*, and this is the case of primary interest in this paper.

In the non-classical case it has been known for a long time that, at least when Γ is co-compact in $G_\mathbb{R}$,

- $H^0(X, L_\mu) = 0$ for any non-trivial μ;
- when μ is sufficiently non-singular,[2] then

$$\begin{cases} H^q(X, L_\mu) = 0, & q \neq q(\mu+\rho) \\ H^{q(\mu+\rho)}(X, L_\mu) \neq 0 \end{cases}$$

where $q(\mu+\rho)$ will be defined in the text.

More precisely, for $k \geqq k_0$ and any non-singular μ

$$\dim H^{q(\mu+\rho)}(X, L_{k\mu}) = \operatorname{vol}(X) \cdot P_\mu(k)$$

where $P_\mu(k)$ is a Hilbert polynomial with leading term $C_\mu k^{\dim D}$ where $C_\mu > 0$ is independent of Γ. Thus, in the non-classical case there is a lot of automorphic cohomology and it does not occur in degree zero. In the classical case, the intensive study of the very rich geometric, Hodge theoretic, arithmetic and representation theoretic properties of automorphic forms has a long and venerable history and remains one of great current interest. In contrast, until recently in the non-classical case the geometric and arithmetic properties of automorphic cohomology have remained largely mysterious.[3]

For three reasons this situation has recently changed. One reason is the works [**Gi**], [**EGW**] that give a general method for interpreting analytic coherent cohomology on a complex manifold as holomorphic de Rham cohomology on an associated

[1] Cf. the Notations and Terminology section below.

[2] *Non-singular*, or *regular*, means that μ is not on the wall of a Weyl chamber; sufficiently non-singular means that μ is at a large enough distance $|\mu|$ from any wall.

[3] Important exceptions are the works Schmid [**Schm1**], Williams [**Wi1**], [**Wi2**], [**Wi3**], [**Wi4**], [**Wi5**], Wells and Wolf [**WW1**], [**WW2**], [**WW3**], and Wolf [**Wo**], some of which will be discussed below. These deal primarily with the representation-theoretic aspects of automorphic cohomology.

correspondence space $\mathcal{W}$.[4] In the two examples of this paper, of which a particular case of the first example is studied in [**Gi**] and [**EGW**], the associated space $\mathcal{W}$ will be seen to have a very rich geometric structure and the relevant holomorphic de Rham cohomology classes will turn out to have canonical representatives. The upshot is that in the situation of this work *automorphic cohomology classes can be "evaluated" at points of* $\mathcal{W}$.

A second reason is the very interesting work [**C1**], [**C2**], [**C3**] of Carayol.[5] In the case $G = \mathcal{U}(2,1)$, a case already considered in [**EGW**], Carayol uses the result in [**EGW**] applied to a diagram

$$\begin{array}{ccc} & \mathcal{W} & \\ {}^{\pi}\swarrow & & \searrow^{\pi'} \\ D & & D', \end{array} \tag{1}$$

where D is non-classical and D' is classical, to construct a Penrose-type transform

$$\mathcal{P} : H^0(D', L'_{\mu'}) \to H^1(D, L_\mu) \tag{2}$$

that relates the classical object $H^0(D', L'_{\mu'})$ to the non-classical object $H^1(D, L_\mu)$. He also shows that (1) and (2) exist on the quotients by Γ. For special choices of μ' the group $H^0(X', L'_{\mu'})$ is interpreted as *Picard automorphic forms*. The construction of $\mathcal{P}$ is via the commutative diagram (the notations are explained below)

$$\begin{array}{ccc} H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes \pi'^{-1} L'_{\mu'})\big) & \dashrightarrow & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi} \otimes \pi^{-1} L_{\mu})\big) \\ \wr\| & & \wr\| \\ H^0(D', L'_{\mu'}) & \xrightarrow{\quad\mathcal{P}\quad} & H^1(D, L_\mu) \end{array} \tag{3}$$

where the vertical isomorphisms are the above mentioned result in [**EGW**].[6]

