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Preface

This book is the second of a two volume collection of proceedings of the fourth summer School in Numerical Analysis, which was held at Lancaster University from 15th July to 3rd August, 1990. The meeting was sponsored by the Science and Engineering Research Council of Great Britain, and was attended by approximately 120 participants from 14 countries. Many of the participants delivered seminars at the meeting, but this volume contains only the contributions of three of the nine invited main speakers.

Each week of the School was devoted to a specific topic within numerical analysis, the topics for 1990 being

- nonlinear partial differential equations
- dynamical systems
- multivariate approximation.

Each main speaker was asked to give a series of five one hour lectures, with the exposition pitched at such a level that reseachers and graduate students could both gain something useful from the courses—a demanding brief, but one which was met very well by all speakers. The three chapters which form this volume are an account of the happenings in week three. The proceedings of weeks one and two are contained in volume I, which bears the title *Advances in Numerical Anaylsis I: Nonlinear Partial Differential Equations and Dynamical Systems*, and is also published by Oxford University Press.

The selection of topics for the School reflects current trends in research in numerical analysis. Since the early 1970s, the approximation theory community has slowly been coming to terms with the particular difficulties associated with problems in two or more dimensions. While the univariate theory was extremely well developed by 1980, such topics as multivariate interpolation, multivariate splines, and the interpolation of scattered data did not possess a unified theory. The decade of the eighties saw a rapid and dramatic change in this situation, and 1990 seemed a particularly appropriate time to review these substantial advances. However, a particularly exciting new development in the form of wavelet theory was also beginning in the latter part of the decade, and by the time the summer school took place, the theory was sufficiently well developed to make a preliminary look at this important area worthwhile. Thus the first chapter of this volume is devoted to wavelet theory in the univariate setting. Before this book appears, wavelet theory will be well and truly established in a second wave of development—that of multivariate wavelets. The second chapter deals with a particular aspect of computer-aided geometric design. There has long been a fruitful interplay between approximation theory and this rapidly expanding and important application of the subject, and the whole area is now far too large to permit satisfactory treatment in one series of 5 lectures. Consequently, Chapter 2 concentrates on the theory of subdivision algorithms. This is a particularly pleasing topic, by virtue of its mathematical elegance as well as its applicability. The final chapter is a truly massive contribution to the new area of radial basis functions. The idea of using radial basis functions for interpolating scattered data was in existence as early as 1970, but the theoretical understandings of this area did not begin to develop until the 1980s. In that decade, a considerable number of able workers turned their attention to the important theoretical underpinnings of the area, and the final chapter gives a good account of much of this activity.

In compiling these proceedings, I have striven for as uniform a presentation as possible, without encroaching on the rights of the individual author to present things as she/he sees fit. A decimal system of notation is adopted for sectioning, with the label 2.3 representing section 3 of Chapter 2. The equation numbering is done within sections, so that equation 2.3 is the third numbered equation in section 2 of the current chapter. Theorems, Lemmas, etc., are numbered sequentially within sections. The postal address of the author is listed at the end of the chapter.

Finally, a few acknowledgements are in order. The organisation of the School was carried out jointly with my good friend and colleague John Gilbert. I thank him for his help and support. Sue Hubbard assisted in the day to day running of the School and did some preliminary 'TEXing' of manuscripts. The Science and Engineering Research Council once again provided generous support. Their contribution covered all the organisational and running costs of the meeting as well as the expenses of the speakers, and the accommodation and subsistence expenses of up to twenty participants each week. The organisation of each weekly scientific programme was undertaken by people who were entitled 'local experts'. These were Professor Charles Elliott, Professor Alastair Spence and Dr John Gregory. There was some puzzlement on the part of participants as to the choice of words in this title, but I am unable to remember whether the questions centred around the use of the word 'expert' or the word 'local'! It is a pleasure to acknowledge their contributions to the overall success of the meeting.

Will Light
Leicester, 1991

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1

Wavelets and Spline Interpolation

Charles K. Chui

Abstract. This is an introductory tutorial of wavelets from a spline theorist's point of view. The concept of integral wavelet transform (IWT) for time-frequency localization is discussed, and a comparison of compactly supported orthonormal wavelets, spline wavelets, and biorthogonal wavelets is given. When spline wavelets are being used, interpolatory splines are readily available for orthogonal wavelet decompositions, yielding the IWT at dyadic locations without performing any integration. In addition, since the dual spline wavelets are used as wavelet window functions, linear-phase filtering is accomplished.

1.1 Introduction

In approximation theory, functions in a Hilbert space such as the space $L^2 = L^2(-\infty, \infty)$ are projected onto a much less complicated subspace V_N to facilitate the solution of an optimization problem or a differential equation, or simply to provide a nice representation of the functions for the purpose of modeling, numerical computation, storage, transmittance, etc. If the projection is a best approximation, then the usual questions of existence, characterization, uniqueness, computational effectiveness, etc., are asked. In addition, if the functions under consideration are restricted to a certain smoothness class, then the problems of the order of approximation and characterization of this smoothness class in terms of this approximation order are studied. In doing so, a nested sequence of approximating subspaces $V_N \supset V_{N-1} \supset \cdots$ is always assumed. However, the question as to how the consecutive differences of the error functions behave is seldom addressed. Note that if the subspaces happen to have "local bases", then this sequence of consecutive differences of error functions give very important information on the approximants, and consequently of the original functions. For instance, information such as locations of different levels of irregularities, singularities, and even chaotic behaviors of the functions can be easily detected from this study. The objective of this writing is to investigate in some details this important point of view.

Let $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$ be a nested sequence of closed subspaces of L^2 such that their union is dense in L^2 and their intersection is the zero function. For each $k \in \mathbb{Z}$, since V_k is subspace of V_{k+1}, we may consider its orthogonal complement W_k. In other words, W_k is also a subspace of V_{k+1}, it is orthogonal to V_k, and the sum of W_k and V_k is all of V_{k+1}. We will use the notation

$$V_{k+1} = W_k \oplus V_k, \tag{1.1}$$

where orthogonality is always assumed in the direct sum. Under the assumptions we made on $\{V_k\}$, it is clear that

$$W_m \perp W_n \quad \text{for} \quad m \neq n, \quad \text{and} \tag{1.2}$$

$$L^2 = \bigoplus_{k\in\mathbb{Z}} W_k. \tag{1.3}$$

Furthermore, for any given $\varepsilon > 0$, and any $f \in L^2$, there exist functions $f_N \in V_N$ and $f_{N-M} \in V_{N-M}$, $M > 0$ such that

$$\begin{cases} \|f - f_N\| < \varepsilon; \\ f_N = g_{N-1} \oplus \cdots \oplus g_{N-M} \oplus f_{N-M}, \quad \text{with } g_k \in W_k; \\ \|f_{N-M}\| < \varepsilon. \end{cases} \tag{1.4}$$

Here, the notation of orthogonal sums is again used. That is, every function in L^2 has a "satisfactory" representation as a finite sum of functions from the mutually orthogonal subspaces W_k with the "remainder" f_{N-M} being usually treated as the "blurred" version of f. If each W_k has a local basis, then the localization of each g_k yields the localization of f at each "orthogonal level", and in particular, gives an analysis of the sequence of consecutive errors:

$$(f_{k+1} - f) - (f_k - f) = f_{k+1} - f_k = g_k.$$

To discuss how this process localizes the different levels of irregularities of f, it is best to study the notion of integral wavelet transform (IWT) of f, which, however, is defined only by means of "dilation" and "translation". For this reason, we will only restrict our attention to a nested sequence of subspaces generated by dilation and translation of a single basis function, although our discussion is sometimes valid in the general situation. In addition, for simplicity, with the exception of a short discussion in Section 1.8, our presentation will be restricted to functions of one variable, as it will certainly be clear to the reader that most of the ideas, techniques, and results presented here easily carry over to the multivariate setting, at least by using tensor products.

