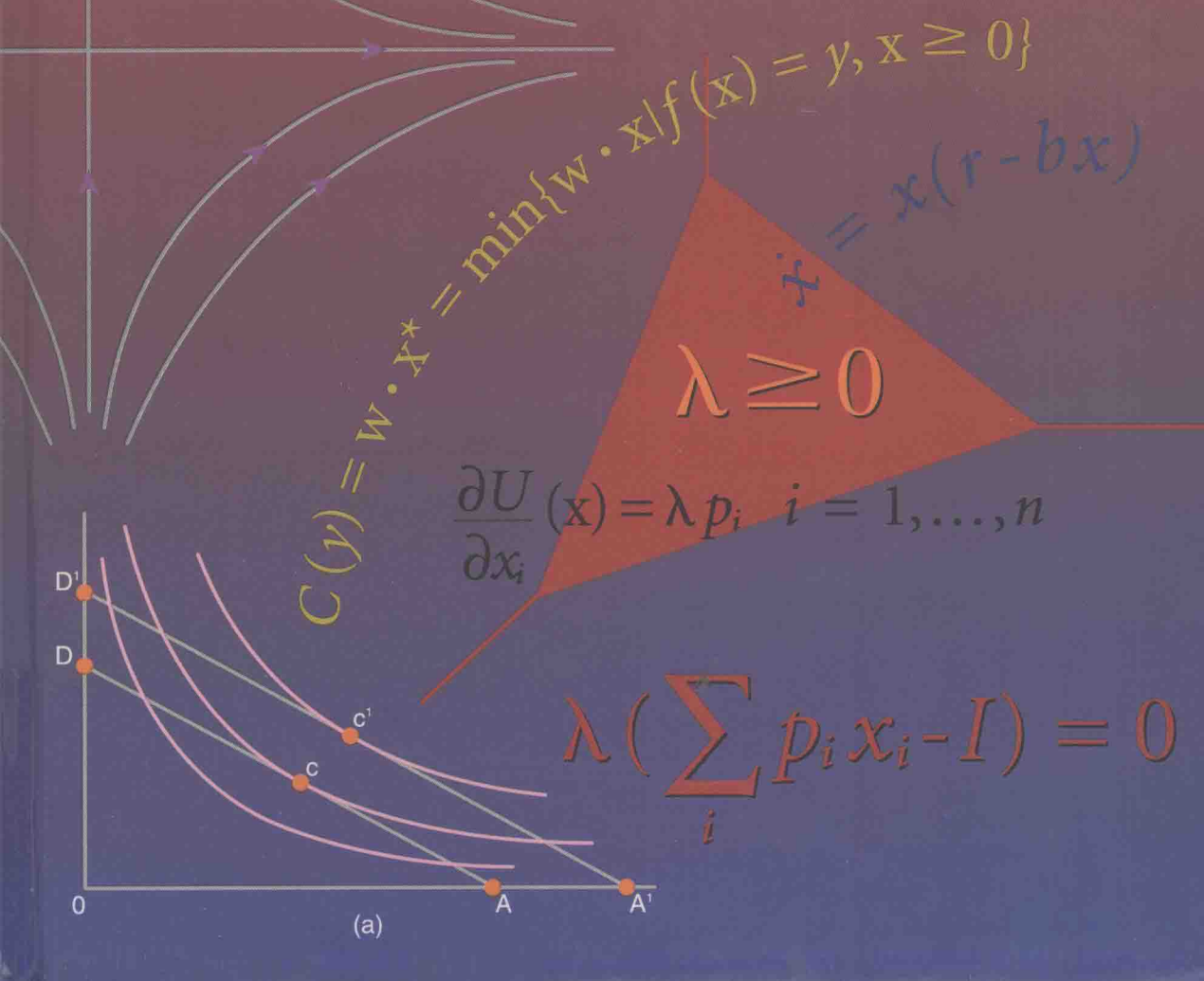


$$\frac{\partial \chi_i}{\partial p}(p, w) = - \frac{\partial^2 \Pi^*}{\partial p \partial w_i} = - \frac{\partial^2 \Pi^*}{\partial w_i \partial p} = - \frac{\partial^2 F}{\partial w_i}(p, w).$$

Mathematics for Economists

CARL P. SIMON - LAWRENCE BLUME



MATHEMATICS FOR ECONOMISTS

Carl P. Simon
and
Lawrence Blume



W • W • NORTON & COMPANY • NEW YORK • LONDON

Copyright © 1994 by W. W. Norton & Company, Inc.

ALL RIGHTS RESERVED
PRINTED IN THE UNITED STATES OF AMERICA
FIRST EDITION

The text of this book is composed in Times Roman, with the display set in Optima. Composition by Integre Technical Publishing Company, Inc. Book design by Jack Meserole.

