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Differential Geometry of Singular Spaces and Reduction of Symmetry

J. Śniatycki

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J. ŚNIATYCKI

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada

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Differential Geometry of Singular Spaces and Reduction of Symmetry

In this book, the author illustrates the power of the theory of subcartesian differential spaces for investigating spaces with singularities. Part I gives a detailed and comprehensive presentation of the theory of differential spaces, including integration of distributions on subcartesian spaces and the structure of stratified spaces. Part II presents an effective approach to the reduction of symmetries.

Concrete applications covered in the text include the reduction of symmetries of Hamiltonian systems, non-holonomically constrained systems, Dirac structures and the commutation of quantization with reduction for a proper action of the symmetry group. With each application, the author provides an introduction to the field in which relevant problems occur.

This book will appeal to researchers and graduate students in mathematics and engineering.

J. Śniatycki is a Professor in the Department of Mathematics and Statistics at the University of Calgary.

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Preface

My first encounter with differential spaces was in the mid 1980s. At a conference in Toruń, I presented the notion of algebraic reduction of symmetries of a Hamiltonian system. After the lecture, Constantin Piron asked me if my reduced spaces were the differential spaces of Sikorski. I had to admit that I did not know what Sikorski's differential spaces were. To this Piron replied something like 'You should be ashamed of yourself! You are a Pole and you do not know what are differential spaces of Sikorski!' During the lunch break I went to the library to consult Sikorski's work. In the afternoon session, I told Piron that the spaces we were dealing with were not the differential spaces of Sikorski. At that time I did not realize that they were differential schemes.

Around the same time, Richard Cushman was working out his examples of singular reduction. I was fascinated by his pictures of reduced spaces with singularities. However, I had not the faintest idea what he was really doing. Since Richard was spending a lot of time in Calgary working on his book with Larry Bates, I had a chance to ask him to explain singular reduction to me. It took me a long time to realize that he was talking the language of differential spaces without being aware of it. From conversations with Richard, it became clear that differential spaces provided a convenient language for the description of the reduction of symmetries for proper actions of symmetry groups.

The next push in the direction of serious investigations of differential spaces came from Ryer Sjamaar and Eugene Lerman. In their *Annals of Mathematics* paper on reduction of symmetries of Hamiltonian systems, they proved a theorem using techniques that are natural to the theory of differential spaces. Studying their proof, I realized that it was very simple and that I could not think of an equally simple proof that would not utilize their techniques. It convinced me that the language of differential spaces facilitated obtaining new results, and I decided to investigate if reduction of symmetries could be completely formulated and analysed within the category of differential spaces.

The theory of differential spaces is essentially differential geometry not restricted to smooth manifolds. Roman Sikorski, who is considered the father of the theory, called his book (in Polish) *Wstęp do Geometrii Różniczkowej*. This translates as 'Introduction to Differential Geometry'. Originally, differential geometry meant the description, in terms of differentiable functions, of curves and surfaces in $\mathbb{R}^n$. Singularities of curves or surfaces under consideration could also be described in terms of smooth functions. Differential geometry evolved in two different directions: the theory of manifolds and singularity theory. Manifolds are smooth spaces not presented as subsets of $\mathbb{R}^n$. Singularity theory is the study of the failure of the manifold structure. Differential geometry in the sense of Sikorski is a reunification of the two theories. It contains the theory of manifolds and also allows the investigation of singularities. It is the investigation of geometry in terms of differentiable functions. Differential geometry, understood in this way, is analogous to algebraic geometry, which is the investigation of geometry in terms of polynomials. The difference between the two theories is in the choice of the space of functions.

I am grateful to Constantin Piron for drawing my attention to Sikorski's book. I greatly appreciate the support and encouragement of Hans Duistermaat. I would like to thank Larry Bates for his support and for bringing Richard Cushman to Calgary, and to thank Jordan Watts for his interest in my work. Above all, I want to thank Richard Cushman for his patience in explaining to me the foundations of his theory of singular reduction and his subsequent collaboration, encouragement and criticism. I also want to thank Cathy Beveridge and Leslie McNab for their help in editing the manuscript. Both Cathy and Leslie have worked hard to make sure that this book is written in proper English. However, I am sure that, in spite of their vigilance, I will have managed to slip in some phrases that go against the proper use of English. Last but not least, I want to thank my wife, Pamela Plummer, without whose support this book would not have been possible.

