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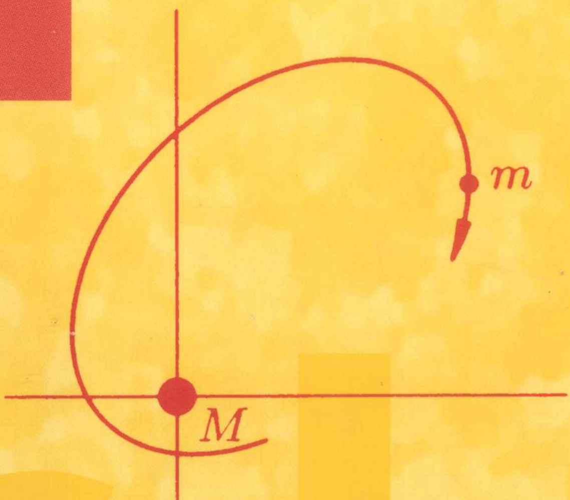
Vladimir A. Zorich

Mathematical Analysis I



$$\int_a^b f(x)\,dx = \mathcal{F}(x)\big|_a^b$$

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Mathematical Analysis I

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Prefaces

Preface to the English Edition

An entire generation of mathematicians has grown up during the time between the appearance of the first edition of this textbook and the publication of the fourth edition, a translation of which is before you. The book is familiar to many people, who either attended the lectures on which it is based or studied out of it, and who now teach others in universities all over the world. I am glad that it has become accessible to English-speaking readers.

This textbook consists of two parts. It is aimed primarily at university students and teachers specializing in mathematics and natural sciences, and at all those who wish to see both the rigorous mathematical theory and examples of its effective use in the solution of real problems of natural science.

Note that Archimedes, Newton, Leibniz, Euler, Gauss, Poincaré, who are held in particularly high esteem by us, mathematicians, were more than mere mathematicians. They were scientists, natural philosophers. In mathematics resolving of important specific questions and development of an abstract general theory are processes as inseparable as inhaling and exhaling. Upsetting this balance leads to problems that sometimes become significant both in mathematical education and in science in general.

The textbook exposes classical analysis as it is today, as an integral part of the unified Mathematics, in its interrelations with other modern mathematical courses such as algebra, differential geometry, differential equations, complex and functional analysis.

Rigor of discussion is combined with the development of the habit of working with real problems from natural sciences. The course exhibits the power of concepts and methods of modern mathematics in exploring specific problems. Various examples and numerous carefully chosen problems, including applied ones, form a considerable part of the textbook. Most of the fundamental mathematical notions and results are introduced and discussed along with information, concerning their history, modern state and creators. In accordance with the orientation toward natural sciences, special attention is paid to informal exploration of the essence and roots of the basic concepts and theorems of calculus, and to the demonstration of numerous, sometimes fundamental, applications of the theory.

For instance, the reader will encounter here the Galilean and Lorentz transforms, the formula for rocket motion and the work of nuclear reactor, Euler's theorem on homogeneous functions and the dimensional analysis of physical quantities, the Legendre transform and Hamiltonian equations of classical mechanics, elements of hydrodynamics and the Carnot's theorem from thermodynamics, Maxwell's equations, the Dirac delta-function, distributions and the fundamental solutions, convolution and mathematical models of linear devices, Fourier series and the formula for discrete coding of a continuous signal, the Fourier transform and the Heisenberg uncertainty principle, differential forms, de Rham cohomology and potential fields, the theory of extrema and the optimization of a specific technological process, numerical methods and processing the data of a biological experiment, the asymptotics of the important special functions, and many other subjects.

Within each major topic the exposition is, as a rule, inductive, sometimes proceeding from the statement of a problem and suggestive heuristic considerations concerning its solution, toward fundamental concepts and formalisms. Detailed at first, the exposition becomes more and more compressed as the course progresses. Beginning *ab ovo* the book leads to the most up-to-date state of the subject.

Note also that, at the end of each of the volumes, one can find the list of the main theoretical topics together with the corresponding simple, but nonstandard problems (taken from the midterm exams), which are intended to enable the reader both determine his or her degree of mastery of the material and to apply it creatively in concrete situations.

More complete information on the book and some recommendations for its use in teaching can be found below in the prefaces to the first and second Russian editions.

Moscow, 2003 *V. Zorich*

Preface to the Fourth Russian Edition

The time elapsed since the publication of the third edition has been too short for me to receive very many new comments from readers. Nevertheless, some errors have been corrected and some local alterations of the text have been made in the fourth edition.

Moscow, 2002 *V. Zorich*

Preface to the Third Russian edition

This first part of the book is being published after the more advanced Part 2 of the course, which was issued earlier by the same publishing house. For the sake of consistency and continuity, the format of the text follows that adopted in Part 2. The figures have been redrawn. All the misprints that were noticed have been corrected, several exercises have been added, and the list of further readings has been enlarged. More complete information on the subject matter of the book and certain characteristics of the course as a whole are given below in the preface to the first edition.

