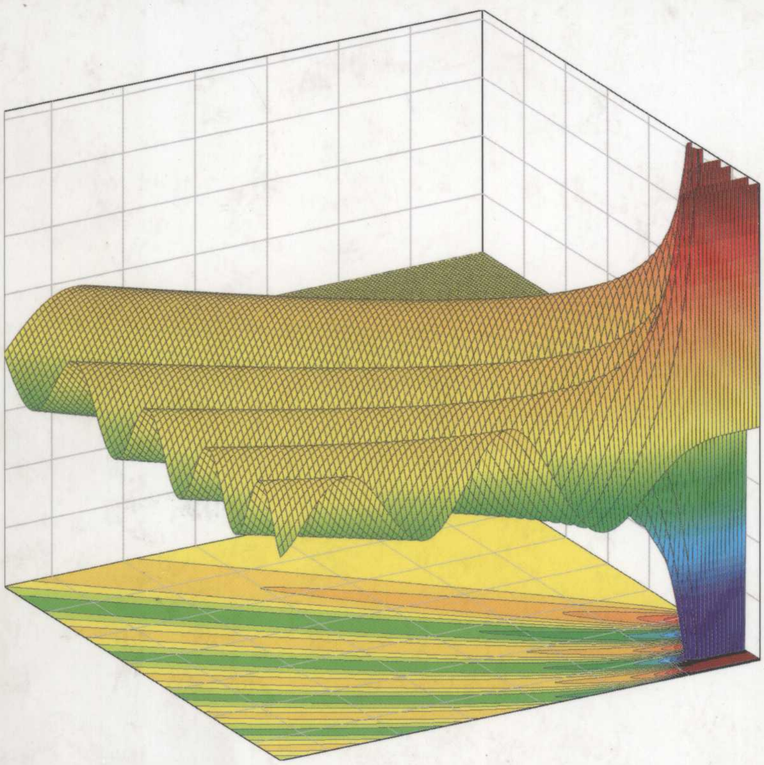
KEITH OLDHAM, JAN MYLAND,
& JEROME SPANIER

AN ATLAS OF FUNCTIONS

SECOND EDITION

WITH *EQUATOR*, THE ATLAS FUNCTION CALCULATOR



CD-ROM

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OVER 300 DIAGRAMS IN COLOR

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PREFACE

The majority of scientists, mathematicians and engineers must consult reference books containing information on a variety of functions. This is because all but the most mundane quantitative work involves relationships that are best described by mathematical functions of various complexities. Of course, the need will depend on the user, but most will require information about the general behavior of the function in question and its mathematical properties, as well as its numerical values at a number of arguments.

The first edition of *An Atlas of Functions*, the product of collaboration between a mathematician and a chemist, appeared during an era when the programmable calculator was the workhorse for the numerical evaluation of functions. That role has now been taken over by the omnipresent computer, and therefore the second edition delegates this duty to *Equator, the Atlas function calculator*. This is a software program that, as well as carrying out other tasks, will calculate values of over 200 functions, mostly with 15 digit precision. There are numerous other improvements throughout this new edition but the objective remains the same: to provide the reader, regardless of his or her discipline, with a succinct compendium of information about all the common mathematical functions in use today.

While relying on *Equator* to generate exact numerical values, the *Atlas of Functions* describes each function graphically and gives ready access to the most important definitions, properties, expansions and other formulas that characterize it, and its relationship to other functions. As well, the utility of the *Atlas* is enhanced by the inclusion of sections that briefly discuss important topics related to specific functions; the new edition has many more such sections. The book is organized into 64 chapters, each of which is devoted to one function or to a family of closely related functions; these appear roughly in order of increasing complexity. A standard format has been adopted for each chapter to minimize the effort needed to locate a sought item of information. A description of how the chapters are sectioned is included as Chapter 0. Several appendices, a bibliography and two comprehensive indices complete the volume.

