$$\frac{\partial x_i}{\partial p}(p,w) = -\frac{\partial^2 \Pi^*}{\partial p \partial w_i} = -\frac{\partial^2 \Pi^*}{\partial w_i \partial p} = -\frac{\partial^2 F}{\partial w_i}(p,w).$$

Mathematics for Economists

CARL P. SIMON • LAWRENCE BLUME

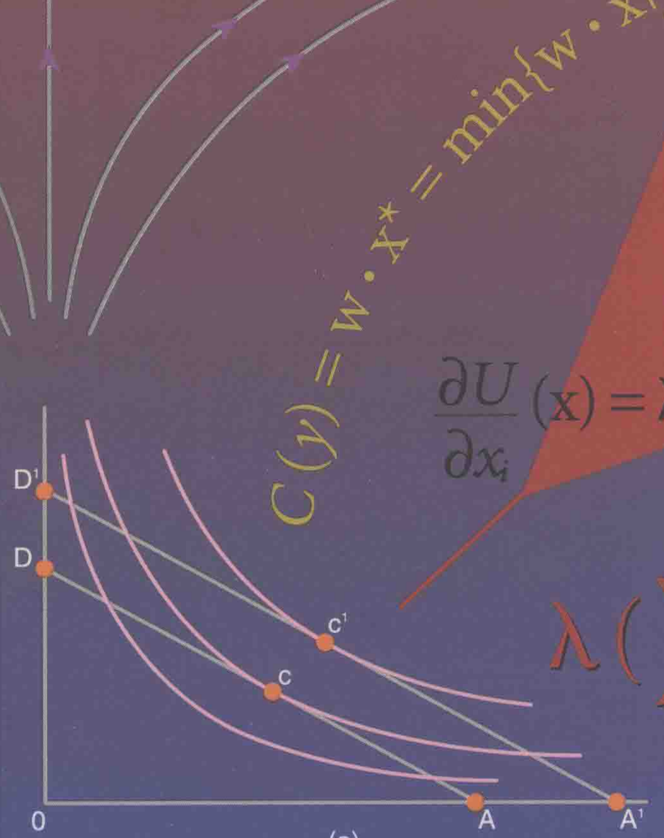
$$C(y) = w \cdot x^* = \min\{w \cdot x \mid f(x) = y, x \geq 0\}$$

$$\dot{x} = x(r - bx)$$

$$\lambda \geq 0$$

$$\frac{\partial U}{\partial x_i}(x) = \lambda p_i \quad i = 1, \ldots, n$$

$$\lambda\left(\sum_i p_i x_i - I\right) = 0$$



MATHEMATICS FOR ECONOMISTS

Carl P. Simon
and
Lawrence Blume

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MATHEMATICS FOR ECONOMISTS

Preface

For better or worse, mathematics has become the language of modern analytical economics. It quantifies the relationships between economic variables and among economic actors. It formalizes and clarifies properties of these relationships. In the process, it allows economists to identify and analyze those general properties that are critical to the behavior of economic systems.

Elementary economics courses use reasonably simple mathematical techniques to describe and analyze the models they present: high school algebra and geometry, graphs of functions of one variable, and sometimes one-variable calculus. They focus on models with one or two goods in a world of perfect competition, complete information, and no uncertainty. Courses beyond introductory micro- and macroeconomics drop these strong simplifying assumptions. However, the mathematical demands of these more sophisticated models scale up considerably. The goal of this text is to give students of economics and other social sciences a deeper understanding and working knowledge of the mathematics they need to work with these more sophisticated, more realistic, and more interesting models.

WHY THIS BOOK?

We wrote this book because we felt that the available texts on mathematics for economists left unfilled some of the basic needs of teachers and students in this area. In particular, we tried to make the following improvements over other texts.

1. Many texts in this area focus on mathematical *techniques* at the expense of mathematical *ideas* and *intuition*, often presenting a "cookbook approach." Our book develops the student's intuition for how and why the various mathematical techniques work. It contains many more illustrations and figures than competing texts in order to build the reader's geometric intuition. It emphasizes the primary role of calculus in approximating a nonlinear function by a linear function or polynomial in order to build a simple picture of the behavior of the nonlinear function — a principle rich in geometric content.

2. Students learn how to use and apply mathematics by working with concrete examples and exercises. We illustrate every new concept and technique with worked-out examples. We include exercises at the end of every section to give students the necessary experience working with the mathematics presented.

3. This is a book on using mathematics to understand the structure of economics. We believe that this book contains more economics than any other

math-for-economists text. Each chapter begins with a discussion of the economic motivation for the mathematical concepts presented. On the other hand, this is a book on mathematics for economists, not a text of mathematical economics. We do not feel that it is productive to learn advanced mathematics and advanced economics at the same time. Therefore, we have focused on presenting an introduction to the mathematics that students need in order to work with more advanced economic models.

4. Economics is a dynamic field; economic theorists are regularly introducing or using new mathematical ideas and techniques to shed light on economic theory and econometric analysis. As active researchers in economics, we have tried to make many of these new approaches available to students. In this book we present rather complete discussions of topics at the frontier of economic research, topics like quasiconcave functions, concave programing, indirect utility and expenditure functions, envelope theorems, the duality between cost and production, and nonlinear dynamics.