In [**C1**], [**C2**] for $\mathcal{U}(2,1)$ the dotted arrow above is constructed by explicit "coordinate calculations", and one of the main purposes of this paper is to give a general, intrinsic geometric construction of such maps. More specifically, the dotted arrow will be seen to be multiplication by the restriction to $\mathcal{W} \subset \check{\mathcal{W}}$ of a canonical

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[4] For some time it has been known that in certain cases the cohomology group $H^{q_0}(D, L_\mu)$ may be realized as a subspace of the space of holomorphic sections of a holomorphic vector bundle over the cycle space $\mathcal{U}$ (cf. [**Schm2**], [**BE**] and [**FHW**]). Moreover this interpretation descends to quotients by Γ. The correspondence space will lie over the cycle space and in a number of ways appears to be a more fundamental object.

[5] The authors would like to thank Wushi Goldring for bringing this work to our attention and for some helpful discussions early in the preparation of this manuscript.

[6] Penrose transforms associated to the diagram

$$\begin{array}{ccc} & \mathcal{I} & \\ \swarrow & & \searrow \\ D & & \mathcal{U}, \end{array} \tag{4}$$

where $\mathcal{I} \subset D \times \mathcal{U}$ is the incidence variety by means of "pull-back and push-down" are classical and over the years have been the subject of extensive work; cf. [**Schm2**], [**BE**], [**EWZ**], [**FHW**] and the references cited therein. The "Penrose-type" transforms we will be discussing in this paper are somewhat different and have more the flavor of the maps on cohomology induced by a correspondence in classical algebraic geometry induced by a cycle on the product of the two varieties.

form

$$\omega \in \Gamma\big(\check{\mathcal{W}}, \Omega^1_\pi \otimes L(\mu, \mu')\big)$$

where $L(\mu, \mu') \to \check{\mathcal{W}}$ is a homogeneous line bundle over $\check{\mathcal{W}} \cong \mathbb{G}_\mathbb{C}/T_\mathbb{C}$ associated to the characters μ, μ' and to the relative positions of the Borel subgroups B and B' associated to D and D'. For the analogous diagram to (3) for a general $H^{q'}(D', L'_\mu)$ and $H^q(D, L_\mu)$ one has

$$\omega = \prod_\alpha \omega^\alpha$$

where the product is over the positive roots associated to B' which change sign when they are considered as roots of B, and ω^α is the dual under the Cartan-Killing form to the root vector X_α. The form ω is invariant under the group action and thus the construction (3) descends to quotients by Γ.[7] As we shall see in section IV.B, the bottom row of (3) is (in this quotient) replaced by a map between two Lie algebra cohomology groups induced by "multiplication by X_α."

A third reason is the recent classification **[GGK1]** of the reductive, $\mathbb{Q}$-algebraic groups that can be realized as a Mumford-Tate group of a polarized Hodge structure and the related classification of the associated Mumford-Tate domains D. Although these domains and their quotients $X = \Gamma\backslash D$ by arithmetic groups arose as target spaces for period mappings $P : S \to X$ where S is a quasi-projective algebraic variety, it has since emerged that their geometry and the cohomology of homogeneous vector bundles over them is of interest in its own right. For the line bundles $L_\mu \to D$ for which the restriction $L_\mu|_S$ is ample, corresponding in the classical case to automorphic forms but for which in the non-classical case $H^0(X, L_\mu) = 0$, the automorphic cohomology $H^{q(\mu)}(X, L_\mu)$, $q(\mu) > 0$, seemed a curiosity of no particular relevance to variations of Hodge structure. It was through the interesting geometry of Mumford-Tate domains that from a Hodge-theoretic perspective automorphic cohomology has emerged as an object of interest.