Moscow, 2001 *V. Zorich*

Preface to the Second Russian Edition

In this second edition of the book, along with an attempt to remove the misprints that occurred in the first edition,[1] certain alterations in the exposition have been made (mainly in connection with the proofs of individual theorems), and some new problems have been added, of an informal nature as a rule.

The preface to the first edition of this course of analysis (see below) contains a general description of the course. The basic principles and the aim of the exposition are also indicated there. Here I would like to make a few remarks of a practical nature connected with the use of this book in the classroom.

Usually both the student and the teacher make use of a text, each for his own purposes.

At the beginning, both of them want most of all a book that contains, along with the necessary theory, as wide a variety of substantial examples

[1] No need to worry: in place of the misprints that were corrected in the plates of the first edition (which were not preserved), one may be sure that a host of new misprints will appear, which so enliven, as Euler believed, the reading of a mathematical text.

of its applications as possible, and, in addition, explanations, historical and scientific commentary, and descriptions of interconnections and perspectives for further development. But when preparing for an examination, the student mainly hopes to see the material that will be on the examination. The teacher likewise, when preparing a course, selects only the material that can and must be covered in the time alloted for the course.

In this connection, it should be kept in mind that the text of the present book is noticeably more extensive than the lectures on which it is based. What caused this difference? First of all, the lectures have been supplemented by essentially an entire problem book, made up not so much of exercises as substantive problems of science or mathematics proper having a connection with the corresponding parts of the theory and in some cases significantly extending them. Second, the book naturally contains a much larger set of examples illustrating the theory in action than one can incorporate in lectures. Third and finally, a number of chapters, sections, or subsections were consciously written as a supplement to the traditional material. This is explained in the sections "On the introduction" and "On the supplementary material" in the preface to the first edition.

I would also like to recall that in the preface to the first edition I tried to warn both the student and the beginning teacher against an excessively long study of the introductory formal chapters. Such a study would noticeably delay the analysis proper and cause a great shift in emphasis.

To show what in fact can be retained of these formal introductory chapters in a realistic lecture course, and to explain in condensed form the syllabus for such a course as a whole while pointing out possible variants depending on the student audience, at the end of the book I give a list of problems from the midterm exam, along with some recent examination topics for the first two semesters, to which this first part of the book relates. From this list the professional will of course discern the order of exposition, the degree of development of the basic concepts and methods, and the occasional invocation of material from the second part of the textbook when the topic under consideration is already accessible for the audience in a more general form.[2]

In conclusion I would like to thank colleagues and students, both known and unknown to me, for reviews and constructive remarks on the first edition of the course. It was particularly interesting for me to read the reviews of A. N. Kolmogorov and V. I. Arnol'd. Very different in size, form, and style, these two have, on the professional level, so many inspiring things in common.

Moscow, 1997 *V. Zorich*

[2] Some of the transcripts of the corresponding lectures have been published and I give formal reference to the booklets published using them, although I understand that they are now available only with difficulty. (The lectures were given and published for limited circulation in the Mathematical College of the Independent University of Moscow and in the Department of Mechanics and Mathematics of Moscow State University.)

From the Preface to the First Russian Edition

The creation of the foundations of the differential and integral calculus by Newton and Leibniz three centuries ago appears even by modern standards to be one of the greatest events in the history of science in general and mathematics in particular.

Mathematical analysis (in the broad sense of the word) and algebra have intertwined to form the root system on which the ramified tree of modern mathematics is supported and through which it makes its vital contact with the nonmathematical sphere. It is for this reason that the foundations of analysis are included as a necessary element of even modest descriptions of so-called higher mathematics; and it is probably for that reason that so many books aimed at different groups of readers are devoted to the exposition of the fundamentals of analysis.

This book has been aimed primarily at mathematicians desiring (as is proper) to obtain thorough proofs of the fundamental theorems, but who are at the same time interested in the life of these theorems outside of mathematics itself.

The characteristics of the present course connected with these circumstances reduce basically to the following:

In the exposition. Within each major topic the exposition is as a rule inductive, sometimes proceeding from the statement of a problem and suggestive heuristic considerations toward its solution to fundamental concepts and formalisms.

Detailed at first, the exposition becomes more and more compressed as the course progresses.

An emphasis is placed on the efficient machinery of smooth analysis. In the exposition of the theory I have tried (to the extent of my knowledge) to point out the most essential methods and facts and avoid the temptation of a minor strengthening of a theorem at the price of a major complication of its proof.

The exposition is geometric throughout wherever this seemed worthwhile in order to reveal the essence of the matter.

The main text is supplemented with a rather large collection of examples, and nearly every section ends with a set of problems that I hope will significantly complement even the theoretical part of the main text. Following the wonderful precedent of Pólya and Szegő, I have often tried to present a beautiful mathematical result or an important application as a series of problems accessible to the reader.

The arrangement of the material was dictated not only by the architecture of mathematics in the sense of Bourbaki, but also by the position of analysis as a component of a unified mathematical or, one should rather say, natural-science/mathematical education.