Partial support from the National Science and Engineering Research Council of Canada is gratefully acknowledged.

List of selected symbols

Upper case Latin alphabet

Ad^*	co-adjoint action
B	open ball
$\mathbb{C}$	complex numbers
$\mathcal{C}^0(S)$	continuous functions on S
$\mathcal{C}^\infty(S)$	smooth functions on S
$\mathcal{C}^\infty(S)^G$	G-invariant functions on S
D	distribution; real part of polarization in Chapter 7
$\mathcal{D}$	space of compactly supported sections
$\mathcal{D}'$	dual of $\mathcal{D}$
$\mathrm{Der}\,\mathcal{C}^\infty(S)$	space of derivations of $\mathcal{C}^\infty(S)$
$\mathfrak{E}$	family of Hamiltonian vector fields
$\mathrm{Exp} : T_pP \to P$	exponential map defined by connection
F	function; polarization in Chapter 7
$\mathcal{F}$	family of functions
$\mathfrak{F}$	family of vector fields
$\mathcal{F}F$	bundle of linear frames of F
$\mathcal{F}T^{\mathbb{C}}M$	bundle of linear frames of $T^{\mathbb{C}}M$
G	Lie group
H	Lie group
$\mathcal{H}$	Hilbert space
I	interval; inclusion map of co-adjoint orbit into $\mathfrak{g}^*$ in Chapter 7
J	momentum map
$\mathcal{J}_0$	ideal generated by components of J
K	manifold
L	manifold
$\mathcal{L}$	prequantization line bundle
M	manifold; stratum

$\mathfrak{M}$	stratification
N	manifold, stratum
N^H	normalizer of H
$\mathcal{N}(S)$	space of functions with vanishing restrictions to S
$\mathfrak{N}$	stratification
O	orbit
$\mathfrak{O}$	partition by orbits
P	manifold, differential space
$\boldsymbol{P}$	prequantization map
P^H	set of points in P fixed by action of H
P_H	set of points in P of symmetry type H
$P_{(H)}$	set of points in P of orbit type H
$\mathfrak{P}(R)$	Poisson vector fields on R
Q	manifold
$\boldsymbol{Q}$	quantization map
R	manifold; differential space; orbit space
$\mathbb{R}$	real numbers
$\boldsymbol{R}$	linear representation
$\mathcal{R}(S)$	space of restrictions to S of functions defined on a larger space
S	manifold; differential space
S_p	slice at p
$\mathcal{S}^\infty(\mathcal{L})$	space of smooth sections of $\mathcal{L}$
$\mathcal{S}^\infty_F(\mathcal{L})$	space of smooth polarized sections of $\mathcal{L}$
$\mathcal{S}^\infty(\mathcal{L})^G$	space of G-invariant smooth sections of $\mathcal{L}$
$T^{\mathbb{C}}M$	complexified tangent bundle of M
$TS,\ T^*S$	tangent and cotangent bundle spaces of S
$T\varphi$	derived map of φ
$T_p^\perp L$	symplectic complement of T_pL
U, V, W	open subsets
$\boldsymbol{U}$	unitary representation
X, Y, Z	global derivations; vector fields; sections of tangent bundle
$X(f)$	evaluation of X on f
$\mathfrak{X}(S)$	family of all vector fields on S

Lower case Latin alphabet

a, b	real numbers
c	complex number
$c : I \to S$	curve in S
d	differential

e	group identity
$\exp : \mathfrak{g} \to G$	exponential map
$\exp(tX)$	local one-parameter local group of diffeomorphisms defined by X
$\exp(tX)(x)$	point on maximal integral curve of X through x
f	function
$f^{-1}(I)$	inverse image of I under f
g	element of group G
$\mathfrak{g}$	Lie algebra of G
$\mathfrak{g}^*$	dual of $\mathfrak{g}$
h	function
$\hbar$	Planck's constant divided by 2π
$\mathfrak{h}$	Lie algebra of H
$\mathfrak{h}^*$	dual of $\mathfrak{h}$
hor TP	horizontal distribution on P
$\boldsymbol{k}$	Riemannian metric
$\mathfrak{m}$	Lie algebra
$\mathfrak{n}$	Lie algebra of N
p	point
q	point
s	parameter
supp f	support of f
t	parameter
u, v	derivation at a point; vector
ver TP	vertical distribution on P
w	derivation at a point; vector
x, y	point
z	point of $\mathcal{L}$