In addition to the traditional book format, an electronic version of *An Atlas of Functions* has also been produced and may even be available through your library or other information center. The chapter content of the paper and electronic editions is identical, but *Equator, the Atlas function calculator* is not included in the latter. The *Equator* CD is included with the print version of the book, and a full description of the software will be found in Appendix C. Because *Equator* is such a useful adjunct to the *Atlas*, stand-alone copies of the *Equator* CD have been made widely available, through booksellers and elsewhere, primarily for the benefit of users of the electronic version of the *Atlas*.

Though the formulas in the *Atlas* and the routines in *Equator* have been rigorously checked, errors doubtless remain. If you encounter an obscurity or suspect a mistake in either the *Atlas* or *Equator*, please let us know at koldham@trentu.ca, jmyland@trentu.ca or jspanier@uci.edu. An *Errata* of known errors and revisions will be found on the publisher's website; please access www.springer.com/978-0-387-48806-6 and follow the links. This will be updated as and if new errors are detected or clarifications are found to be needed. Use of the *Atlas of Functions* or *Equator, the Atlas function calculator* is at your own risk. The authors and the publisher disclaim liability for any direct or consequential damage resulting from use of the *Atlas* or *Equator*.

It is a pleasure to express our gratitude to Michelle Johnston, Sten Engblom, and Trevor Mace-Brickman for their help in the creation of the *Atlas* and *Equator*. The frank comments of several reviewers who inspected an early version of the manuscript have also been of great value. We give sincere thanks to *Springer*, and particularly to Ann Kostant and Oona Schmid, for their commitment to the lengthy task of carrying the concept of *An Atlas of Functions* through to reality with thoroughness, enthusiasm, skill, and even some humor. Their forbearance in dealing with the authors is particularly appreciated.

We hope you will enjoy using *An Atlas of Functions* and *Equator*, and that they will prove helpful in your work or studies.

January 2008

Keith B. Oldham
Jan C. Myland
Jerome Spanier

CONTENTS

Every chapter has sections devoted to: notation, behavior, definitions, special cases, intrarelationships, expansions, particular values, numerical values, limits and approximations, operations of the calculus, complex argument, generalizations, and cognate functions. In addition, each chapter has the special features itemized below its title.

CHAPTER 0

GENERAL CONSIDERATIONS

Functions are operators that accept numbers as input and generate other numbers as output. The simplest kinds receive one number, usually named the *argument*, and produce another number, called the *value* of the function

0:0:1
$$\textit{argument} \xrightarrow{\text{input}} \boxed{\text{function}} \xrightarrow{\text{output}} \textit{value}$$

Functions that need only one argument to trigger an output are *univariate* functions. *Bivariate* functions require two input numbers; one of these two *variables* generally retains the name "argument", whereas the second variable goes by another name, such as *order, index*, *modulus*, *coefficient*, *degree*, or *parameter*. There are also *trivariate*, *quadrivariate*, and even *multivariate* functions. Likewise certain functions may give multiple outputs; the number of such output values may be finite or infinite. Such *multivalued* functions are often conventionally restricted to deliver a single output, a so-called *principal value*, and this is the standard used in the *Atlas*.

In this chapter are collected some considerations that relate to all, or most, functions. The general organization of the *Atlas* is also explained here. Thus, this could be a good starting point for the reader. However, the intent of the authors is that the information in the *Atlas* be immediately available to an unprepared reader. There are no special codes that must be mastered in order to use the book, and the only conventions that we adopt are those that are customary in scientific writing.

Each chapter in the *Atlas* is devoted to a single function or to a small number of intimately related functions. The preamble to the chapter exposes any such relationships and introduces special features of the subject function.

0:1 NOTATION

The *nomenclature* and *symbolism* of mathematical functions are bedeviled by ambiguities and inconsistencies. Several names may attach to a single function, and one symbol may be used to denote several functions. In the first section of each chapter the reader is alerted to such sources of possible confusion.