Library of Congress Cataloging-in-Publication Data

Blume, Lawrence.

Mathematics for economists / Lawrence Blume and Carl Simon.

p. cm.

1. Economics, Mathematical. I. Simon, Carl P., 1945- .

II. Title.

HB135.B59 1994

510/.24339—dc20 93-24962

ISBN 0-393-95733-0

W. W. Norton & Company, Inc., 500 Fifth Avenue, New York, N.Y. 10110
www.wwnorton.com

W. W. Norton & Company Ltd., Castle House, 75/76 Wells Street,
London W1T 3QT

MATHEMATICS FOR ECONOMISTS

Preface

For better or worse, mathematics has become the language of modern analytical economics. It quantifies the relationships between economic variables and among economic actors. It formalizes and clarifies properties of these relationships. In the process, it allows economists to identify and analyze those general properties that are critical to the behavior of economic systems.

Elementary economics courses use reasonably simple mathematical techniques to describe and analyze the models they present: high school algebra and geometry, graphs of functions of one variable, and sometimes one-variable calculus. They focus on models with one or two goods in a world of perfect competition, complete information, and no uncertainty. Courses beyond introductory micro- and macroeconomics drop these strong simplifying assumptions. However, the mathematical demands of these more sophisticated models scale up considerably. The goal of this text is to give students of economics and other social sciences a deeper understanding and working knowledge of the mathematics they need to work with these more sophisticated, more realistic, and more interesting models.

WHY THIS BOOK?

We wrote this book because we felt that the available texts on mathematics for economists left unfilled some of the basic needs of teachers and students in this area. In particular, we tried to make the following improvements over other texts.

1. Many texts in this area focus on mathematical *techniques* at the expense of mathematical *ideas* and *intuition*, often presenting a “cookbook approach.” Our book develops the student’s intuition for how and why the various mathematical techniques work. It contains many more illustrations and figures than competing texts in order to build the reader’s geometric intuition. It emphasizes the primary role of calculus in approximating a nonlinear function by a linear function or polynomial in order to build a simple picture of the behavior of the nonlinear function — a principle rich in geometric content.

2. Students learn how to use and apply mathematics by working with concrete examples and exercises. We illustrate every new concept and technique with worked-out examples. We include exercises at the end of every section to give students the necessary experience working with the mathematics presented.

3. This is a book on using mathematics to understand the structure of economics. We believe that this book contains more economics than any other

math-for-economists text. Each chapter begins with a discussion of the economic motivation for the mathematical concepts presented. On the other hand, this is a book on mathematics for economists, not a text of mathematical economics. We do not feel that it is productive to learn advanced mathematics and advanced economics at the same time. Therefore, we have focused on presenting an introduction to the mathematics that students need in order to work with more advanced economic models.

4. Economics is a dynamic field; economic theorists are regularly introducing or using new mathematical ideas and techniques to shed light on economic theory and econometric analysis. As active researchers in economics, we have tried to make many of these new approaches available to students. In this book we present rather complete discussions of topics at the frontier of economic research, topics like quasiconcave functions, concave programming, indirect utility and expenditure functions, envelope theorems, the duality between cost and production, and nonlinear dynamics.

5. It is important that students of economics understand what constitutes a solid proof — a skill that is learned, not innate. Unlike most other texts in the field, we try to present careful proofs of nearly all the mathematical results presented — so that the reader can understand better both the logic behind the math techniques used and the total structure in which each result builds upon previous results. In many of the exercises, students are asked to work out their own proofs, often by adapting proofs presented in the text.

An important motivation for understanding what constitutes a careful proof is the need for students to develop the ability to read an argument and to decide for themselves whether or not the conclusions really do follow from the stated hypotheses. Furthermore, a good proof tells a story; it can be especially valuable by laying bare the underlying structure of a model in such a way that one clearly sees which of the model's component parts are responsible for producing the behavior asserted in the statement of the economic principle. Some readers of this text will go on to draw conclusions from economic models in their own research. We hope that the experience of working with proofs in this text will be a valuable guide to developing one's own ability to read and write proofs.

WHAT'S IN THIS BOOK?

At the core of modern microeconomics is the hypothesis that economic agents consciously choose their most preferred behavior according to the alternatives available to them. The area of mathematics most relevant to such a study is the maximization or minimization of a function of several variables in which the variables are constrained by equalities and inequalities. This mathematical problem in all the necessary generality, sometimes called the Lagrange multiplier problem, is a focal point of this book. (See especially Chapters 16 to 19.) The chapters of this book are arranged so that this material can be reached quickly and efficiently.

This text begins with overviews of one-variable calculus (Chapters 2 to 4) and of exponentials and logarithms (Chapter 5). One can either cover this material during the first weeks of the class or, more commonly we believe, can ask students to read it on their own as a review of the calculus they have taken. The examples and exercises in these earliest chapters should make either process relatively simple.