Lower case Greek alphabet

α	1-form
β	1-form
δ_{ij}	Kronecker δ
ζ, η	elements of Lie algebra
θ	1-form
ϑ	cotangent bundle projection
λ, μ, ν	elements of co-adjoint orbit
$\lambda : \mathcal{L} \to P$	complex line bundle projection
ξ	element of Lie algebra

π	projection map
ϖ	form; distributional symplectic form in Chapter 8
ρ	map
ρ^*	pull-back by ρ
ρ_*	push-forward by ρ
σ	section
σ^*	pull-back by σ
σ_*	push-forward by σ
τ	map; tangent bundle projection
τ^*	pull-back by τ
φ	map
φ^*	pull-back by φ
φ_*	push-forward by φ
ω	symplectic form

Upper case Greek alphabet

Λ	Lagrangian submanifold
Π	projection of G-invariant section
Σ	restriction of section
Φ	action
Ψ	action
Ω	symplectic form of co-adjoint orbit
$\mathbf{\Omega}^k_K(S)$	space of Koszul k-forms on S
$\mathbf{\Omega}^k_M(S)$	space of Marshall k-forms on S
$\mathbf{\Omega}^k_Z(S)$	space of Zariski k-forms on S

Non-alphabetic symbols

∇	covariant derivative
$\langle \cdot \mid \cdot \rangle$	evaluation; sesquilinear form on a line bundle
$\sqrt{\lvert \wedge^n F \rvert}$	half-densities on F
$\lrcorner$	left interior product
$[\cdot, \cdot]$	Lie bracket
$\{\cdot, \cdot\}$	Poisson bracket
$_{\mid}$	restriction
$(\cdot \mid \cdot)$	scalar product on a Hilbert space

Contents

Preface — *page* vii
List of selected symbols — ix

1 Introduction — 1

PART I DIFFERENTIAL GEOMETRY OF SINGULAR SPACES

2 Differential structures — 15
2.1 Differential spaces — 15
2.2 Partitions of unity — 21

3 Derivations — 25
3.1 Basic properties — 25
3.2 Integration of derivations — 31
3.3 The tangent bundle — 37
3.4 Orbits of families of vector fields — 44

4 Stratified spaces — 52
4.1 Stratified subcartesian spaces — 52
4.2 Action of a Lie group on a manifold — 56
4.3 Orbit space — 67
4.4 Action of a Lie group on a subcartesian space — 81

5 Differential forms — 91
5.1 Koszul forms — 91
5.2 Zariski forms — 94
5.3 Marshall forms — 99

PART II REDUCTION OF SYMMETRIES

6 Symplectic reduction 105
6.1 Symplectic manifolds with symmetry 105
6.1.1 Co-adjoint orbits 105
6.1.2 Symplectic manifolds 108
6.1.3 Poisson algebra 110
6.2 Poisson reduction 111
6.3 Level sets of the momentum map 114
6.4 Pre-images of co-adjoint orbits 125
6.5 Reduction by stages for proper actions 126
6.6 Shifting 129
6.7 When the action is free 134
6.8 When the action is improper 135
6.9 Algebraic reduction 136

7 Commutation of quantization and reduction 150
7.1 Review of geometric quantization 151
7.1.1 Prequantization 152
7.1.2 Polarization 155
7.1.3 Examples of unitarization 156
7.2 Commutation of quantization and singular reduction at $J = 0$ 161
7.3 Special cases 184
7.3.1 The results of Guillemin and Sternberg 184
7.3.2 Kähler polarization without compactness assumptions 185
7.3.3 Real polarization 187
7.4 Non-zero co-adjoint orbits 190
7.5 Commutation of quantization and algebraic reduction 203
7.5.1 Quantization of algebraic reduction 203
7.5.2 Kähler polarization 206
7.5.3 Real polarization 207
7.5.4 Improper action 208

8 Further examples of reduction 211
8.1 Non-holonomic reduction 211
8.2 Dirac structures 214
8.2.1 Symmetries of the Pontryagin bundle 215
8.2.2 Free and proper action 217
8.2.3 Proper non-free action 223

References 228
Index 233

1

Introduction

This book is written for researchers and graduate students in the field of geometric mechanics, especially the theory of systems with symmetries. A wider audience might include differential geometers, algebraic geometers and singularity theorists. The aim of the book is to show that differential geometry in the sense of Sikorski is a powerful tool for the study of the geometry of spaces with singularities. We show that this understanding of differential geometry gives a complete description of the stratification structure of the space of orbits of a proper action of a connected Lie group G on a manifold P. We also show that the same approach can handle intersection singularities; see Section 8.2.