1.2 Integral Wavelet Transforms

It is well know that in many applications, the behavior of a function $f(t)$ can only be observed from its Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt. \tag{2.1}$$

For instance, if t and ω represent the "time" and "frequency" variables, respectively, then a "signal" $f(t)$ in the time-domain is detected by measuring its "spectrum" $\hat{f}(\omega)$ in the frequency domain. The spectral representation of a signal f by using the Fourier transform $\hat{f}$ in (2.1), however, is usually not effective, since the integral is determined by the *global information* of the signal. In particular, time-evolution of frequencies is not reflected by this representation. This defect of the Fourier transform was already observed by D. Gabor, who, in his 1946 paper [28], applied a Gaussian function to "window" the Fourier transform. Since the Fourier transform of a Gaussian function is again a Gaussian function, an application of the Plancherel Identity shows that the spectrum $\hat{f}$ is also windowed. In other words, both f and $\hat{f}$ are localized. In general, a function $g \in L^2$ is called a *window function*, provided that it has finite standard deviation, namely:

$$\Delta_g := \left[\frac{\int_{-\infty}^{\infty} (t-t_0)^2 (g(t))^2\,dt}{\int_{-\infty}^{\infty} (g(t))^2\,dt} \right]^{\frac{1}{2}} < \infty, \tag{2.2}$$

where t_0 is the "center" of the window function defined by

$$t_0 := \frac{\int_{-\infty}^{\infty} t(g(t))^2\,dt}{\int_{-\infty}^{\infty} (g(t))^2\,dt}. \tag{2.3}$$

Hence, with center at t_0, the quantity $2\Delta_g$ can be used to represent the width of this window function. In other words, we will view the window provided by the function g as an interval $[t_0 - \Delta_g, t_0 + \Delta_g]$. If both g and its Fourier transform $\hat{g}$ are window functions with centers at t_0, ω_0 and widths $2\Delta_g, 2\Delta_{\hat{g}}$, respectively, then the Plancherel Identity yields:

$$\int_{-\infty}^{\infty} f(\tau)\overline{g(\tau + t_0 - t)}e^{-i\tau(\omega-\omega_0)}d\tau$$
$$= e^{-i(\omega-\omega_0)(t-t_0)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta)\overline{\hat{g}(\eta + \omega_0 - \omega)}e^{i\eta(t-t_0)}d\eta. \tag{2.4}$$

That is, when a signal f in the Fourier transform is windowed at t, its spectrum $\hat{f}$ in the inverse-Fourier transform is also windowed at ω. In fact, the time-frequency window in the (t,ω)-plane for (2.4) is given by

$$[t - \Delta_g, t + \Delta_g] \times [\omega - \Delta_{\hat{g}}, \omega + \Delta_g]. \tag{2.5}$$

We remark that by the *Heisenberg Uncertainty Principle*, the window in (2.5) cannot be designed to have area smaller than $1/\pi$, since

$$\Delta_g \Delta_{\hat{g}} \geq \frac{1}{4\pi} \tag{2.6}$$

for any g, and equality in (2.6) holds if and only if g is a Gaussian function $g(t) = e^{-at^2}$ for any $a > 0$. A very important observation here is that the size of the window in (2.5) is independent of the location of its center (t, ω), so that the *same* window is used to localize both high and low frequencies. Consequently, a short-lived high-frequency signal cannot be accurately detected if Δ_g is not small enough, while a small value of Δ_g certainly results in inefficiency in the study of low-frequency behavior.

The integral wavelet transform (IWT) to be discussed next has the capability of zooming in on short-lived high-frequency phenomena and zooming out in low-frequency environment. In other words, although the area of the time-frequency window is constant as governed by the Heisenberg Uncertainty Principle, it automatically narrows when its center is moved upward in the upper (t, ω)-plane and widens at low frequencies ω. This is exactly what is needed to study irregularities and to locate singularities. As introduced by Grossmann and Morlet [30], for any window function ψ, the integral wavelet transform of any $f \in L^2$ is defined by

$$(W_\psi f)(b, a) := \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt. \tag{2.7}$$

Note that this definition only depends on dilation and translation of the window function ψ, and hence it is somewhat easier to compute $W_\psi f$ than the window Fourier transform. The importance is that if an appropriate ψ is chosen as the window function in (2.7), then we will see in Section 1.3 that there is no need to compute $W_\psi f$, at least at the dyadic points $(j/2^k, 2^{-k})$, $j, k \in \mathbb{Z}$. A so-called "decomposition algorithm" for this purpose will be discussed in Section 1.4. The following extra conditions on ψ are needed in the definition of the IWT:

(i) $\widehat{\psi}$ is also a window function with center at some $\omega_0 > 0$, and

(ii) $c_\psi := \int_0^\infty \left(|\widehat{\psi}(\omega)|^2/|\omega|\right) d\omega < \infty$ for real ψ, and $\int_{-\infty}^{\infty} \left(|\widehat{\psi}(\omega)|^2/|\omega|\right) d\omega < \infty$ in general.

The reason for assuming (i) is that we have to localize frequencies with the zoom in and zoom out capability, and (ii) is needed for the reconstruction of f from the values of $(W_\psi f)(b, a)$. Indeed, for real-valued ψ, if (ii) is satisfied, then for every $f \in L^2$, we have

$$f(t) = \frac{2\pi}{c_\psi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} (W_\psi f)(b, a) \psi\left(\frac{t-b}{a}\right) db \right\} \frac{da}{a^{5/2}}. \tag{2.8}$$

The proof of this formula is straightforward and we only refer the interested reader to [9,25,29]. Observe the similarity with the classical results on singular integral operators (cf., [4]). Again, for an appropriate choice of ψ, f can be recovered from $W_\psi f$ at the dyadic points $(j/2^k, 2^{-k})$, $j, k \in \mathbb{Z}$, by using an infinite "wavelet series"; and in fact, a so-called "reconstruction algorithm" will be introduced in Section 1.4 for this purpose.

Now, by applying the Plancherel Identity, the integral wavelet transform in (2.7) becomes

$$(W_\psi f)(b, a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{ib\omega} \hat{f}(\omega) \overline{\widehat{\psi}(a\omega)} d\omega. \tag{2.9}$$

Hence, by setting

$$g(\omega) := \widehat{\psi}(\omega + \omega_0), \tag{2.10}$$

where $\omega_0 > 0$ is the center of $\widehat{\psi}$ (cf., (i)), we have

$$\widehat{\psi}(a\omega) = g\left(\frac{\omega - \frac{\omega_0}{a}}{a^{-1}}\right),$$

so that (2.9) is equivalent to

$$(W_\psi f)(b, a) = \frac{\sqrt{a}}{\pi} \int_{-\infty}^{\infty} e^{ib\omega} \hat{f}(\omega) \overline{g\left(\frac{\omega - \frac{\omega_0}{a}}{a^{-1}}\right)} d\omega. \tag{2.11}$$

Observe that while the width of the time-window function $\psi((t-b)/a)$ in the definition of the IWT in (2.7) is $2a\Delta_\psi$, the width of the frequency-window function $g((\omega - \omega_0/a)/a^{-1})$ in (2.11) is $2a^{-1}\Delta_{\widehat{\psi}}$ (cf., (2.10)). Hence, if the center of ψ is located at t_0, then the time-frequency window of the IWT $W_\psi(b, a)$ in the time-scale plot ((t, a)-plane) is

$$[b - t_0 - a\Delta_\psi, b - t_0 + a\Delta_\psi] \times \left[\frac{\omega_0}{a} - a^{-1}\Delta_{\widehat{\psi}}, \frac{\omega_0}{a} + a^{-1}\Delta_{\widehat{\psi}}\right]. \tag{2.12}$$

Here, although the area of the window is a constant given by $4\Delta_\psi \Delta_{\widehat{\psi}}$ as expected, the size changes according to the values of a. If we set $\frac{1}{a}$ to be (a positive constant multiple of) the frequency ω, then the time-frequency window (2.12) in the (t, ω)-plane narrows at high frequencies and widens at low frequencies. This is the zoom in and zoom out effect of the IWT $W_\psi f$. The choice of a positive number ω_0 as the center of $\widehat{\psi}$ in (i) enables us to study positive frequencies by using the window (2.12), since the scaling factor a is positive. For more details, see [9,10].