Just how interpreting automorphic cohomology as global holomorphic objects might be related to period mappings is a matter yet to be explored. More specifically, representation theory and complex geometry associate to $\Gamma\backslash D$ natural objects. Except in the classical case, pulling these objects back under $P : S \to \Gamma\backslash D$ generally gives zero. The constructions in this paper suggest a diagram

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \Gamma\backslash\mathcal{W} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \Gamma\backslash D, \end{array}$$

where $\widetilde{S}$ is a complex manifold with $\dim \widetilde{S} = 2 \dim S$ which has a mixed function-theoretic/algebro-geometric character, and where automorphic cohomology pulls back naturally to the above diagram. We hope to pursue this further in a future work.

[7]One will notice the similarity to the classical Borel-Weil-Bott theorem. This is of course not accidental and will be discussed below where the form ω will be seen to have a representation-theoretic interpretation; cf. the appendix to section III.D.

In this work we shall especially focus on examples.[8] One will be the *basic example*, essentially $\mathbb{P}^1$. Although the most elementary of cases, many of the main features of the general situation already arise here. The other two will be referred to as *example one* and *example two*. Example one will be the $\mathcal{U}(2,1)$ case; here we shall use the correspondence space from [**EGW**] and shall formulate intrinsically and reprove some of the results from [**C1**], [**C2**].[9] This suggests how the general case might go. In order to test the validity of this suggestion, we shall work out our second example of Sp(4). Here, a main step is to construct the correspondence space $\mathcal{W}$ for Sp(4), a construction that turns out to involve the concept of *Lagrange quadrilaterals*. In fact, from the two examples it is clear how the Penrose transform can be defined once one has the correspondence space $\mathcal{W}$ in hand. Although there is now a general construction of $\mathcal{W}$ and an analysis of its properties which will be given in a separate work, we have chosen to here focus on the two examples, in part because of the very beautiful geometry associated to each and in part because understanding them points to the way the general case should go.

The term *correspondence space* arises from the following consideration: The equivalence classes of homogeneous complex structures on $G_{\mathbb{R}}/T$ are indexed by the cosets in W/W_K where W is the Weyl group of $G_{\mathbb{C}}$ and W_K is the Weyl group of the maximal compact subgroup K of $G_{\mathbb{R}}$. We label these as D_w where $w \in W/W_K$. The correspondence space is then "universal" for maps $\mathcal{W} \to D_w$ and leads to diagrams

$$\begin{array}{ccc} & \mathcal{W} & \\ \swarrow & & \searrow \\ D_w & & D_{w'} \end{array}$$

giving rise to Penrose transforms between $H^q(D_w, L_\mu)$'s and $H^{q'}(D_{w'}, L_{\mu'})$'s. In particular, when one of the D_w is classical, which implies that $G_{\mathbb{R}}$ is of Hermitian type, this should lead to an identification of at least some non-classical automorphic cohomology with a classical object. This insight appears in [**C1**] and [**C2**] and is one hint that automorphic cohomology has a richer structure than previously thought.

We mention that as homogeneous complex manifolds for the complex Lie group $G_{\mathbb{C}}$ all of the domains D_w have a common *compact dual* $\check{D} \cong G_{\mathbb{C}}/B$ where B is a Borel subgroup. The D_w's are the $G_{\mathbb{R}}$-equivalence classes of the open $G_{\mathbb{R}}$-orbits in $\check{D}$. The correspondence space for the compact dual is $\check{\mathcal{W}} \cong G_{\mathbb{C}}/T_{\mathbb{C}}$, and $\mathcal{W} \subset \check{\mathcal{W}}$ turns out to be an open subset that is somewhat subtle to define.[10] In particular, it seems to be a somewhat new type of object; one that fibres over the cycle space $\mathcal{U}$, which has many of the characteristics of a bounded domain of holomorphy in $\mathbb{C}^N$, with affine algebraic varieties as fibres. It thus has a mixed complex function theoretic/algebro-geometric character. As mentioned above, this will be treated in the separate work [**GG**].

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[8] The main reason for this is that the examples suggest how the general case might go. For instance, based on this work the general definition and properties of the correspondence space is given in [**GG**]. A second reason is that the examples reveal what is to us a very nice geometry.