In content. This course is being published in two books (Part 1 and Part 2).

The present Part 1 contains the differential and integral calculus of functions of one variable and the differential calculus of functions of several variables.

In differential calculus we emphasize the role of the differential as a linear standard for describing the local behavior of the variation of a variable. In addition to numerous examples of the use of differential calculus to study functional relations (monotonicity, extrema) we exhibit the role of the language of analysis in writing simple differential equations – mathematical models of real-world phenomena and the substantive problems connected with them.

We study a number of such problems (for example, the motion of a body of variable mass, a nuclear reactor, atmospheric pressure, motion in a resisting medium) whose solution leads to important elementary functions. Full use is made of the language of complex variables; in particular, Euler's formula is derived and the unity of the fundamental elementary functions is shown.

The integral calculus has consciously been explained as far as possible using intuitive material in the framework of the Riemann integral. For the majority of applications, this is completely adequate.[3] Various applications of the integral are pointed out, including those that lead to an improper integral (for example, the work involved in escaping from a gravitational field, and the escape velocity for the Earth's gravitational field) or to elliptic functions (motion in a gravitational field in the presence of constraints, pendulum motion.)

The differential calculus of functions of several variables is very geometric. In this topic, for example, one studies such important and useful consequences of the implicit function theorem as curvilinear coordinates and local reduction to canonical form for smooth mappings (the rank theorem) and functions (Morse's lemma), and also the theory of extrema with constraint.

Results from the theory of continuous functions and differential calculus are summarized and explained in a general invariant form in two chapters that link up naturally with the differential calculus of real-valued functions of several variables. These two chapters open the second part of the course. The second book, in which we also discuss the integral calculus of functions of several variables up to the general Newton–Leibniz–Stokes formula thus acquires a certain unity.

We shall give more complete information on the second book in its preface. At this point we add only that, in addition to the material already mentioned, it contains information on series of functions (power series and Fourier series included), on integrals depending on a parameter (including the fundamental solution, convolution, and the Fourier transform), and also on asymptotic expansions (which are usually absent or insufficiently presented in textbooks).

We now discuss a few particular problems.

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[3] The "stronger" integrals, as is well known, require fussier set-theoretic considerations, outside the mainstream of the textbook, while adding hardly anything to the effective machinery of analysis, mastery of which should be the first priority.

On the introduction. I have not written an introductory survey of the subject, since the majority of beginning students already have a preliminary idea of differential and integral calculus and their applications from high school, and I could hardly claim to write an even more introductory survey. Instead, in the first two chapters I bring the former high-school student's understanding of sets, functions, the use of logical symbolism, and the theory of a real number to a certain mathematical completeness.

This material belongs to the formal foundations of analysis and is aimed primarily at the mathematics major, who may at some time wish to trace the logical structure of the basic concepts and principles used in classical analysis. Mathematical analysis proper begins in the third chapter, so that the reader who wishes to get effective machinery in his hands as quickly as possible and see its applications can in general begin a first reading with Chapter 3, turning to the earlier pages whenever something seems nonobvious or raises a question which hopefully I also have thought of and answered in the early chapters.

On the division of material. The material of the two books is divided into chapters numbered continuously. The sections are numbered within each chapter separately; subsections of a section are numbered only within that section. Theorems, propositions, lemmas, definitions, and examples are written in italics for greater logical clarity, and numbered for convenience within each section.

On the supplementary material. Several chapters of the book are written as a natural extension of classical analysis. These are, on the one hand, Chapters 1 and 2 mentioned above, which are devoted to its formal mathematical foundations, and on the other hand, Chapters 9, 10, and 15 of the second part, which give the modern view of the theory of continuity, differential and integral calculus, and finally Chapter 19, which is devoted to certain effective asymptotic methods of analysis.

The question as to which part of the material of these chapters should be included in a lecture course depends on the audience and can be decided by the lecturer, but certain fundamental concepts introduced here are usually present in any exposition of the subject to mathematicians.

In conclusion, I would like to thank those whose friendly and competent professional aid has been valuable and useful to me during the work on this book.

The proposed course was quite detailed, and in many of its aspects it was coordinated with subsequent modern university mathematics courses – such as, for example, differential equations, differential geometry, the theory of functions of a complex variable, and functional analysis. In this regard my contacts and discussions with V. I. Arnol'd and the especially numerous ones with S. P. Novikov during our joint work with the so-called "experimental student group in natural-science/mathematical education" in the Department of Mathematics at MSU, were very useful to me.

I received much advice from N. V. Efimov, chair of the Section of Mathematical Analysis in the Department of Mechanics and Mathematics at Moscow State University.

I am also grateful to colleagues in the department and the section for remarks on the mimeographed edition of my lectures.

Student transcripts of my recent lectures which were made available to me were valuable during the work on this book, and I am grateful to their owners.

I am deeply grateful to the official reviewers L. D. Kudryavtsev, V. P. Petrenko, and S. B. Stechkin for constructive comments, most of which were taken into account in the book now offered to the reader.

Moscow, 1980 *V. Zorich*

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