1.3 Multiresolution Analysis and Wavelets

We will follow Mallat [33] and Meyer [36] in our introduction of wavelets by using the notion of *multiresolution analysis* (cf., [34,35] for the motivation of this terminology). As in Section 1.1, we consider a nested sequence of closed subspaces $\{V_n\}$ of L^2:

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \tag{3.1}$$

and will assume that $\{V_n\}$ satisfies the following conditions:

(a) $\mathrm{clos}_{L^2}(\bigcup_{n\in\mathbb{Z}} V_n) = L^2$;

(b) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;

(c) $f \in V_n \Rightarrow f(2\cdot) \in V_{n+1},\ n \in \mathbb{Z}$; *and*

(d) *there exists a* $\phi \in V_0$ *such that* $\{\phi(\cdot - n)\colon n \in \mathbb{Z}\}$ *is an unconditional basis of* V_0, *in the sense that there exist positive constants* A, B *such that*

$$A\|\{c_n\}\|_{\ell^2}^2 \le \left\|\sum_{n\in\mathbb{Z}} c_n\phi(\cdot - n)\right\|_{L^2}^2 \le B\|\{c_n\}\|_{\ell^2}^2$$

for all $\{c_n\} \in \ell^2$.

If (a)–(d) are satisfied, then we say that the nested sequence $\{V_n\}$ of closed subspaces of L^2 yields a multiresolution analysis of L^2, or equivalently, the function ϕ in (d) generates a multiresolution analysis of L^2.

A typical example is the sequence of spline spaces defined as follows: Let m be any positive integer and consider

$$\mathcal{S}_m = \{f \in C^{m-2}\colon f|_{[n,n+1]} \in \pi_{m-1}, n \in \mathbb{Z}\}, \tag{3.2}$$

where π_{m-1} denotes the collection of all polynomials with degree no greater than $m-1$. $\mathcal{S}_m$ is called the polynomial spline space of order m (or degree $m-1$), with the set $\mathbb{Z}$ of integers as the knot sequence. It is well known (cf. [41]) that the m^{th} order B-spline N_m defined by

$$N_m(t) = (N_{m-1} * N_1)(t) = \int_0^1 N_{m-1}(t-x)dx, \tag{3.3}$$

with $N_1 = \chi_{[0,1)}$, is in $\mathcal{S}_m$, and in fact, it generates all of $\mathcal{S}_m$ in the sense that every $f \in \mathcal{S}_m$ can be represented by a spline series

$$f = \sum_{n\in\mathbb{Z}} a_n N_m(\cdot - n),$$

where $\{a_n\}$ is an arbitrary sequence. Here, pointwise convergence is certainly guaranteed since at each point the series is only a finite sum due to the simple fact that

$$\mathrm{supp}\ N_m = [0, m]. \tag{3.4}$$

Now, for each $k \in \mathbb{Z}$, let

$$V_k = \mathrm{clos}_{L^2}\langle N_m(2^k \cdot - n)\colon n \in \mathbb{Z}\rangle,$$

where the standard notation for linear span is used. Then V_k is the closed spline space of order m and with knot sequence $2^{-k}\mathbb{Z}$, and hence it is clear that we have a nested sequence $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$. This nested sequence certainly satisfied (a)–(c). In addition, (d) is satisfies with A and B being the supremum and infimum, respectively, of the spectrum of the positive definite symmetric banded Toeplitz matrix $[a_{i-j}]$, where

$$a_j = \int_{-\infty}^{\infty} N_m(t)N_m(t-j)dt = N_{2m}(m+j). \tag{3.5}$$

Let us now return to the general setting where ϕ is any generator of a multiresolution analysis of L^2. Then by defining

$$\phi_{k,n} = \phi(2^k \cdot - n), \tag{3.6}$$

it is clear that for each $k \in \mathbb{Z}$, $\{\phi_{k,n}\colon n \in \mathbb{Z}\}$ is an unconditional basis of V_k, with

$$2^{-k}A\|\{c_n\}\|_{\ell^2}^2 \le \left\|\sum_{n\in\mathbb{Z}} c_n\phi_{k,n}\right\|_{L^2}^2 \le 2^{-k}B\|\{c_n\}\|_{\ell^2}^2$$

for all $\{c_n\} \in \ell^2$, where A and B are given in (d). Next, for every $k \in \mathbb{Z}$, let us consider the orthogonal complementary subspace W_k of V_{k+1} relative to V_k as defined in (1.1). As already observed in Section 1.1, the subspaces W_k of L^2, $k \in \mathbb{Z}$, are mutually orthogonal and yield an orthogonal decomposition of L^2 as indicated in (1.3). From a more practical point of view, the (approximate) finite orthogonal decomposition (1.4) is also achieved for any $\varepsilon > 0$. We will delay our discussion of the decomposition and reconstruction algorithms for this finite orthogonal decomposition to the next section.

In Section 1.5, we will see by using an orthogonalization argument that there exists a function $\psi \in W_0$ that generates all the orthogonal complementary subspaces W_k in the sense that

$$W_k = \mathrm{clos}_{L^2}\langle \psi_{k,n}\colon n \in \mathbb{Z}\rangle, \tag{3.7}$$

where

$$\psi_{k,n} = \psi(2^k \cdot -n), \quad k,n \in \mathbb{Z}. \tag{3.8}$$

In fact, under very mild conditions, there exists a ψ with exponential decay. In any case, ψ is called a wavelet corresponding to the multiresolution analysis generated by ϕ. The importance of the pair (ϕ,ψ) should be clear: ϕ is used for approximation while ψ is used for analysing the errors (cf., Section 1.1). Of course, $\phi \in V_0$ and $\psi \in W_0$ are usually not unique. The minimally supported $\phi \in W_0$ and $\psi \in W_0$ are called the (generalized) B-spline and B-wavelet for this multiresolution analysis under consideration. We will not pursue the study of minimally supported ϕ and ψ in this account but only refer the interested reader to [20] where generalized B-splines and B-wavelets are characterized. In general, since ϕ is used for approximation, it determines the order of approximation of this multiresolution analysis. In addition, since $\phi \in V_0 \subset V_1$ and $\{\phi(2\cdot -n)\colon n \in \mathbb{Z}\}$ is an unconditional basis of V_1, there exists a unique sequence $\{p_n\} \in \ell^2$ such that

$$\phi(t) = \sum_{n\in\mathbb{Z}} p_n \phi(2t-n). \tag{3.9}$$

The Fourier transform equivalence of (3.9) is

$$\hat{\phi}(\omega) = P(e^{-i\omega/2})\hat{\phi}\left(\frac{\omega}{2}\right), \tag{3.10}$$

where

$$P(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}} p_n z^n \tag{3.11}$$

is called the two-scale symbol of ϕ. Of course, it is always desirable to have a finite sequence $\{p_n\}$, or by a simple shift, a polynomial two-scale symbol $P(z)$. Note that if $P(z)$ is a polynomial with $P(0) \neq 0$ and degree $P = N$, then it is easy to show that

$$\text{supp}\ \phi = [0, N],$$

(cf. [26]). For instance, the two-scale symbol of the m^{th} order B-spline N_m is $2^{-m}(1+z)^m$ and the support of N_m is $[0,m]$.

The following result relating the order of approximation of ϕ and the oscillatory property of the wavelet ψ is useful.

Theorem 3.1. *Let (ϕ,ψ) be a pair that generates a multiresolution analysis $\{V_n\}$ and the orthogonal complementary spaces $\{W_n\}$ as described above. Then the following statements are equivalent:*

(1°) *The order of approximation of ϕ is m, in the sense that*

$$\inf_{g\in V_n} \|f-g\| = O\left(\left(\frac{1}{2^n}\right)^m\right)$$

for all $f \in C^m \cap L^2$.

(2°) $D^j\hat{\phi}(2\pi\ell) = 0,\ \ell \in \mathbb{Z}\backslash\{0\},\ j = 0,\dots,m-1.$

(3°) *The commutator of ϕ is of order m, in the sense that*

$$[g \mid \phi] := \sum_{j\in\mathbb{Z}} g(j)\phi(\cdot - j) - \sum_{j\in\mathbb{Z}} \phi(j) g(\cdot - j)$$

is identically zero for all polynomials g of degree $\le m-1$.