For the sake of standardization, we have imposed certain conventions relating to symbols. Though this has meant sometimes adopting unfamiliar notation, each function in the *Atlas* has its own unique symbol, as listed in the *Symbol Index*. We have eschewed boldface and similar typographical niceties for symbolizing functions on the grounds that they are difficult to reproduce by pencil on paper. We reserve the use of italics to represent numbers (such as function arguments, x (or y); constants, c; and coefficients, a_1, a_2, a_3, $\cdots$, a_n) and avoid their use in symbolizing functions. When a variable is necessarily an integer it is represented by n (or m), rather than x, and

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often appears subscript following the function's symbol, instead of within parentheses. Variables that are often integers, but are not necessarily so, are represented by ν (or μ). Arguments that are frequently interpreted as angles may be represented by θ (or ϕ). Occasionally, as in Chapter 64, some symbol other than x is used unexpectedly to represent the variable, x being reserved to serve as the argument of a more general function. For the same reason, we may avoid using "argument" as the name of the variable in such cases. Notice that a roman f symbol is used to represent an arbitrary function, or as a stand-in for a group of specified function symbols, but the italic f is employed to signify the numerical value of $\mathrm{f}(x)$ corresponding to a specific x. Thus the axes of cartesian graphs may be labeled x and f, rather than the customary x,y found in texts dealing with analytical geometry.

We generally avoid the use of primes to represent differentiation but, where they have become part of the established symbolism, as in Section 52:7 and Chapter 56, this usage is followed. Elsewhere a notation such as f' merely connotes "another f".

In Sections 46:14 and 46:15, the z symbol serves as a cartesian coordinate. Elsewhere the symbol z is reserved to denote a *complex variable* equal to $x+iy$. All other variables are implicitly real, unless otherwise noted.

The names of most functions end with the word "function" (the error function, the Hurwitz function), but three other terminal words are commonly encountered. Some univariate functions that accept, exclusively or primarily, integer arguments are called "numbers" (Fibonacci numbers, lambda numbers); a few bivariate functions are named similarly (Stirling numbers). Functions defined as power series of finite length generally take the name "polynomials" (Chebyshev polynomials, exponential polynomial). The word "integral" often ends the name of functions that are defined as integrals (hyperbolic cosine integral, Dawson's integral). Yet other functions have unique names that don't fit into the general pattern (dilogarithm, binomial coefficient). Adjectives relate some functions to a parent function (associated Laguerre function, incomplete gamma function, auxiliary Fresnel integral). An index of function symbols will be found following the appendices, while the names of all our functions are included in the Subject Index that concludes the *Atlas*.

0:2 BEHAVIOR

This section reveals how the function changes in value as its variables change, thereby exposing the general "shape" of the function. This information is conveyed by a verbal description, supplemented by graphics.

There are several styles of figure that amplify the text in Sections 2 and elsewhere. The first is a cartesian line-graph of the function's values f plotted straightforwardly versus its argument x. Frequently there are several lines, representing different functions, plotted in different colors on the same graph. The second style of figure, particularly suitable for bivariate functions, is a three-dimensional orthographic view of the surface, showing how the function varies in magnitude as each of the variables changes over a restricted range. Pairs of such graphics are often used in Sections 11 to represent the real and imaginary parts of complex-valued functions. Not infrequently, complex-valued functions are inherently multivalued and, to convert such a function into a single-valued counterpart, it is necessary to "cut" the surface; such a cut appears as a grey "cliff" on the three-dimensional figure. Our three-dimensional graphics are colored, but the color plays only a subsidiary role. Some bivariate functions have *discontinuities,* such as a sudden change in value from $+\infty$ to $-\infty$, and when there are several of these, three-dimensional figures become so confused as to be unhelpful. In these circumstances, we sometimes resort to a third style of graphic, that we call a *projection graph.* In this perspective representation, a three-dimensional image is combined with a two-dimensional display, the axes of which correspond to two variables, with color being used to indicate the magnitude of the function at each point in the rectangular space. With trivariate and quadrivariate functions, graphical representation ceases to be useful and the behaviors of such functions may be described in this *Atlas* without the aid of graphics. You will encounter figures of other kinds, too, each designed to be helpful in the local context.