The analysis of solutions to optimization problems usually involves studying the solutions to the systems of equations given by the first-order conditions. The first half of this book focuses on the study of such systems of equations. We first develop a rather complete theory of the solutions of *linear* systems, focusing on such questions as: Does a solution exist? How many are there? What happens to the solution as the equations change a little? (Chapters 6 to 10.) We then turn to the study of the more realistic and more complex *nonlinear* systems (Chapters 11 to 15). We apply the metaprinciple of calculus to this study of nonlinear systems: the best way to study the behavior of the solutions of a nonlinear system is to examine the behavior of a closely related *linear* system of equations. Finally, we pull all this material together in Chapters 16 to 19 in our discussion of optimization problems — unconstrained and constrained — that is the heart of this text.

Chapters 20 through 25 treat two other basic mathematical issues that arise in the study of economic models. Chapters 20 and 21 give an in-depth presentation of *properties* of economic relationships, such as homogeneity, concavity, and quasiconcavity, while Chapter 22 illustrates how these properties arise naturally in economic models. Furthermore, there are often natural *dynamics* in economic processes: prices adjust, economies grow, policies adapt, economic agents maximize over time. Chapters 23, 24, and 25 introduce the mathematics of dynamic systems, focusing on the eigenvalues of a matrix, linear difference equations, and linear and nonlinear differential equations.

This book is laid out so that one can get to the fundamental results and consequences of constrained optimization problems as quickly as possible. In some cases, for example, in the study of determinants, limits of sequences, and compact sets, there are important topics that are slightly off the beaten path to the study of constrained optimization problems. To keep the presentation as flexible as possible, we have placed the description of these topics in the last five chapters of this book. Chapter 26 presents details about the properties of determinants outlined in Chapter 9. Chapter 27 completes the application of matrix algebra in Chapters 7 and 8 to the determination of the size of the set of solutions of a linear system, ending with a discussion of the Fundamental Theorem of Matrix Algebra. Chapter 28 presents economic applications of the Fundamental Theorem. Chapter 29 does some fine-tuning on the study of sets and sequences introduced in Chapter 12. Chapter 30 collects some of the more complex proofs of the multivariable analysis presented in Chapters 13, 14, and 15. In classroom presentations the material in any of these last five chapters can be presented: 1) right after the corresponding material in the earlier chapter, 2) at the end of the course, or 3) not at all, depending on the amount of time available or the needs of the students.

COORDINATION WITH OTHER COURSES

Often the material in this course is taught concurrently with courses in advanced micro- and macroeconomics. Students are sometimes frustrated with this arrangement because the micro and macro courses usually start working with constrained optimization or dynamics long before these topics can be covered in an orderly mathematical presentation.

We suggest a number of strategies to minimize this frustration. First, we have tried to present the material so that a student can read each introductory chapter in isolation and get a reasonably clear idea of how to work with the material of that chapter, even without a careful reading of earlier chapters. We have done this by including a number of worked exercises with descriptive figures in every introductory chapter.

Often during the first two weeks of our first course on this material, we present a series of short modules that introduces the language and formulation of the more advanced topics so that students can easily read selected parts of later chapters on their own, or at least work out some problems from these chapters.

Finally, we usually ask students who will be taking our course to be familiar with the chapters on one-variable calculus and simple matrix theory before classes begin. We have found that nearly every student has taken a calculus course and nearly two-thirds have had some matrix algebra. So this summer reading requirement — sometimes supplemented by a review session just before classes begin — is helpful in making the mathematical backgrounds of the students in the course more homogeneous.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the valuable suggestions and comments of our colleagues, students and reviewers: colleagues such as Philippe Artzner, Ted Bergstrom, Ken Binmore, Dee Dechert, David Easley, Leonard Herk, Phil Howrey, John Jacquez, Jan Kmenta, James Koopman, Tapan Mitra, Peter Morgan, John Nachbar, Scott Pierce, Zvi Safra, Hal Varian, and Henry Wan; students such as Kathleen Anderson, Jackie Coolidge, Don Dunbar, Tom George, Kevin Jackson, David Meyer, Ann Simon, David Simon, and John Wooders, and the countless classes at Cornell and Michigan who struggled through early drafts; reviewers such as Richard Anderson, Texas A & M University; James Bergin, Queen's University; Brian Binger, University of Arizona; Mark Feldman, University of Illinois; Roger Folsom, San Jose State University; Femida Handy, York University; John McDonald, University of Illinois; Norman Obst, Michigan State University; John Riley, University of California at Los Angeles; and Myrna Wooders, University of Toronto. We appreciate the assistance of the people at W.W. Norton, especially Drake McFeely, Catherine Wick and Catherine Von Novak. The order of the authors on the cover of this book merely reflects our decision to use different orders for different books that we write.