(4°) *$P(z)$ is divisible by $(1+z)^m$.*

(5°) $\sum_{n\in\mathbb{Z}} (-1)^n n^j p_n = 0,\ j = 0,\dots,m-1.$

(6°) $\int_{-\infty}^{\infty} x^j \psi(x)dx = 0,\ j = 0,\dots,m-1.$

Of course, in the above statements, one must impose conditions on ϕ an ψ such that all the infinite series and improper integrals exist. Conditior such as ϕ and ψ having decay faster than $O(|x|^{-m-1})$ are sufficient; but i most practical purposes to be discussed later, they have exponential decε or even compact supports. Statement (2°) is due to Schoenberg (cf. [41 but is usually called the Strang and Fix condition, since it was first studi in the multivariate setting in [43] (cf. [7] for a somewhat detailed discussi and a study of the commutator). Condition (4°) was discussed in [42], aı (5°) easily follows from (4°). The oscillatory property (6°) of the wavel ψ can be explained from the fact that $\psi \in W_0$ and $W_0 \perp V_0$, and in view (3°), polynomials of degree $\le m-1$ can be generated, at least locally, $\phi \in V_0$.

Let us now discuss the relation between wavelets as introduced in t section and the integral wavelet transform (IWT) studied in Section 1 To do so, we need the notion of duals introduced in [19,20], as follows function $\tilde{\phi} \in V_0$ is said to be dual to $\phi \in V_0$ if

$$\langle\tilde{\phi}(\cdot - m), \phi(\cdot - n)\rangle := \int_{-\infty}^{\infty} \tilde{\phi}(t-m)\overline{\phi(t-n)}dt = \delta_{m,n} \qquad (3.$$

for all $m,n \in \mathbb{Z}$. Similarly, a function $\tilde{\psi} \in W_0$ is said to be dual to $\psi \in$ if

$$\langle\tilde{\psi}(\cdot - m), \psi(\cdot - n)\rangle = \delta_{m,n}, \quad m,n \in \mathbb{Z}. \qquad (3$$

Using the condition of unconditional bases, it is clear that $\tilde{\phi}$ and $\tilde{\psi}$ unique. Now, recall from (1.3) that every function $f \in L^2$ has a un orthogonal decomposition:

$$f = \sum_{k\in\mathbb{Z}} g_k, \quad g_k \in W_k;$$

and in view of (3.7) and (3.8), we have

$$g_k = \sum_{j\in\mathbb{Z}} d_j^k \psi_{k,j} \tag{3.14}$$

where again the coefficients d_j^k are unique. That is, we have a (unique) wavelet series representation

$$f(t) = \sum_{k,j} d_j^k \psi_{k,j}(t) \tag{3.15}$$

for every $f \in L^2$. Now, by the definition of the dual in (3.13) and the fact that $\widetilde{\psi}_{k,j} \in W_k$ and $W_k \perp W_n$ for all $k \neq n$, we have

$$\langle f, \widetilde{\psi}_{k,j}\rangle = d_j^k \langle \widetilde{\psi}_{k,j}, \psi_{k,j}\rangle = 2^{-k} d_j^k. \tag{3.16}$$

On the other hand, from the definition of the IWT in (2.7), we also have

$$\langle f, \widetilde{\psi}_{k,j}\rangle = \int_{-\infty}^{\infty} f(t)\overline{\widetilde{\psi}\left(\frac{t-j2^{-k}}{2^{-k}}\right)}dt = 2^{-k/2} W_{\widetilde{\psi}}(j2^{-k}, 2^{-k}). \tag{3.17}$$

Hence, by putting (3.16) and (3.17) into (3.15), we have:

$$f(t) = \sum_{k,j\in\mathbb{Z}} 2^{k/2} W_{\widetilde{\psi}}(j2^{-k}, 2^{-k})\psi_{k,j}(t). \tag{3.18}$$

That is, with the exception of the multiplicative constant $2^{k/2}$, the values of the IWT of f with window function $\widetilde{\psi}$ at the dyadic time-scale positions $((j/2^k), (1/2^k))$ constitute the wavelet coefficients of the wavelet series representation (3.15) or (3.18) of f using the wavelet ψ. Consequently, to find the IWT of any $f \in L^2$ at $(b,a) = ((j/2^k), (1/2^k))$, $j,k \in \mathbb{Z}$, we may simply find the coefficients $\{2^{-k/2} d_j^k\}$. In the next section, we will discuss a so-called *decomposition algorithm* for computing d_j^k via an approximant f_N of f from V_N. In addition, a so-called *reconstruction algorithm* for recovering f_N from the IWT at $((j/2^k), (1/2^{-k}))$ and a "blurred" version f_{N-M}, $M > 0$, of f_N, will also be studied.

1.4 Algorithms and Linear Phase Filtering

Let ϕ generate a multiresolution analysis of L^2 and ψ be a corresponding wavelet. Recall that ϕ is uniquely determined by its two-scale relation (3.9), and the two-scale symbol of the sequence $\{p_n\}$ that dictates this formulation is defined by $P(z)$ in (3.11), with a factor of $1/2$ to facilitate the Fourier transform formulation (3.10). Similarly, since $\psi \in W_0 \subset V_1$ and $\{\phi(2\cdot -n)\colon n \in \mathbb{Z}\}$ is an unconditional basis of V_1, there exists a unique sequence $\{q_n\} \in \ell^2$ such that

$$\psi(t) = \sum_{n\in\mathbb{Z}} q_n \phi(2t-n). \tag{4.1}$$

The Fourier transform equivalence of (4.1) is given by

$$\widehat{\psi}(\omega) = Q(e^{-i\omega/2})\hat{\phi}\left(\frac{\omega}{2}\right), \tag{4.2}$$

with

$$Q(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}} q_n z^n. \tag{4.3}$$

For convenience, we will also call (4.1) the two-scale formula for ψ and $Q(z)$ the corresponding two-scale symbol of ψ.

Next, let us consider the orthogonal sum $V_1 = W_0 \oplus V_0$. Since $\{\phi(\cdot - n)\colon n \in \mathbb{Z}\}$ and $\{\psi(\cdot - n)\colon n \in \mathbb{Z}\}$ are unconditional bases of V_0 and W_0, respectively, there exist four (unique) sequences $\{a_{-2n}\}, \{b_{-2n}\}, \{a_{1-2n}\}$, and $\{b_{1-2n}\}$, $n \in \mathbb{Z}$, such that

$$\begin{cases} \phi(2t) &= \sum_{n\in\mathbb{Z}}[a_{-2n}\phi(t-n) + b_{-2n}\psi(t-n)] \\ \phi(2t-1) &= \sum_{n\in\mathbb{Z}}[a_{1-2n}\phi(t-n) + b_{1-2n}\psi(t-n)]. \end{cases} \tag{4.4}$$

Hence, for any $\ell \in \mathbb{Z}$, we have

$$\phi(2t-\ell) = \sum_{n\in\mathbb{Z}}[a_{\ell-2n}\phi(t-n) + b_{\ell-2n}\psi(t-n)]. \tag{4.5}$$

So, since the functions $\phi_{1,\ell} = \phi(2\cdot - \ell)$, $\ell \in \mathbb{Z}$, form an unconditional basis of V_1, the two sequences $\{a_n\}$ and $\{b_n\}$, $n \in \mathbb{Z}$, in (4.5) uniquely determine the orthogonal decomposition $V_1 = V_0 \oplus W_0$. Let us also consider their symbols:

$$G(z) = \sum_{n\in\mathbb{Z}} a_{-n} z^n \tag{4.6}$$

and

$$H(z) = \sum_{n\in\mathbb{Z}} b_{-n} z^n. \tag{4.7}$$

Observe that we have not multiplied by the factor 1/2 in (4.6) and (4.7), and in addition, a negative sign in the index is used.

Now, by using the notation in (3.6) and (3.8), the decomposition formula (4.5) can be written as

$$\phi_{k+1,\ell} = \sum_{n\in\mathbb{Z}} [a_{\ell-2n}\phi_{k,n} + b_{\ell-2n}\psi_{k,n}], \tag{4.8}$$

for all $k, \ell \in \mathbb{Z}$, and the two two-scale formulas (3.9) and (4.1) can be expressed as

$$\begin{cases} \phi_{k,\ell} &= \sum_{n\in\mathbb{Z}} p_{n-2\ell}\phi_{k+1,n}; \\ \psi_{k,\ell} &= \sum_{n\in\mathbb{Z}} q_{n-2\ell}\phi_{k+1,n}, \end{cases} \tag{4.9}$$

for all $k, \ell \in \mathbb{Z}$. Since (4.9) will be used for deriving the reconstruction algorithm, the two-scale sequences $\{p_n\}$ and $\{q_n\}$ are also called *reconstruction sequences*.