Some functions are defined for all values of their variable(s), from $-\infty$ to $+\infty$. For other functions there are restrictions, such as $-1 < x \le 1$ or $n = 1,2,3,\cdots$, on those values that specify the *domain* of each variable and thereby the *range* of the function. Likewise, the function itself may be restricted in range and may be real valued, complex valued, or each of these in different domains. Such considerations are discussed in the second section of each chapter. In this context, the idea of *quadrants* is sometimes useful. Borrowed from the graphical representation of the function f, this concept facilitates separate discussion of the properties of a function f according to the signs of x and f, as in the table.

Quadrant	x	f
first	+	+
second	−	+
third	−	−
fourth	+	−

0:3 DEFINITIONS

Often there are several formulas relating a function to its variable(s), although they may not all apply over the entire range of the function. These various interrelationships are listed in the third section of each chapter under the heading "Definitions" even though, from a strictly logical viewpoint, some might prefer to select one as the unique definition and cite the others as "equivalences" or "representations".

Several types of definition are encountered in Sections 3. For example, a function may be defined:

(a) by an equation that explicitly defines the function in terms of simpler functions and algebraic operations;

(b) by a formula relating the function to its variable(s) through a finite or an infinite number of arithmetic or algebraic operations;

(c) as the derivative or indefinite integral of a simpler function;

(d) as an *integral transform* of the form

0:3:1 $$f(x) = \int_{t_0}^{t_1} g(x,t)\,dt$$

where g is a function having one more variable than f, t_0 and t_1 being specified limits of integration;

(e) through a *generating function*, G(x,t), that defines a family of functions $f_j(x)$ via the expansion

0:3:2 $$G(x,t) = \sum_j f_j(x)\, g_j(t)$$

where $g_j(t)$ is a simpler set of functions such as t^j;

(f) as the *inverse* of another function F(x) so that the implicit equation

0:3:3 $$F\big(f(x)\big) = x$$

is used to define f(x) [this is graphically equivalent to reflecting the function F(x) in a straight line of unity slope through the origin, as elaborated in Section 14:15];

(g) as a special case or a limiting case of a more general function;

(h) parametrically through a pair of equations that separately relate the function f(x) and its argument x to a third variable;

(i) implicitly via a differential equation [Section 24:14], the solution (or one of the solutions) of which is the subject function;

(j) through concepts borrowed from geometry or trigonometry; and

(k) by *synthesis*, the application of a sequence of algebraic and differintegration [Section 12:14] operations applied to a simpler function, as described in Section 43:14.

0:4 SPECIAL CASES

If the function reduces to a simpler function for special values of the variable(s), this is noted in the fourth section of each chapter.

0:5 INTRARELATIONSHIPS

An equation linking the two functions f(x) and g(x) is an *interrelationship* between them. In contrast, one speaks of an *intrarelationship* if there is a formula that provides a link between instances of a single function at two or more values of one of its variables, for example, between f(x_1) and f(x_2). In this *Atlas* intrarelationships will be found in Section 5 of each chapter, interrelationships mainly in Sections 3 and 12.

An equation expressing the relationship between f($-x$) and f(x) is called a *reflection formula*. Less commonly there exist reflection formulas relating f($a-x$) to f($a+x$) for nonzero values of a.

A second class of intrarelationships are *translation formulas*; these relate f($x+a$) to f(x). The most general translation formula, in which a is free to vary continuously, becomes an *argument-addition formula* that relates f($x+y$) to f(x) and f(y). However, many translation formulas are restricted to special values of a such as $a = 1$ or $a = n\pi$; the relationships are then known as *recurrence relations* or *recursion formulas*. Such relationships are common in bivariate functions; a recursion formula then normally relates f(ν, x) to f($\nu-1$, x) or to both f($\nu-1$, x) and f($\nu-2$, x). A very general argument-addition formula is provided by the *Taylor expansion* (Brook Taylor, English mathematician and physicist, 1685 – 1731):

0:5:1
$$\mathrm{f}(y \pm x) = \mathrm{f}(y) \pm x\frac{\mathrm{df}}{\mathrm{d}x}(y) + \frac{x^2}{2!}\frac{\mathrm{d}^2\mathrm{f}}{\mathrm{d}x^2}(y) \pm \frac{x^3}{3!}\frac{\mathrm{d}^3\mathrm{f}}{\mathrm{d}x^3}(y) + \cdots$$

Expressions for the remainder after this series is truncated to a finite number of terms are provided by Abramowitz and Stegun [Section 3.6], and by Jeffrey [page 79].