[9] As will be explained below, for a given choice of positive Weyl chamber $S\mathcal{U}(2,1)/T_S$ and $\mathcal{U}(2,1)/T$ are the same as complex manifolds but are not the same as *homogeneous* complex manifolds. For Hodge-theoretic purposes the latter is more important.

[10] $\check{\mathcal{W}}$ is sometimes referred to as the "enhanced flag variety" in the representation theory literature.

In section III.C we shall discuss our *basic example*, the case of $G = \mathrm{SL}_2$. Although it is certainly "elementary", looking at it from the point of view of the correspondence space and Penrose transform gives new perspective on this simplest of cases and already suggests some aspects of what turns out to be the general mechanism. Of note is the *canonical* identification of the group $H^1(\Omega^1_{\mathbb{P}^1}(k))$, $k \leqq 0$, with *global* holomorphic data; this is a harbinger of a fairly general situation.

The general mechanism was also suggested in part by formulating the calculations in the setting of *moving frames*. The elements of $G_{\mathbb{C}}$ may be identified as frames adopted to the geometry of the situation. The points of $G_{\mathbb{C}}/T_{\mathbb{C}}$ are the corresponding projective frames, and then the equations of the moving frame and their integrability conditions, the *Maurer-Cartan equations*, reveal the computational framework for the Penrose transform and suggest what the form ω above should be. Interpreting the formulas in terms of the roots of $G_{\mathbb{C}}$, $G_{\mathbb{R}}$ and the Borel subgroups B_w corresponding to D_w gives the suggested general prescription for ω that was mentioned above.

Associated to a domain is its *cycle space* $\mathcal{U}$, defined in this paper to be the set of $G_{\mathbb{C}}$-translates $Z = gZ_0$ of the maximal compact subvariety $Z_0 = K/T$ that remain in the open domain $D \subset \check{D}$. There is a comprehensive treatment of cycle spaces in [**FHW**], where they give a more general definition of the cycle space. It is known (loc. cit.) that in the non-classical case

$$\mathcal{U} \subset \check{\mathcal{U}} := G_{\mathbb{C}}/K_{\mathbb{C}}$$

where $\mathcal{U}$ is an open Stein domain in the affine algebraic variety $\check{\mathcal{U}}$.[11] There is the incidence diagram (4) but $\mathcal{I}$ is not Stein so the [**EGW**] method does not apply to this picture.[12] There is however a surjective map $\mathcal{W} \to \mathcal{U}$ where, in first approximation, the fibre lying over a point in $\mathcal{U}$ corresponding to $Z \cong K/T \subset D$ is the correspondence space $K_{\mathbb{C}}/T_{\mathbb{C}}$ for the homogeneous projective variety Z. For instance, in both the examples we shall consider we will have $Z \cong \mathbb{P}^1$ and the corresponding fibre will be $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \{\text{diagonal}\}$. In some sense one may think of $\mathcal{W}$ as a common Stein refinement of the cycle spaces $\mathcal{U}_w$ for all the domains D_w, a refinement to which the methods of [**EGW**] apply for all the D_w's.[13]

The cycle spaces will enter in an essential way in the proof of the injectivity of the Penrose transform for certain ranges of μ and μ'. Basically, the idea is that non-injectivity leads to an equation

$$F\omega = d_\pi G \tag{5}$$

where G is a holomorphic section of a line bundle $L(\mu, \mu') \to \mathcal{W}$. The equation (5) gives differential restrictions on G, and with these it is shown that G lives on a quotient variety $\mathcal{J}$ of $\mathcal{W}$ and that $\mathcal{J}$ is covered by the lifts $\widetilde{Z}$ of compact subvarieties $Z \subset D$. Then is is shown that for the range of weights μ of interest and for all such Z the restriction

$$G\big|_{\tilde{Z}} = 0\,.$$

[11] The substantive statements here are (i) that $K_{\mathbb{C}} = \{g \in G_{\mathbb{C}} : gZ_0 = Z_0\}$, (ii) that $\mathcal{U}$ is Stein, and (iii) $\mathcal{U}$ is Kobayashi hyperbolic.