5. It is important that students of economics understand what constitutes a solid proof — a skill that is learned, not innate. Unlike most other texts in the field, we try to present careful proofs of nearly all the mathematical results presented — so that the reader can understand better both the logic behind the math techniques used and the total structure in which each result builds upon previous results. In many of the exercises, students are asked to work out their own proofs, often by adapting proofs presented in the text.

An important motivation for understanding what constitutes a careful proof is the need for students to develop the ability to read an argument and to decide for themselves whether or not the conclusions really do follow from the stated hypotheses. Furthermore, a good proof tells a story; it can be especially valuable by laying bare the underlying structure of a model in such a way that one clearly sees which of the model's component parts are responsible for producing the behavior asserted in the statement of the economic principle. Some readers of this text will go on to draw conclusions from economic models in their own research. We hope that the experience of working with proofs in this text will be a valuable guide to developing one's own ability to read and write proofs.

WHAT'S IN THIS BOOK?

At the core of modern microeconomics is the hypothesis that economic agents consciously choose their most preferred behavior according to the alternatives available to them. The area of mathematics most relevant to such a study is the maximization or minimization of a function of several variables in which the variables are constrained by equalities and inequalities. This mathematical problem in all the necessary generality, sometimes called the Lagrange multiplier problem, is a focal point of this book. (See especially Chapters 16 to 19.) The chapters of this book are arranged so that this material can be reached quickly and efficiently.

This text begins with overviews of one-variable calculus (Chapters 2 to 4) and of exponentials and logarithms (Chapter 5). One can either cover this material during the first weeks of the class or, more commonly we believe, can ask students to read it on their own as a review of the calculus they have taken. The examples and exercises in these earliest chapters should make either process relatively simple.

The analysis of solutions to optimization problems usually involves studying the solutions to the systems of equations given by the first-order conditions. The first half of this book focuses on the study of such systems of equations. We first develop a rather complete theory of the solutions of *linear* systems, focusing on such questions as: Does a solution exist? How many are there? What happens to the solution as the equations change a little? (Chapters 6 to 10.) We then turn to the study of the more realistic and more complex *nonlinear* systems (Chapters 11 to 15). We apply the metaprinciple of calculus to this study of nonlinear systems: the best way to study the behavior of the solutions of a nonlinear system is to examine the behavior of a closely related *linear* system of equations. Finally, we pull all this material together in Chapters 16 to 19 in our discussion of optimization problems — unconstrained and constrained — that is the heart of this text.

Chapters 20 through 25 treat two other basic mathematical issues that arise in the study of economic models. Chapters 20 and 21 give an in-depth presentation of *properties* of economic relationships, such as homogeneity, concavity, and quasiconcavity, while Chapter 22 illustrates how these properties arise naturally in economic models. Furthermore, there are often natural *dynamics* in economic processes: prices adjust, economies grow, policies adapt, economic agents maximize over time. Chapters 23, 24, and 25 introduce the mathematics of dynamic systems, focusing on the eigenvalues of a matrix, linear difference equations, and linear and nonlinear differential equations.

This book is laid out so that one can get to the fundamental results and consequences of constrained optimization problems as quickly as possible. In some cases, for example, in the study of determinants, limits of sequences, and compact sets, there are important topics that are slightly off the beaten path to the study of constrained optimization problems. To keep the presentation as flexible as possible, we have placed the description of these topics in the last five chapters of this book. Chapter 26 presents details about the properties of determinants outlined in Chapter 9. Chapter 27 completes the application of matrix algebra in Chapters 7 and 8 to the determination of the size of the set of solutions of a linear system, ending with a discussion of the Fundamental Theorem of Matrix Algebra. Chapter 28 presents economic applications of the Fundamental Theorem. Chapter 29 does some fine-tuning on the study of sets and sequences introduced in Chapter 12. Chapter 30 collects some of the more complex proofs of the multivariable analysis presented in Chapters 13, 14, and 15. In classroom presentations the material in any of these last five chapters can be presented: 1) right after the corresponding material in the earlier chapter, 2) at the end of the course, or 3) not at all, depending on the amount of time available or the needs of the students.

COORDINATION WITH OTHER COURSES

Often the material in this course is taught concurrently with courses in advanced micro- and macroeconomics. Students are sometimes frustrated with this arrangement because the micro and macro courses usually start working with constrained optimization or dynamics long before these topics can be covered in an orderly mathematical presentation.

We suggest a number of strategies to minimize this frustration. First, we have tried to present the material so that a student can read each introductory chapter in isolation and get a reasonably clear idea of how to work with the material of that chapter, even without a careful reading of earlier chapters. We have done this by including a number of worked exercises with descriptive figures in every introductory chapter.

Often during the first two weeks of our first course on this material, we present a series of short modules that introduces the language and formulation of the more advanced topics so that students can easily read selected parts of later chapters on their own, or at least work out some problems from these chapters.

Finally, we usually ask students who will be taking our course to be familiar with the chapters on one-variable calculus and simple matrix theory before classes begin. We have found that nearly every student has taken a calculus course and nearly two-thirds have had some matrix algebra. So this summer reading requirement — sometimes supplemented by a review session just before classes begin — is helpful in making the mathematical backgrounds of the students in the course more homogeneous.

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We dedicate this book to our wives Susan and Maralyn.

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