A third class of intrarelationships are *argument-multiplication formulas* that relate f(nx) to f(x). More rarely there exist *function-multiplication formulas* or *function-addition formulas* that provide expressions for f(x) f(y) and f(x) + f(y), respectively.

Yet other intrarelationships are those provided by finite and infinite series. With bivariate and multivariate functions there may be a great number of such formulas, and functions other than f may be involved.

0:6 EXPANSIONS

The sixth section of each chapter is devoted to ways in which the function(s) may be expressed as a finite or infinite array of terms. Such arrays are normally series, products, or continued fractions.

Notation such as

0:6:1
$$\mathrm{f}(x) = \sum_{j=0}^{\infty} \mathrm{g}_j(x)$$

is used to represent a *convergent infinite series*, where g is a function of j and x. Unless otherwise qualified, 0:6:1 implies that, for values of x in a specified range, the numerical value of the finite sum

0:6:2
$$\mathrm{g}_0(x) + \mathrm{g}_1(x) + \mathrm{g}_2(x) + \cdots + \mathrm{g}_j(x) + \cdots + \mathrm{g}_J(x)$$

can be brought indefinitely close to f(x) by choosing J to be a large enough integer.

Frequently encountered are convergent series whose successive terms, for sufficiently large j, decrease in magnitude and alternate in sign. We shall loosely call such series *alternating series*. A valuable property of such alternating series enables the remainder after a finite number of terms are summed to be estimated in terms of the first omitted term. When $\sum(-)^j g_j(x)$ is used to represent an alternating series, this result:

0:6:3 $$\left|\sum_{j=0}^{\infty}(-)^j g_j(x) - \sum_{j=0}^{J}(-)^j g_j(x)\right| < \left|g_{J+1}(x)\right|$$

plays an important role in the design of many algorithms.

In contrast to 0:6:1, the symbolism

0:6:4 $$f(x) \sim \sum_j g_j(x) \qquad j = 0,1,2,\cdots,J \qquad x \to \infty$$

which is reserved for *asymptotic series*, implies that, for every J, the numerical value of 0:6:2 can be brought indefinitely close to f(x) by making x, not J, sufficiently large. It is this restriction on the magnitude of x that makes an asymptotic expansion, though of great utility in many applications, rather treacherous for the incautious user [see Hardy].

If the function $g_j(x)$ in 0:6:1 or 0:6:4 can be written as the product $c_j x^{\alpha+\beta j}$, where c_j is independent of x while α and β are constants, then the expansions 0:6:1 and 0:6:4 are called *Frobenius series* (Ferdinand Georg Frobenius, Prussian mathematician, 1849–1917). In the case of an asymptotic series, β is often negative. When $\alpha = 0$ and $\beta = 1$, the name *power series* [Section 10:13] is used if the series is infinite, or polynomial [Chapter 17] if it is finite.

The *infinite product* notation

0:6:5 $$f(x) = \prod_{j=0}^{\infty} g_j(x)$$

implies that the numerical value of the finite product

0:6:6 $$g_0(x)g_1(x)g_2(x)\cdots g_J(x)$$

approaches f(x) indefinitely closely as J takes larger and larger integer values.