Let $f_{k+1} \in V_{k+1}$ be decomposed as

$$f_{k+1} = f_k + g_k \tag{4.10}$$

where $f_k \in V_k$ and $g_k \in W_k$, and for any $n \in \mathbb{Z}$, write

$$\begin{cases} f_n &= \sum_j c_j^n \phi_{n,j} \\ \mathbf{c}^n &= \{c_j^n\} \end{cases} \tag{4.11}$$

and

$$\begin{cases} g_n &= \sum_j d_j^n \psi_{n,j} \\ \mathbf{d}^n &= \{d_j^n\} \end{cases} \tag{4.12}$$

as in (3.14). Then by applying (4.8), the left-hand side of (4.10) can be written as

$$\sum_j c_j^{k+1}\phi_{k+1,j} = \sum_j c_j^{k+1} \sum_n [a_{j-2n}\phi_{k,n} + b_{j-2n}\psi_{k,n}]$$
$$= \sum_j \left[\left(\sum_n a_{n-2j}c_n^{k+1}\right)\phi_{k,j} + \left(\sum_n b_{n-2j}c_n^{k+1}\right)\psi_{k,j}\right],$$

while the right-hand side of (4.10) is simply

$$\sum_j [c_j^k \phi_{k,j} + d_j^k \psi_{k,j}].$$

Hence, it follows from $V_0 \perp W_0$ and the ℓ^2-linear independence of $\{\phi_{k,j}\}_{j\in\mathbb{Z}}$ and $\{\psi_{k,j}\}_{j\in\mathbb{Z}}$ that

$$\begin{cases} c_j^k &= \sum_n a_{n-2j}c_n^{k+1} \\ d_j^k &= \sum_n b_{n-2j}c_n^{k+1}. \end{cases} \tag{4.13}$$

This yields the following wavelet decomposition (pyramid) algorithm:

$$\begin{array}{ccccccccc} & & \mathbf{d}^{N-1} & & \mathbf{d}^{N-2} & & & & \mathbf{d}^{N-M} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ \mathbf{c}^N & \longrightarrow & \mathbf{c}^{N-1} & \longrightarrow & \mathbf{c}^{N-2} & \longrightarrow & \cdots & \longrightarrow & \mathbf{c}^{N-M} \end{array}$$

for computing the orthogonal decomposition

$$f_N = g_{N-1} \oplus \cdots \oplus g_{N-M} \oplus f_{N-M} \tag{4.14}$$

described in (1.4).

On the other hand, by applying (4.9), the right-hand side of (4.10) can be written as

$$\sum_j \sum_n [c_j^k p_{n-2j}\phi_{k+1,n} + d_j^k q_{n-2j}\phi_{k+1,n}]$$
$$= \sum_j \left[\sum_n p_{j-2n}c_n^k + \sum_n q_{j-2n}d_n^k\right]\phi_{k+1,j}$$

while the left-hand side of (4.10) is

$$\sum_j c_j^{k+1}\phi_{k+1,j}.$$

Hence, it follows from the ℓ^2-linear independence of $\{\phi_{k+1,j}\}$, $j \in \mathbb{Z}$, that

$$c_j^{k+1} = \sum_n (p_{j-2n}c_n^k + q_{j-2n}d_n^k). \tag{4.15}$$

This yields the following reconstruction (pyramid) algorithm:

$$\begin{array}{ccccccccc} \mathbf{d}^{N-M} & & \mathbf{d}^{N-M+1} & & & & \mathbf{d}^{N-1} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \mathbf{c}^{N-M} & \longrightarrow & \mathbf{c}^{N-M+1} & \longrightarrow & \cdots & \longrightarrow & \mathbf{c}^{N-1} & \longrightarrow & \mathbf{c}^N \end{array}$$

for computing f_N from $g_{N-1}, \ldots, g_{N-M}$ and f_{N-M} in (4.14).

Of course, in applications, the orthogonal wavelet components

$$g_{N-1}, \ldots, g_{N-M}$$

and the blurred component f_{N-M} of f_N are "reduced" for storage or transmittance, or they might be "filtered", before applying the reconstruction algorithm. In any case, the efficiency of the decomposition algorithm (4.13) and the reconstruction algorithm (4.15) depends on the sequences $\{p_n\}$, $\{q_n\}$, $\{a_n\}$ and $\{b_n\}$. The shorter these sequences and the smaller number of decimal places in their terms, the more efficient are the algorithms. For the purpose of implementation, any infinite sequence must be truncated and any irrational term must be replaced by a reasonable rational number. Hence, in choosing (ϕ, ψ), the above discussion must be taken into consideration. Another aspect that should also be taken into consideration, especially in many applications in signal analysis, is the problem of "distortion". As is well known in signal processing (cf., Oppenheim and Shaffer[39]), distortion can be completely avoided if the filtering function (or sequence) have linear phase or at least generalized linear phase.

A filtering function $f(t) \in L^1 \cap L^2$ is said to have *generalized linear phase* if its Fourier transform has the formulation:

$$\hat{f}(\omega) = A(\omega)e^{i(a\omega+b)},$$

where $A(\omega)$ is real-valued and a, b are real constants. It is said to have *linear phase* if $A(\omega) = |\hat{f}(\omega)|$ and $b = 0$ or π; that is,

$$\hat{f}(\omega) = \pm|\hat{f}(\omega)|e^{ia\omega}.$$

Similarly, a filtering sequence $\{f_n\} \in \ell^1$ is said to have generalized linear phase if its discrete Fourier transform

$$F(\omega) := \sum_n f_n e^{in\omega}$$

can be written as

$$F(\omega) = A(\omega)e^{i(a\omega+b)},$$

where $A(\omega)$ is real-valued and a, b are real constants. It is said to have linear phase if

$$F(\omega) = \pm|F(\omega)|e^{ia\omega}.$$

In Chui and Wang[20], the following characterization result was established. For simplicity, we only consider real-valued $f(t)$ and $\{f_n\}$.

Lemma 4.1. *Let $f(t) \in L^1 \cap L^2$ and $\{f_n\} \in \ell^1$ be real-valued. Then the following statements hold.*

(1°) *$f(t)$ has generalized linear phase if and only if it is either symmetric or antisymmetric about some t_0.*

(2°) *$\{f_n\}$ has generalized linear phase if and only if it is either symmetric or antisymmetric about some $n_0 \in \frac{1}{2}\mathbb{Z}$.*

(3°) *$\{f_n\}$ has linear phase if and only if there is some $n_0 \in \mathbb{Z}$ such that $F(\omega)e^{-in_0\omega}$ is real-valued, even, and has no sign changes.*

To apply this lemma to studying the phase properties of wavelets, it is best only to consider wavelets with minimum supports, since the increase in support allows too much flexibility in changing the phases. The following result was obtained in [20].

Theorem 4.2. *Let ϕ generate a multiresolution analysis of L^2 such that the two-scale sequence $\{p_n\}$ of ϕ is finite and real-valued. Also, let ψ be a corresponding wavelet with minimum support and let $\tilde{\psi}$ be the dual of ψ. Then*

(1°) *if $\{p_n\}$ has generalized linear phase, both ψ and $\tilde{\psi}$ have generalized linear phases, and*

(2°) *if $\{p_n\}$ has linear phase, both ψ and $\tilde{\psi}$ have linear phases.*

For more details, the reader is referred to [20].

1.5 Orthogonal Wavelets

As usual, let ϕ generate a multiresolution analysis $\{V_n\}$ of L^2 and ψ be any corresponding wavelet that generates the orthogonal complementary subspaces $\{W_n\}$. Then it follows that for all j and $\ell \in \mathbb{Z}$, we have $\psi_{k,j} \perp \psi_{n,\ell}$ whenever $k \neq n$. Note, however, that in general, $\psi_{k,j}$ is not orthogonal to $\psi_{k,\ell}$ for $j \neq \ell$. If $\psi(\cdot - j)$ is orthogonal to $\psi(\cdot - \ell)$, $j \neq \ell$, and $\|\psi\|_{L^2} = 1$, we say that ψ is an orthonormal wavelet. In other words, ψ is an orthonormal wavelet if and only if $\{2^{\frac{k}{2}}\psi_{k,j}\}$, $k, j \in \mathbb{Z}$, is an orthonormal basis of L^2.

From ϕ, it is quite easy to find an orthonormal wavelet. Let us first establish the following two lemmas.