[12] As noted above, it is this picture to which much of the classical literature on Penrose transforms, given by "pull-back and push-down", pertains.

[13] It is a non-trivial consequence of Matsuki duality that the $\mathcal{U}_w$ are all the *same* open set in $\check{\mathcal{U}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, but the compact subvarieties of D_w parametrized by $\mathcal{U}_w$ are different. This *universality property*, which is closely related to Matsuki duality, will play an important role in the subsequent definition and analysis of the properties of $\mathcal{W}$ given in [**GG**].

Since $\mathcal{J}$ is covered by the $\widetilde{Z}$'s, this implies that $G = 0$.[14]

As mentioned above one primary objective of this work is to formulate and illustrate the general method of Penrose-type transforms. A second objective is to use this method to define and derive results about one *arithmetic aspect* of automorphic cohomology. Informally stated, the result is that, *up to a transcendental factor that depends only on the CM type, arithmetic automorphic cohomology classes assume arithmetic values at CM points in* $\mathcal{W}$.

To explain this, a first observation is that the compact dual $\check{D}$ is a homogeneous, rational projective variety defined over a number field k. We shall say that a complex vector space V has an arithmetic structure in case there is a number field L with an embedding $L \hookrightarrow \mathbb{C}$ together with an L-vector space $V_L \subset V$ such that $V \cong \mathbb{C} \otimes_L V_L$. In all cases considered below the arithmetic structures will be natural in a sense that we hope will be clear from the context. For example, for any number field $L \supset k$, at an L-rational point of $\check{D}$ the fibres of $G_{\mathbb{C}}$-homogeneous vector bundles that are defined over k will have a natural arithmetic structure.

A second type of arithmetic structure arises when we realize D as a Mumford-Tate domain. There are then defined the set of complex multiplication, or CM, points $\varphi \in D$. The action of the CM field L_φ on the fibres at φ of the Hodge bundles then gives an arithmetic structure to these vector spaces. A basic result [**GGK1**] is that this arithmetic structure is comparable with the previously mentioned one for $\varphi \in \check{D}$, in the sense that there is a number field L' with $k \subset L'$, $L_\varphi \subset L'$ and such that when tensored with L' these two arithmetic structures coincide.

In our two examples, via the Penrose transform with the resulting natural isomorphism of the images of cuspidal automorphic forms[15]

$$H^1_o(X, L_\mu) \cong H^0_o(X', L'_{\mu'}) \tag{6}$$

an arithmetic structure on the RHS will induce one on the LHS. For the $\mathcal{U}(2,1)$ and Sp(4) examples and for a special choice of μ', the RHS consists of cuspidal *Picard*, respectively *Siegel modular forms*. If $H \xrightarrow{\pi} \Gamma\backslash H =: Y$ is the quotient of the Hermitian symmetric domain H to which X' maps, then the RHS is the cuspidal subspace of $H^0(Y, \omega_Y^{\otimes l/3})$.

It is known that Y has a *canonical model*, which is a projective variety defined over a number field $\mathbf{k}$ with homogeneous coordinate ring $\underset{l \geqq 0}{\oplus} H^0(Y, \omega_Y^{\otimes l/3})$ that is defined over $\mathbf{k}$.[16] The vector space $H^0(Y(\mathbf{k}), \omega_{Y(\mathbf{k})}^{\otimes l/3}) := H^0(Y, \omega_Y^{\otimes l/3})_{\mathbf{k}}$ are the *modular forms of weight l defined over* $\mathbf{k}$. For $y \in Y(\mathbf{k})$ a k-rational point, the fibre $\omega_{Y,y}$ of $\mathbb{C} \otimes_k \omega_{Y(\mathbf{k}),y}$ at y is defined over $\mathbf{k}$, and if $\psi \in H^0(Y, \omega_Y^{\otimes l/3})_{\mathbf{k}}$ then the value

$$\psi(y) \in \omega_{Y(\mathbf{k}),y}^{\otimes l/3}.$$

[14] This method will not apply to $\mathcal{W}$ itself since, being Stein, it contains no compact subvarieties. In order for it to apply we must quotient $\mathcal{W}$ on the right by parabolic subgroups $P_{\mathbb{C}}$ with $T_{\mathbb{C}} \subset P_{\mathbb{C}} \subset K_{\mathbb{C}}$. The roots of $P_{\mathbb{C}}$ are the ones that appear in the definition of the form ω mentioned above. Again, it is the examples discussed in this paper that suggest the general mechanism. These extensions are currently under investigation by a number of people.