The notation

0:6:7 $$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\,\frac{\alpha_4}{\beta_4 + \cdots}$$

is a standard abbreviation for the *continued fraction*

0:6:8 $$\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cfrac{\alpha_3}{\beta_3 + \cfrac{\alpha_4}{\beta_4 + \cdots}}}}$$

in which each α_j and β_j may denote constants or variables. A continued fraction may serve as a representation of some function f(x). Continued fractions may be infinite, as denoted in 0:6:7, or finite (or "terminated"):

0:6:9 $$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\cdots\frac{\alpha_{J-1}}{\beta_{J-1} +}\,\frac{\alpha_J}{\beta_J}$$

though the former are most common in this *Atlas*. Of great utility in working with continued fractions is the equivalence

0:6:10 $$\beta_0 + \frac{\alpha_1}{\beta_1 +}\,\frac{\alpha_2}{\beta_2 +}\,\frac{\alpha_3}{\beta_3 +}\cdots\frac{\alpha_n}{\beta_n} = \beta_0 + \frac{\gamma_1\alpha_1}{\gamma_1\beta_1 +}\,\frac{\gamma_1\gamma_2\alpha_2}{\gamma_2\beta_2 +}\,\frac{\gamma_2\gamma_3\alpha_3}{\gamma_3\beta_3 +}\cdots\frac{\gamma_{n-1}\gamma_n\alpha_n}{\gamma_n\beta_n}$$

In what we shall call the "standard" form of a continued fraction, the variable x appears only in the numerators, that is, only in the α portions of 0:6:9. However, other forms exist in which x is part of β, or both α and β, as in the left-hand side of the identity

0:6:11 $$\frac{1}{\gamma_0-}\,\frac{\gamma_0 x}{\gamma_1+x-}\,\frac{\gamma_1 x}{\gamma_2+x-}\,\frac{\gamma_2 x}{\gamma_3+x-}\cdots\frac{\gamma_{n-1}x}{\gamma_n+x}=\frac{1}{\gamma_0}+\frac{x}{\gamma_0\gamma_1}+\frac{x^2}{\gamma_0\gamma_1\gamma_2}+\cdots+\frac{x^n}{\gamma_0\gamma_1\gamma_2\cdots\gamma_n}$$

which demonstrates the interchangeability of continued fractions and polynomials. Lozenge diagrams [Section 10:14] can facilitate such as interchange.

0:7 PARTICULAR VALUES

If certain values of the variable(s) of a function generate noteworthy function values, these are cited in the seventh section of each chapter, often as a table. The entry "$-\infty|+\infty$" in such a table, or elsewhere in the *Atlas*, or in the output of *Equator*, implies that the function has a discontinuity and, moreover, that at an argument slightly more negative than the argument in question, the function's value is large and negative; whereas, at an argument slightly more positive, the function is large and positive. Entries such as "$+\infty|+\infty$" similarly provide information about the sign of the function's value on either side of a discontinuity.

In Section 7 of many chapters we include information about those arguments that lead to inflections, minima, maxima, and particularly zeros of the subject function f(x). The term *extremum* is used to mean either a local maximum or a local minimum.

An *inflection* of a function occurs at a value of its argument at which the second derivative of the function is zero; that is:

0:7:1 $$\frac{d^2f}{dx^2}(x_i)=0 \qquad f(x_i)=\text{inflection of } f(x)$$

A local *minimum* and a local *maximum* of a function are characterized respectively by

0:7:2 $$\frac{df}{dx}(x_m)=0,\quad \frac{d^2f}{dx^2}(x_m)>0 \qquad f(x_m)=\text{minimum of } f(x)$$

and

0:7:3 $$\frac{df}{dx}(x_M)=0,\quad \frac{d^2f}{dx^2}(x_M)<0 \qquad f(x_M)=\text{maximum of } f(x)$$

A *zero* of a function is a value of its argument at which the function vanishes; that is, if

0:7:4 $$f(r)=0 \quad\text{then}\quad r=\text{a zero of } f(x)$$

Equivalent to the phrase "a zero of f(x)" is "a *root* of the equation f(x) = 0." A *double zero* or a *double root* occurs at a value r of the argument such that

0:7:5 $$f(r)=\frac{df}{dx}(r)=0 \qquad r=\text{a double zero of } f(x)$$

The concept extends to multiple zeros or repeated roots; thus, if

0:7:6 $$f(r)=\frac{df}{dx}(r)=\cdots=\frac{d^n f}{dx^n}(r)=0 \qquad r=\text{a zero of } f(x) \text{ of multiplicity } n+1$$

A value r of x that satisfies 0:7:4 but not 0:7:5 corresponds to a *simple zero* or a *simple root*. The graphical significance of a root, a maximum, a minimum, and an inflection, is evident from Figure 0-1.