Lemma 5.1. . *Let $0 < A \leq B < +\infty$ and $\xi \in L^2$. Then the following statements are equivalent:*

(1°) *For any $\mathbf{c} = \{c_n\} \in \ell^2$,*

$$A\|\mathbf{c}\|_{\ell^2}^2 \leq \left\| \sum_{n\in\mathbb{Z}} c_n \xi(\cdot - n) \right\|_{L^2}^2 \leq B\|\mathbf{c}\|_{\ell^2}^2,$$

and hence, $\{\xi(\cdot - n)\colon n \in \mathbb{Z}\}$ is an unconditional basis of the L^2-closure of its span.

(2°) *The Fourier transform* $\hat{\xi}$ *of* ξ *satisfies*

$$A \le \sum_{n\in\mathbb{Z}} |\hat{\xi}(\omega + 2\pi n)|^2 \le B \quad \text{almost everywhere.}$$

(Of course if $\xi \in L^1 \cap L^2$*, then* $\hat{\xi}$ *is continuous so that the above inequalities hold for all* ω*.)*

Proof. For any $\mathbf{c} = \{c_n\} \in \ell^2$, we denote its discrete Fourier transform by

$$C(\omega) = \sum_n c_n e^{in\omega}.$$

Then by the Plancherel Identity, we have

$$\begin{aligned}\left\| \sum_{n\in\mathbb{Z}} c_n \xi(\cdot + n) \right\|_{L^2}^2 &= \frac{1}{2\pi} \|C\hat{\xi}\|_{L^2}^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} |C(\omega)|^2 \sum_{n\in\mathbb{Z}} |\hat{\xi}(\omega + 2\pi n)|^2 d\omega. \quad (5.1)\end{aligned}$$

Hence, since

$$\|\mathbf{c}\|_{\ell^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} |C(\omega)|^2 d\omega,$$

(1°) clearly follows from (2°). The converse is also an easy consequence of (5.1), since $C(\omega)$ is an arbitrary 2π-periodic function in $L^2(0, 2\pi)$. ■

Lemma 5.2. *For any* $\xi \in L^2$*, the following statements are equivalent:*

(1°) $\{\xi(\cdot - n)\colon n \in \mathbb{Z}\}$ *is orthonormal in the sense that*

$$\int_{-\infty}^{\infty} \xi(t - m)\overline{\xi(t - n)} dt = \delta_{m,n}$$

for all $m, n \in \mathbb{Z}$.

(2°) *For all* $k \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\xi}(\omega)|^2 e^{ik\omega} d\omega = \delta_{k,0}.$$

(3°) *For almost all* ω,

$$\sum_{n\in\mathbb{Z}} |\hat{\xi}(\omega + 2\pi n)|^2 = 1.$$

Proof. Consider the nonnegative 2π-periodic function

$$F(\omega) := \sum_{n\in\mathbb{Z}} |\hat{\xi}(\omega + 2\pi n)|^2,$$

which is clearly in $L^1(0, 2\pi)$. In fact, its k^{th} Fourier coefficient is given by

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} F(\omega) e^{-ik\omega} d\omega &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |\hat{\xi}(\omega + 2\pi n)|^2 e^{-ik\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\xi}(\omega)|^2 e^{-ik\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\xi}(\omega)\overline{\hat{\xi}(\omega) e^{ik\omega}} d\omega \\ &= \int_{-\infty}^{\infty} \xi(t)\overline{\xi(t-k)} dt \\ &= \int_{-\infty}^{\infty} \xi(t-m)\overline{\xi(t-n)} dt,\end{aligned}$$

where $k = n - m$. This sequence of identities also completes the proof of the lemma. ■

We now return to an arbitrary $\phi \in L^1 \cap L^2$ that generates a multiresolution analysis of L^2, so that $\{\phi(\cdot - n)\colon n \in \mathbb{Z}\}$ is an unconditional basis of V_0. By Lemma 5.1, the function ϕ^*, whose Fourier transform is defined by

$$\widehat{\phi^*}(\omega) := \frac{\hat{\phi}(\omega)}{\left(\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2\right)^{1/2}}, \quad (5.2)$$

is also in $L^1 \cap L^2$, and from (5.2) it is easy to verify that ϕ lies in the L^2 closure of the linear span of

$$\{\phi^*(\cdot - n)\colon n \in \mathbb{Z}\}. \quad (5.3)$$

Since $\hat{\phi}^*$ clearly satisfies (3°) in Lemma 5.2 for all ω, the family (5.3) is an orthonormal basis of V_0, so that ϕ^* generates the same multiresolution analysis of L^2 as ϕ. In particular, ϕ^* has a two-scale formula:

$$\phi^*(t) = \sum_{n\in\mathbb{Z}} p_n^* \phi^*(2t - n), \quad (5.4)$$

where $\{p_n^*\} \in \ell^2$ is uniquely determined by ϕ^*. We are now able to construct an orthonormal wavelet ψ^* as follows:

$$\psi^*(t) := \sum_{n\in\mathbb{Z}} (-1)^n \overline{p_{1-n}^*}\phi^*(2t-n). \tag{5.5}$$

Before we show that ψ^* is indeed an orthonormal wavelet for this multiresolution analysis, let us first establish an identity for the two-scale symbol

$$P^*(z) := \frac{1}{2}\sum_{n\in\mathbb{Z}} p_n^* z^n$$

of the orthonormal multiresolution analysis generator ϕ^*.

Lemma 5.3. *We have*

$$|P^*(z)|^2 + |P^*(-z)| = 1, \qquad |z| = 1. \tag{5.6}$$

Proof. By Lemma 5.2, since ϕ^* is orthonormal, we have, for all $z = e^{-i\omega/2}$,

$$\begin{aligned}
1 &= \sum_{n\in\mathbb{Z}} |\hat{\phi}^*(\omega+2\pi n)|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|P^*(e^{-i(\frac{\omega}{2}+\pi n)})\hat{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|P^*((-1)^n z)\hat{\phi}^*\left(\frac{\omega}{2}+\pi n\right)\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|P^*(z)\hat{\phi}^*\left(\frac{\omega}{2}+2\pi k\right)\right|^2 \\
&\quad + \sum_{k\in\mathbb{Z}} \left|P^*(-z)\hat{\phi}^*\left(\frac{\omega}{2}+\pi+2\pi k\right)\right|^2 \\
&= |P^*(z)|^2 + |P^*(-z)|^2. \qquad \blacksquare
\end{aligned}$$

We are now ready to verify that ψ^* is an orthonormal wavelet.

Theorem 5.4. *Let ϕ^* be defined as in (5.2) and suppose that ϕ^* generates the same multiresolution analysis $\{V_n\}$ of L^2 as ϕ. Then ψ^*, as defined in (5.5), generates the orthogonal complementary subspaces W_n.*

Proof. The proof of this theorem is divided into three steps as follows.

(i) To verify that $\psi^* \in W_0$, let $n \in \mathbb{Z}$ be arbitrarily chosen. Then we have, from (5.4) and (5.5),

$$\begin{aligned}
\langle \phi^*(\cdot - n), \psi^* \rangle &= \sum_{j,k\in\mathbb{Z}} p_j^*(-1)^k p_{1-k}^* \langle \phi^*(2\cdot -j), \phi^*(2\cdot +2n-k)\rangle \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} (-1)^k p_{-2n+k}^* p_{1-k}^* \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}} (-1)^{1-j+2n} p_{1-j}^* p_{-2n+j}^*,
\end{aligned}$$

which must be zero, since the last quantity, which is obtained by the change of index $j = 1-k-2n$, is the negative of the quantity preceding it. This elegant observation, which holds for all ℓ^2 sequences $\{p_n^*\}$ regardless of the orthogonality property, is given in [42].

(ii) To show that

$$\{\psi^*(\cdot - n) \colon n \in \mathbb{Z}\}$$

is orthonormal, we first note that the Fourier transform formulation of (5.5) is

$$\widehat{\psi}^*(\omega) = \left(\frac{1}{2}\sum_{n\in\mathbb{Z}} (-1)^n \overline{p_{1-n}^*} e^{-in\omega/2}\right) \hat{\phi}^*\left(\frac{\omega}{2}\right),$$

so that, with $z = e^{-\omega/2}$, we have

$$|\widehat{\psi}^*(\omega)| = |P^*(-z)| \left|\hat{\phi}^*\left(\frac{\omega}{2}\right)\right|.$$

Hence, following the same proof as that of Lemma 5.3 and applying (5.6), we have

$$\sum_{n\in\mathbb{Z}} |\psi^*(\omega+2\pi n)|^2 = 1,$$

or $\{\psi^*(\cdot - n) \colon n \in \mathbb{Z}\}$ is orthonormal.