[15] Cf. section IV.A for the notation and terminology. For $\Gamma \subset G$ co-compact, the subscript "o" may be dropped on both sides.

[16] For the examples considered in this work, the boundary components of Y in the Baily-Borel compactification will have codimension at least two, so finiteness conditions at the cusps are not necessary.

In our two examples, H will be realized as a Mumford-Tate domain and the notion of a CM point $h \in H$ is well-defined. As noted above, the fibres $\mathbb{F}^p_h$ of the Hodge bundles $\mathbb{F}^p \to H$ then have an arithmetic structure, so that there is a number field L and an L-vector space $\mathbb{F}^p_{h,L} \subset \mathbb{F}^p_h$ with $\mathbb{F}^p_h = \mathbb{C} \otimes_L \mathbb{F}^p_{h,L}$. The canonical bundle ω_H is constructed from the Hodge bundles, and therefore at a CM point h we have $\omega^{\otimes l/3}_{H,h,L} \subset \omega^{\otimes l/3}_{H,h}$. A classical result [**Shi**] is that *there is a fixed transcendental factor* $\Delta \in \mathbb{C}^*/\overline{\mathbb{Q}}^*$ *that depends only on the CM field associated to* h*, together with a choice of positive embeddings of the field, and a finite extension* $L' \supset L$ *such that for* $\psi \in H^0(Y, \omega^{\otimes l/3}_H)_{\mathbf{k}}$

$$\Delta^{-l}(\pi^*\psi)(h) \in \omega^{\otimes l/3}_{H,h;L'}.$$

In other words, in the sense just explained up to the factor Δ arithmetic automorphic forms assume arithmetic values at CM points.

Using the isomorphism (6), for suitable characters μ we may define an arithmetic structure on the cuspidal automorphic cohomology group $H^1_o(X, L_\mu)$. Using the [**EGW**] method we may then evaluate an automorphic cohomology class $\alpha \in H^1_o(X, L_\mu)$ in the fibres of bundles at $\mathbf{w} \in \mathcal{W}$ constructed from the Hodge bundles. At a CM point of $\mathcal{W}$ these vector spaces have arithmetic structures, and our result (IV.D.3) is that *at such a point* $\mathbf{w}$ *whose CM structure is compatible with that on its image* $\psi \in H$*, up to a fixed transcendental factor as above the value* $\alpha(\mathbf{w})$ *is arithmetic.* Moreover, these points are dense in the analytic topology.[17]

We remark that this work is one dealing primarily with the complex geometry and coherent cohomology of Mumford-Tate domains and their quotients by discrete groups. It is written from the perspective of the geometry of a class of interesting locally homogeneous complex manifolds that independently arise from Hodge theory and from representation theory. The deeper geometric and cohomological aspects of representation theory are treated here only superficially. We refer to the paper [**Schm3**] for an exposition of some of these aspects that will be used in the sequel to this paper [**GG**] where the general properties of correspondence spaces will be discussed. We also refer to the introduction to [**CK**] for a lucid overview of some related aspects of arithmetic automorphic representation theory and the role of TDLDS's in this theory.[18] One of our main points is that different coherent cohomology groups may be associated to the same representation, either finite