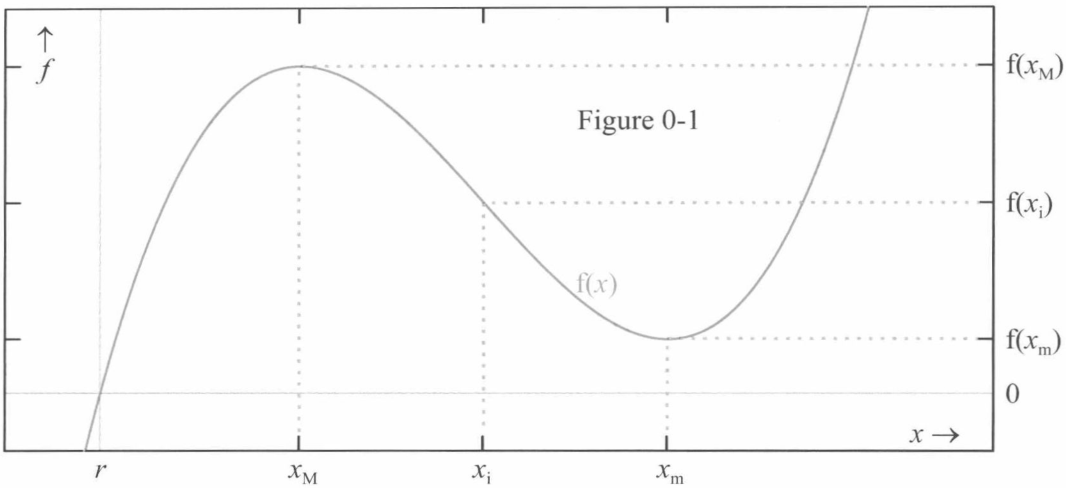

Figure 0-1

0:8 NUMERICAL VALUES

Very few tables of numerical values are found in the *Atlas* because the compact disk that accompanies the print edition of this book is designed to obviate that need. As described in Appendix C, the disk provides access to *Equator, the Atlas function calculator*. As well as carrying out certain other tasks, *Equator* is able to calculate the numerical values of over two hundred mathematical functions.

The computational methods employed by *Equator* are so diverse and interconnected that it is impractical to present the code or algorithms. Nevertheless, the mathematical basis of the calculations is explained in Section 8 of the relevant chapter. Generally, Section 8 reveals the domain(s) of the variable(s) within which *Equator* operates but the user must appreciate that not all of the variable combinations that lie within these domains will necessarily generate a function value. For a variety of reasons, including overflow or underflow during a computation, or an inadequacy of residual precision at some stage of the calculation, no numerical output may be possible. Our goal is that any answer generated be significant to the number of digits cited in the output. See Appendix C for further information about *Equator*.

0:9 LIMITS AND APPROXIMATIONS

Often, as the argument or another variable of the function approaches a particular number, such as zero or infinity, its behavior comes to approximate that of some simpler function as a limit. Such instances are noted in Sections 9, either verbally or with the help of an equation. The symbol ≈ indicates approximate equality.

Limiting behaviors can often serve as valuable approximations, and these may be presented in Sections 9. Whenever some approximation, not necessarily arising from a limit, is particularly noteworthy, it is reported in this section, too. The symbol → is used to indicate approach.

0:10 OPERATIONS OF THE CALCULUS

Some of the most important properties of functions are associated with their behavior when subjected to the various operations of the calculus. Accordingly, the tenth is often one of the largest sections of a chapter.