(iii) Finally, to show that $\{\psi^*(\cdot - n) \colon n \in \mathbb{Z}\}$ spans W_0, we will do a little more by exhibiting the decomposition relationship of $V_1 = V_0 \oplus W_0$ as follows. Let us set

$$\begin{cases} a_n := \frac{1}{2}\overline{p_n^*} \\ b_n := \frac{1}{2}(-1)^n p_{1-n}^* \end{cases}. \tag{5.7}$$

Then it is sufficient to establish:

$$\phi^*(2t-k) = \sum_{n\in\mathbb{Z}} a_{k-2n}\phi^*(t-n) + \sum_{n\in\mathbb{Z}} b_{k-2n}\psi^*(t-n) \tag{5.8}$$

for all $k \in \mathbb{Z}$. Note that with $z := e^{-i\omega/2}$ the Fourier transform equivalence of (5.8) is given by

$$\frac{1}{2} z^k \hat{\phi}^*\left(\frac{\omega}{2}\right) = \left(\sum_{n\in\mathbb{Z}} a_{k-2n} z^{2n}\right) \hat{\phi}^*(\omega) + \left(\sum_{n\in\mathbb{Z}} b_{k-2n} z^{2n}\right) \widehat{\psi}^*(\omega). \tag{5.9}$$

On the other hand, recall from (5.4) and (5.5) that

$$\begin{cases} \hat{\phi}^*(\omega) &= P^*(z)\hat{\phi}^*\left(\frac{\omega}{2}\right) \\ \widehat{\psi}^*(\omega) &= Q^*(z)\hat{\phi}^*\left(\frac{\omega}{2}\right) \end{cases},$$

where $z = e^{-i\omega/2}$ and

$$Q^*(z) := \frac{1}{2} \sum_{n\in\mathbb{Z}} \overline{p^*_{1-n}}(-z)^n = -z\overline{P^*(-z)}. \tag{5.10}$$

Hence, by using the definition of $\{a_n\}$ and $\{b_n\}$ in (5.7), the identity (5.9) is equivalent to

$$\begin{aligned} 1 &= \left(\sum_n \overline{p^*_{k-2n}} z^{2n-k}\right) P^*(z) - z \left(\sum_n p^*_{1-k+2n}(-z)^{2n-k}\right) \overline{P^*(-z)} \\ &= 2\overline{P^*_k(z)}P^*(z) + 2P^*_{k-1}(-z)\overline{P^*(-z)}, \end{aligned} \tag{5.11}$$

where

$$P^*_k(z) := \begin{cases} \dfrac{P^*(z) + P^*(-z)}{2} & \text{for even } k \\ \dfrac{P^*(z) - P^*(-z)}{2} & \text{for odd } k \end{cases}$$

Since it is clear that (5.11) is equivalent to the identity (5.6) for $|z| = 1$, we have established (5.8). ■

In the above derivation, we have also obtained the following result.

Corollary 5.5. *If a multiresolution analysis generator ϕ is orthonormal, then an orthonormal wavelet ψ can be chosen such that the two-scale (or reconstruction) sequences are $\{p_n\}$ and $\{(-1)^n \bar{p}_{1-n}\}$, and the corresponding decomposition sequences are $\{\bar{p}_n/2\}$ and $\{(1/2)(-1)^n p_{1-n}\}$.*

For more details, see [9,24,25,33,38]. Let us now discuss some specific examples.

Example (1°) *The Battle-Lemarié wavelets.*

By using the m^{th} order B-spline N_m as ϕ in (5.2), the orthonormal wavelet $\psi^*_m := \psi^*$ in (5.5) is uniquely determined. In fact, an explicit formula for $\widehat{\psi^*_m}$ can be found in Lemarié [32] and Battle [2]. Battle's derivation, however, is quite different: instead of orthogonalizing $\phi = N_m$ to yield ϕ^* and consequently ψ^*_m as (5.2) and (5.5), ψ^*_m is obtained by first orthogonalizing the scale levels, using the so-called "block spin" assignments, and then orthogonalizing the translates. It should be remarked, however, that with the exception of the Haar (wavelet) function which is obtained from the first order B-spline N_1, there is no explicit formula for any of the other orthonormal spline wavelets ψ^*_m, although a very nice (but fairly complicated) expression of their Fourier transforms $\widehat{\psi^*_m}$ is given in [2] and [32]. Also, although the orthonormal spline wavelets do not have compact supports, they all have exponential decay. Of course, as is well known from the Phase-Space Localization theorem, $\widehat{\psi^*_m}$ cannot decay exponentially.

Example (2°) *The Meyer wavelets.*

The wavelets ψ constructed by Meyer [37], on the other hand, have the property that their Fourier transforms $\widehat{\psi}$ have compact support. In fact, Meyer's construction is based on $\hat{\phi}$ instead of ϕ.

Let $0 < \varepsilon \le \frac{\pi}{3}$ and define $\hat{\eta}(\omega) = 0$ for $|\omega| \ge \pi + \varepsilon$, and $\hat{\eta}(\omega) = 1$ for $|\omega| \le \pi - \varepsilon$. There is a lot of freedom for choosing $\hat{\eta}(\omega), \pi - \varepsilon < |\omega| < \pi + \varepsilon$. We pick two arbitrary constants A, B with $0 < A < 1 < B < \infty$ and select $\hat{\eta} \in C^N$ (where $0 \le N \le \infty$ is also arbitrarily preassigned), such that

$$A \le \sum_{n\in\mathbb{Z}} |\hat{\eta}(\omega + 2\pi n)|^2 \le B$$

for all ω. Then the function $\hat{\phi}$ defined by

$$\hat{\phi}(\omega) = \frac{\hat{\eta}(\omega)}{\left(\sum_{n\in\mathbb{Z}} |\hat{\eta}(\omega + 2\pi n)|^2\right)^{1/2}} \tag{5.12}$$

clearly satisfies

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 = 1$$

for all ω, and supp $\hat{\phi} = [-\pi - \varepsilon, \pi + \varepsilon]$. In addition, by Lemma 5.2, $\{\phi(\cdot - n) \colon n \in \mathbb{Z}\}$ is an orthonormal set. Now, define

$$P(e^{-i\omega/2}) := \sum_{n\in\mathbb{Z}} \hat{\phi}(\omega + 4\pi n)$$

and the corresponding sequence $\{p_n\}$ so that

$$P(z) = \frac{1}{2} \sum_{n\in\mathbb{Z}} p_n z^n. \tag{5.13}$$

We claim that

$$\hat{\phi}(\omega) = P(e^{-i\omega/2})\hat{\phi}\left(\frac{\omega}{2}\right) \tag{5.14}$$

which is, of course, equivalent to

$$\phi(t) = \sum_n p_n \phi(2t-n). \tag{5.15}$$

To verify (5.14), we first note that for $\omega \notin$ supp ϕ, then

$$P(e^{-i\omega/2}) = \sum_{n\neq 0} \hat{\phi}(\omega + 4\pi n)$$

so that either $\hat{\phi}(\frac{\omega}{2}) = 0$ or else $P(e^{-i\omega/2}) = 0$. That is, (5.14) is satisfied with $0 = 0$. On the other hand, if $\omega \in$ supp $\hat{\phi}$, then we have $P(e^{-i\omega/2}) = \hat{\phi}(\omega)$ as well as $\hat{\eta}(\omega + 2\pi n) = 0$ for all $n \neq 0$ which yields, using (5.12), $\hat{\phi}((\omega/2)) = 1$. Hence (5.14) holds.