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[17] We remark that a geometrically more natural "evaluation" of automorphic cohomology classes in $H^1_0(X, L_\mu)$ would be to classes in $H^1(S, \mathcal{O}_S(L_\mu))$

$$S = \Gamma_S \cap \mathcal{H}$$

where $\mathcal{H} \subset D$ is an equivariantly embedded copy of the upper half plane $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ arising from an inclusion $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow G$ and $\Gamma_S = \Gamma \cap \mathrm{SL}_2(\mathbb{Q})$ is an arithmetic group. The notation "S" stands for *Shimura curve.* In general, equivariantly embedded Hermitian symmetric domains in non-classical Mumford-Tate domains have independently arisen from a number of perspectives ([**FL**], [**R**] and the above) and would seem to be objects worthy of further study. Some comments about this issue will be given in the forthcoming CBMS volume [**GGK2**]. In a related vein, [**KP**] using the framework in this paper has given a general setting and extension of results in [**C3**], which in particular provide another definition for arithmeticity of automorphic cohomology classes.

[18] As will be noted below, there is a to us striking similarity between the groups $H^d(\Gamma\backslash D, L_{-\rho})$, where $H^d(D, L_{-\rho})$ is the Harish-Chandra module associated to a TDLDS, and to special divisors of degree $g-1$ on an algebraic curve of genus g. In both cases, Euler characteristics are zero and deeper methods must be used to get at the geometry.

dimensional of $G_{\mathbb{C}}$ in the compact case or infinite dimensional of $G_{\mathbb{R}}$ in the non-compact case (including both $G_{\mathbb{R}}/T$'s and $\Gamma\backslash G_{\mathbb{R}}/T$'s), and that in some generality the connection between these different manifestations may be realized geometrically. Although we here have informally mentioned some of these general results, for the reasons stated above we have in this work focused on our examples.

It is the authors' pleasure to thank Sarah Warren for a marvelous job of converting an at best barely legible handwritten manuscript into mathematical text.

Outline

The following is an outline of the contents of the various sections of this paper.

We begin in section I.A with a general discussion of the homogeneous complex manifolds that will be considered in this work. Here, and later, we emphasize the distinction between equivalence of *homogenous* complex manifolds and *homogeneous* vector bundles over them, rather than just equivalence as complex manifolds and holomorphic vector bundles.

In section I.B we discuss our first example, which is the non-classical complex structure on $\mathcal{U}(2,1)/T := D$, realized as one of the three open orbits of $\mathcal{U}(2,1)$ acting on the homogeneous projective variety of flags $(0) \subset F_1 \subset F_2 \subset F_3 = \mathbb{C}^3$ where $\dim F_i = i$. Here $\mathbb{C}^3$ has the important additional structure of being the complexification of $\mathbb{F}^3$ where $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ is a quadratic imaginary number field. It is this additional structure that leads to the realization of D as a Mumford-Tate domain, thereby bringing Hodge theory into the story. The other two open $G_{\mathbb{R}}$-orbits D' and D'' are classical and may also be realized as Mumford-Tate domains, or what is more relevant to this work, the set of Hodge flags asssociated to Mumford-Tate domains consisting of polarized Hodge structures of weight one with additional structure.

All three of the above domains have three descriptions: geometric, group-theoretic and Hodge-theoretic. The interplay between these different perspectives is an important part of the exposition. Especially important is the book-keeping between the tautological, root and weight, and Hodge theoretic descriptions of the $\mathcal{U}(2,1)$-homogeneous line bundles over the domains, which is given in section II.B.

In our first example the three domains may be pictured as

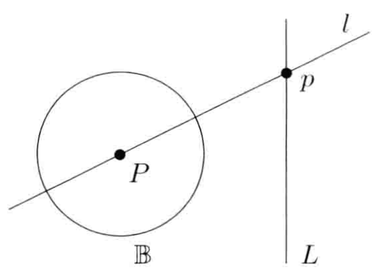


FIGURE 1

where $\mathbb{B}$ is the unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$ defined by the Hermitian form with matrix diag$(1, 1 - 1)$ and

$$\begin{cases} D &= \{(p, l)\} \\ D' &= \{(P, l)\} \\ D'' &= \{(p, L)\} \ . \end{cases}$$