We assume here that the reader has a working knowledge of differential geometry and the topology of manifolds, and we use theorems in these fields freely without giving proofs or references. On the other hand, the material on differential spaces is developed from scratch. The results on differential spaces are proved in detail. This should make the book accessible to graduate students.

The book is split into two parts. In Part I, we introduce the reader to the differential geometry of singular spaces and prove some results, which are used in Part II to investigate concrete systems. The technique of differential geometry presented here is fairly straightforward, and the reader might get a false impression that the scope of the theory does not differ much from that of the geometry of manifolds. However, the examples given in Part I will serve as warnings that such an impression is false. Part II is devoted to applications of the general theory. Each chapter in this part may be considered as an extensive example of the use of differential geometry to deal with singularities in concrete problems. Since these problems occur in various theories, each chapter begins with a section introducing elements of the underlying theory, in order to show the reader the relevance of the problem under consideration.

The book contains no exercises, because the actual techniques involved are very simple. In addition to the standard techniques of the differential geometry of manifolds, we use techniques of algebraic geometry for rings of smooth functions. The fact that algebraically defined derivations of smooth functions admit integral curves is the main difference between differential and algebraic geometry.

The technical details of the presentation are based on the TEX style file chosen for the preparation of this book. Displayed results are labelled by the number of the chapter, the number of the section in the chapter and the number of the result within the section. For example, 'Lemma 2.1.3' stands for Lemma 1.3 in Chapter 2; it can also be read as the third lemma in Section 2.1. Displayed equations are referenced by the number of the chapter and the number of the equation within the chapter. For example, 'equation (3.21)' stands for equation 21 in Chapter 3.

This book is based on several years of research. Some of the results presented here were obtained by the author. Some other results have been taken directly from the work of other researchers. The remainder corresponds to an adaptation and reformulation of the work of other authors so that it fits into the theory presented here. In order to keep the flow of the presentation in the subsequent chapters free from obstructions, we give below a detailed description of the content of the book and the references to the literature.

Part I is devoted to a comprehensive presentation of the current status of the differential geometry of singular spaces. A comprehensive bibliography of the literature on differential spaces during the period 1965–1992 was published in 1993 by Buchner, Heller, Multarzyński and Sasin (Buchner *et al.*, 1993). According to these authors, the first paper on differential spaces was Sikorski (1967). In the same year, at a meeting of the American Mathematical Society, Aronszajn presented an extensive programme of differential-geometric study of subcartesian spaces in terms of singular charts. Aronszajn's subcartesian spaces included arbitrary subspaces of $\mathbb{R}^n$ (see Aronszajn, 1967). In 1973, Walczak showed that subcartesian spaces are special cases of differential spaces (see Walczak, 1973).

In Section 2.1, we describe the basic definitions and constructions of Sikorski's theory following his book (see Sikorski, 1972). The fundamental notion of this theory is the differential structure $C^\infty(S)$ of a space S, consisting of functions on S deemed to be smooth. The differential structure of a space carries all information about the geometry of the space. In particular, a map $\varphi : S \to T$ is smooth if it pulls back smooth functions to smooth functions. A diffeomorphism is an invertible smooth map with a smooth inverse. As in topology, subsets, products and quotients of differential spaces are differential spaces. However, the quotient differential space need not have the quotient topology. Proposition 2.1.11, which gives conditions for equivalence of the quotient differential-space topology and the quotient topology, is taken from the work of Pasternak-Winiarski (1984) .

A differential space S is subcartesian if every point of S has a neighbourhood diffeomorphic to a subset of some Cartesian space $\mathbb{R}^n$. The category of subcartesian differential spaces is the main object of our study. Manifolds are subcartesian spaces that are locally diffeomorphic to open subsets of $\mathbb{R}^n$. If M is a manifold, the collection of all local diffeomorphisms to open subsets of $\mathbb{R}^n$ forms the maximal atlas on M. Differential geometry, understood as the study of the geometry of a space in terms of the ring of smooth functions on that space, naturally extends from manifolds to subcartesian spaces. We do not go beyond subcartesian spaces, because a differential space which is not subcartesian need not have a locally finite dimension.