In addition, since $\hat{\phi}(0) = 1$ and $\hat{\phi}(2\pi n) = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, it follows from Theorem 3.1 that ϕ generates a multiresolution analysis of L^2. Recall that $\{\phi(\cdot - n)\colon n \in \mathbb{Z}\}$ is orthonormal, and therefore is an unconditional basis of V_0. Now, the corresponding orthonormal wavelet ψ has Fourier transform

$$\widehat{\psi}(\omega) = \left(\frac{1}{2}\sum_n (-1)^n \bar{p}_{1-n} e^{-in\omega/2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

where $\{p_n\}$ is defined in (5.13). Hence, we have

$$\text{supp } \widehat{\psi} = [-2\pi - 2\varepsilon, 2\pi + 2\varepsilon].$$

We remark that $\psi(t)$ has decay $O(|t|^{-\alpha})$, where $\alpha > 0$ is arbitrarily large if we require $\eta \in C^\infty$.

Example (3°) *The Daubechies wavelets.*

While Meyer's orthonormal wavelets have compactly supported Fourier transforms, the orthonormal wavelets constructed by Daubechies in [24] are compactly supported. The basic strategy in Daubechies' construction is to solve the two two-scale equation

$$\phi(t) = \sum_{n\in\mathbb{Z}} p_n \phi(2t-n)$$

subject to the following constraints:

(i) $\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t)dt = 1$;

(ii) $\{p_n\}$ is a *finite* sequence;

(iii) $\sum_k p_{2n+k}\bar{p}_k = 2\delta_{n,0},\ n \in \mathbb{Z}$;

(iv) the infinite product

$$\prod_{k=1}^{\infty} P(e^{-i2^{-k}\omega}),$$

where $P(z) = \frac{1}{2}\sum p_n z^n$ denotes the two-scale symbol of $\{p_n\}$, converges uniformly on compact subintervals of $\mathbb{R}$.

It is easy to see that the support of ϕ is the convex hull of the support of $\{p_n\}$. In addition, condition (iii) is equivalent to (5.6) which is a necessary condition for the orthogonality of $\{\phi(\cdot - n)\colon n \in \mathbb{Z}\}$. Of course, the convergence of the infinite product is essential, since under the condition (i), this limit is the required $\hat{\phi}(\omega)$. Once $\hat{\phi}(\omega)$ and $P(z)$ are determined, the compactly supported orthonormal wavelet ψ is defined in the same manner as (5.5). Of course, one cannot expect to have an explicit formulation of Daubechies' wavelets. However, efficient recursive computational methods are available in the literature such as [3]. An interesting result in [24] is that with the exception of the Haar function, Daubechies' compactly supported orthonormal wavelets are not symmetric nor antisymmetric about any point; and hence, from Lemma 4.1, they do not even have generalized linear phases.

1.6 Spline Wavelets and Biorthogonal Bases

When wavelets ψ were introduced in Section 1.3, it was already implicit that wavelets with different scales are always orthogonal, namely:

$$\langle \psi(2^k \cdot - \ell), \psi(2^n \cdot - j)\rangle = 0$$

for all $\ell, j, k, n \in \mathbb{Z}$ as long as $k \neq n$. In order to relate the wavelet series with the integral wavelet transform (IWT) introduced in Section 1.2, dual wavelets $\widetilde{\psi}$, defined by $\psi, \widetilde{\psi} \in W_0$ and the biorthogonal relationship (3.13), were introduced. Of course, if ψ is orthonormal, then ψ is self-dual, namely: $\widetilde{\psi} = \psi$. In an attempt to give explicit and very simple expressions of $\widetilde{\psi}$, we bypass the orthogonalization procedure (5.2) to construct ψ more directly. This point of view was first considered by Chui and Wang in [18] for polynomial splines using fundamental cardinal splines; and in [19], compactly supported spline wavelets and their duals were first constructed. A general theory that characterizes wavelets for an arbitrary multiresolution analysis is given in [20].

Recall that in the discussion of orthonormal wavelets, the single most important quantity is the two-scale symbol $P(z)$ of $\{p_n\}$, since from $P(z)$, the two-scale symbol for the corresponding orthonormal wavelet is given by

$$Q(z) = -z\overline{P(-z)}, \qquad |z| = 1; \tag{6.1}$$

and the symbols of the decomposition sequences $\{a_n\}$ and $\{b_n\}$ are simply

$$\begin{cases} G(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n} = \overline{P(z)}, & |z| = 1; \\ H(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n} = -\bar{z}P(-z), & |z| = 1, \end{cases} \tag{6.2}$$

(cf., (5.7)-(5.11) for the derivation).

In the general setting, however, we seem to have some freedom in choosing $Q(z)$, but the decomposition symbols $G(z)$ and $H(z)$ are then governed by $P(z)$ and $Q(z)$ through the identities:

$$\begin{cases} P(z)G(z) + Q(z)H(z) = 1, & |z| = 1; \\ P(z)G(-z) + Q(z)H(-z) = 0, & |z| = 1. \end{cases} \tag{6.3}$$

These identities are consequences of the two-scale formulas (3.10), (4.2) and the decomposition formula (4.5). For simplicity and all practical purposes, we only restrict our attention to finite two-scale sequences $\{p_n\}$; that is, we only consider polynomial symbols

$$P(z) = \frac{1}{2}\sum_{n=0}^{N} p_n z^n, \quad p_0 \neq 0,\ p_N \neq 0, \tag{6.4}$$

where a shift has been applied to ϕ. For instance, for the m^{th} order B-splines N_m, the two-scale symbol clearly satisfies

$$\left(\frac{1-z^2}{i\omega}\right)^m = P_m(z)\left(\frac{1-z}{i\omega/2}\right)^m,$$

where $z = e^{-i\omega/2}$, or

$$P_m(z) = \left(\frac{1+z}{2}\right)^m.$$

So, how much freedom do we really have in the choice of $Q(z)$, given a two-scale symbol $P(z)$? To answer this question, we need the notion of the *generalized Euler-Frobenius polynomial* Π_ϕ of the multiresolution analysis generator ϕ introduced in Chui and Wang [20] as follows. Let

$$\gamma_\phi(n) = \int_{-\infty}^{\infty} \phi(n+x)\overline{\phi(x)}dx.$$

Then it is clear that $\gamma_\phi(-n) = \overline{\gamma_\phi(n)}$. Also, since it follows from the two-scale formula of ϕ and (6.4) that supp $\phi = [0, N]$, it is also clear that

$$\text{supp } \gamma_\phi \subseteq [-N, N].$$

Let $0 \le k_\phi \le N$ be such that supp $\gamma_\phi = [-k_\phi, k_\phi]$. Then the generalized Euler-Frobenius polynomial of ϕ was defined in [20] by

$$\Pi_\phi(z) = \sum_{n=0}^{2k_\phi} \gamma_\phi(n - k_\phi)z^n. \tag{6.5}$$

For instance, if $\phi = N_m$ is the m^{th} order B-spline, then ϕ_{N_m} is the classical Euler-Frobenius polynomial multiplied by a factor of $[(2m-1)!]^{-1}B$:

$$\Pi_{2m-1}(z) := (2m-1)! \sum_{j=0}^{2m-2} N_{2m}(j+1)z^j\ ; \tag{6.6}$$

and for any orthonormal ϕ, we have $\Pi_\phi(z) = 1$, since $k_\phi = 0$. Consider the polynomial

$$\beta_\phi(z) := z^{N-k_\phi-1}\Pi_\phi(z)\check{P}(z), \tag{6.7}$$

where $\check{P}(z)$ denotes the reciprocal polynomial of $P(z)$, and factorize β_ϕ into

$$\beta_\phi(z) = \mu_\phi(z)\lambda_\phi(z^2), \tag{6.8}$$

where μ_ϕ, λ_ϕ are polynomials with $\lambda_\phi(1) = 1, \mu_\phi(z)$ not divisible by z^2, and $\mu_\phi(z)$ has no non-zero symmetric root in the sense that $\mu_\phi(z_0) = 0$ and $z_0 \neq 0$ implies $\mu_\phi(-z_0) \neq 0$. The following result was established in [20].

Theorem 6.1. *The class of all two-scale symbols for the wavelets that generate the orthogonal complementary subspaces W_n is given by*

$$Q(z) = \mu(-z)W(z^2), \tag{6.9}$$

where W is analytic in a neighborhood of $|z| = 1$.

For instance, let us again consider the spline example $\phi = N_m$, where β_ϕ in (6.7) becomes

$$\beta_m(z) = \frac{1}{(2m-1)!}\left(\frac{1+z}{2}\right)^m \Pi_{2m-1}(z) = \mu(z)$$