All of thse domains are quotients of the correspondence variety $\mathcal{W}$ given by the set of configurations

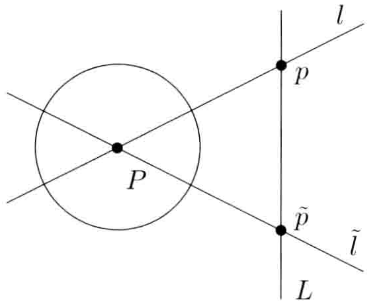
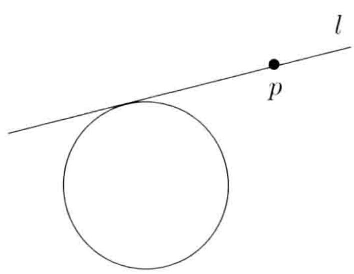
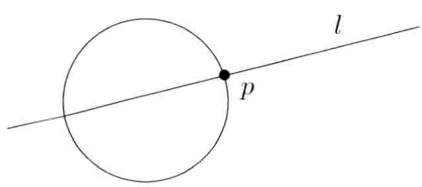


FIGURE 1a

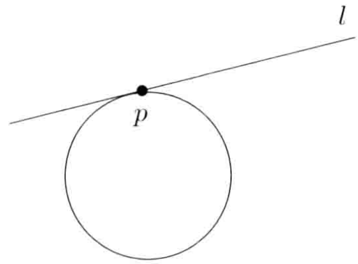
Correspondence varieties play a central role in the theory.[19]

Although it will not be needed for the present work, we mention that three other non-open orbits of the action of $G_{\mathbb{R}}$ on the flag manifold may be pictured as





[19]We observe that $\mathcal{W}$ may be described as the set of projective frames $(p, \tilde{p}, P)$ in the above figure; this is a general phenomenon. We note that $\mathcal{W}$ is an open domain in the set $\check{\mathcal{W}} \cong \mathbb{G}_{\mathbb{C}}/T_{\mathbb{C}}$, which is the *enhanced flag variety.* As noted in the introduction, motivated by the examples below is a general phenomenon as will be proved in [**GG**].



where the third is the unique closed orbit. These and their Mutsuki dual $K_{\mathbb{C}}$-orbits will play a central role in the sequel.[20]

The second example is the non-classical complex structure on $\mathrm{Sp}(4,\mathbb{R})/T := D$ where D is realized as the period domain of polarized Hodge structures of weight $n = 3$ and with all Hodge numbers $h^{p,q} = 1$, an example that arises in the study of the mirror-quintic Calabi-Yau varieties. In this case there are four inequivalent complex structures, of which two, the D mentioned above and one classical one D', will play important roles in this work. Again, the three descriptions — geometric, group-theoretic and Hodge-theoretic — and their interplay are important in this work.

The geometric description of the domains D and D' will be given by configurations

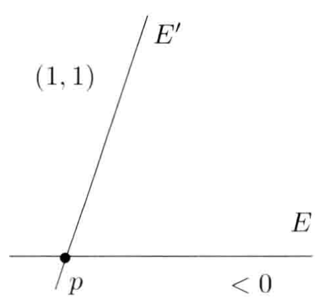


FIGURE 2

where

$$D = \{(p, E')\}$$
$$D' = \{(p, E)\}\,.$$

Here we are given a non-degenerate alternating form Q and conjugation σ on a four dimensional complex vector space V with $\mathbb{P}^3 = \mathbb{P}V$. The form Q defines the Hermitian form $H(u,v) = iQ(u,\sigma v)$. In the above figure, E and E' are Lagrange lines and the $(1,1)$ and < 0 denote the signature of H restricted to them. The

[20] In the above example the pictured orbits in ∂D will be seen to have Hodge-theoretic significance in terms of the Kato-Usui theory [**KU**] of limiting mixed Hodge structures and in representation theory where the TDLDS is constructed by parabolic inductions from the unique closed $G_{\mathbb{R}}$-orbit given by the third figure above (cf. [**KP**] and [**GGK2**]).