In Section 2.2, we show that subcartesian spaces admit partitions of unity. The importance of partitions of unity stems from the fact that they enable us to globalize collections of local data. The existence of partitions of unity on locally compact and paracompact differential spaces was first proved by Cegiełka (1974). Here, we follow the proof of Marshall (1975a).

In Chapter 3, we discuss vector fields on subcartesian spaces. A vector field on a manifold M can be described either as a derivation of a ring $C^\infty(M)$ of smooth functions on M or as a generator of a local one-parameter group of local diffeomorphisms of M. These two notions are equivalent if M is a manifold. However, they may be inequivalent on a subcartesian space S that is not a manifold.

In Section 3.1, we study the basic properties of derivations of the differential structure $C^\infty(S)$ of a subcartesian space S. We show that every derivation X of $C^\infty(S)$ can be locally extended to a derivation of $C^\infty(R^n)$. This result allows the study of ordinary differential equations on subcartesian spaces, which we discuss in Section 3.2. The existence and uniqueness theorem for integral curves of derivations on a subcartesian space was first proved by Śniatycki (2003a).

In Section 3.3, we discuss the tangent bundle space TS of S, defined as the space of derivations of $C^\infty(S)$ at points of S. In the literature, TS is also called the tangent pseudobundle or the Zariski tangent bundle. Following Watts (2006), we define the regular component S_{reg} of S as the set of all points p of S at which $\dim T_pS$ is locally constant, and prove that S_{reg} is open and dense in S and that the restriction TS_{reg} of TS to S_{reg} is locally spanned by global derivations; see Lusala *et al.* (2010). Example 3.3.12, taken from Epstein and

Śniatycki (2006), shows that a differential space that is regular everywhere need not be a manifold.

In Section 3.4, we study global derivations of S that generate local one-parameter groups of local diffeomorphisms. We call such global derivations vector fields. We show that the orbits of any family of vector fields on a subcartesian space S are smooth manifolds immersed in S. This result, first proved by Śniatycki (2003b), is a generalization of some theorems of Sussmann (1973) and Stefan (1974). In particular, it implies that orbits of the family $\mathfrak{X}(S)$ of all vector fields on S give a partition of S by smooth manifolds. Therefore, every subcartesian space S has a minimal partition by smooth manifolds. This result gives us an alternative interpretation of the strata of a minimal stratification of a subcartesian space, which we study in Chapter 4.

In Chapter 4, we discuss stratified spaces, first investigated by Whitney (1955), who called them 'manifold collections'. The term 'stratification' is due to Thom (1955–56). A stratified space is usually described as a topological space partitioned in a special way by smooth manifolds. Here, we restrict our considerations to stratified spaces that are also subcartesian differential spaces.

In Section 4.1, we discuss stratified subcartesian spaces following the work of Śniatycki (2003b) and Lusala and Śniatycki (2011). A stratified space is, by definition, partitioned by smooth manifolds. The results of Chapter 3 show that a subcartesian space is also partitioned by smooth manifolds, which are orbits of the family of all vector fields. We show that if a stratified space S is subcartesian and the stratification of S is locally trivial, then the partition of S by orbits of the family of all vector fields is also a stratification of S. Moreover, this second stratification of S is coarser than the original stratification. If the original stratification is minimal, then it is the same as the stratification given by the orbits of the family of all vector fields. In other words, a minimal locally trivial stratification of a subcartesian space is completely determined by its differential structure.

In Section 4.2, we describe the orbit type stratification $\mathfrak{M}$ of a manifold P given by a proper action on P of a connected Lie group G. This stratification is not minimal, because the union of all the strata is the manifold P. The presentation adopted here borrows from the presentations of the same topic in the books by Cushman and Bates (1997), Duistermaat and Kolk (2000), and Pflaum (2001).

Section 4.3 is devoted to a discussion of the structure of the orbit space $R = P/G$. We show that the projection to the orbit space R of the strata of $\mathfrak{M}$ is a locally trivial and minimal stratification of R. This is called the orbit type stratification of the orbit space R. We also show that R is a subcartesian space.