We dedicate this book to our wives Susan and Maralyn.

Contents

Preface xxi

P A R T I Introduction

1	Introduction	3
1.1	MATHEMATICS IN ECONOMIC THEORY	3
1.2	MODELS OF CONSUMER CHOICE	5
	Two-Dimensional Model of Consumer Choice	5
	Multidimensional Model of Consumer Choice	9
2	One-Variable Calculus: Foundations	10
2.1	FUNCTIONS ON \mathbf{R}^1	10
	Vocabulary of Functions	10
	Polynomials	11
	Graphs	12
	Increasing and Decreasing Functions	12
	Domain	14
	Interval Notation	15
2.2	LINEAR FUNCTIONS	16
	The Slope of a Line in the Plane	16
	The Equation of a Line	19
	Polynomials of Degree One Have Linear Graphs	19
	Interpreting the Slope of a Linear Function	20
2.3	THE SLOPE OF NONLINEAR FUNCTIONS	22
2.4	COMPUTING DERIVATIVES	25
	Rules for Computing Derivatives	27

2.5	DIFFERENTIABILITY AND CONTINUITY	29
	A Nondifferentiable Function	30
	Continuous Functions	31
	Continuously Differentiable Functions	32
2.6	HIGHER-ORDER DERIVATIVES	33
2.7	APPROXIMATION BY DIFFERENTIALS	34
3	One-Variable Calculus: Applications	39
3.1	USING THE FIRST DERIVATIVE FOR GRAPHING	39
	Positive Derivative Implies Increasing Function	39
	Using First Derivatives to Sketch Graphs	41
3.2	SECOND DERIVATIVES AND CONVEXITY	43
3.3	GRAPHING RATIONAL FUNCTIONS	47
	Hints for Graphing	48
3.4	TAILS AND HORIZONTAL ASYMPTOTES	48
	Tails of Polynomials	48
	Horizontal Asymptotes of Rational Functions	49
3.5	MAXIMA AND MINIMA	51
	Local Maxima and Minima on the Boundary and in the Interior	51
	Second Order Conditions	53
	Global Maxima and Minima	55
	Functions with Only One Critical Point	55
	Functions with Nowhere-Zero Second Derivatives	56
	Functions with No Global Max or Min	56
	Functions Whose Domains Are Closed Finite Intervals	56
3.6	APPLICATIONS TO ECONOMICS	58
	Production Functions	58
	Cost Functions	59
	Revenue and Profit Functions	62
	Demand Functions and Elasticity	64
4	One-Variable Calculus: Chain Rule	70
4.1	COMPOSITE FUNCTIONS AND THE CHAIN RULE	70
	Composite Functions	70
	Differentiating Composite Functions: The Chain Rule	72
4.2	INVERSE FUNCTIONS AND THEIR DERIVATIVES	75
	Definition and Examples of the Inverse of a Function	75
	The Derivative of the Inverse Function	79
	The Derivative of $x^{m/n}$	80

5	Exponents and Logarithms	82
5.1	EXPONENTIAL FUNCTIONS	82
5.2	THE NUMBER e	85
5.3	LOGARITHMS	88
	Base 10 Logarithms	88
	Base e Logarithms	90
5.4	PROPERTIES OF EXP AND LOG	91
5.5	DERIVATIVES OF EXP AND LOG	93
5.6	APPLICATIONS	97
	Present Value	97
	Annuities	98
	Optimal Holding Time	99
	Logarithmic Derivative	100

P A R T I I Linear Algebra

6	Introduction to Linear Algebra	107
6.1	LINEAR SYSTEMS	107
6.2	EXAMPLES OF LINEAR MODELS	108
	Example 1: Tax Benefits of Charitable Contributions	108
	Example 2: Linear Models of Production	110
	Example 3: Markov Models of Employment	113
	Example 4: IS-LM Analysis	115
	Example 5: Investment and Arbitrage	117
7	Systems of Linear Equations	122
7.1	GAUSSIAN AND GAUSS-JORDAN ELIMINATION	122
	Substitution	123
	Elimination of Variables	125
7.2	ELEMENTARY ROW OPERATIONS	129
7.3	SYSTEMS WITH MANY OR NO SOLUTIONS	134
7.4	RANK — THE FUNDAMENTAL CRITERION	142
	Application to Portfolio Theory	147
7.5	THE LINEAR IMPLICIT FUNCTION THEOREM	150

8	Matrix Algebra	153
8.1	MATRIX ALGEBRA	153
	Addition	153
	Subtraction	154
	Scalar Multiplication	155
	Matrix Multiplication	155
	Laws of Matrix Algebra	156
	Transpose	157
	Systems of Equations in Matrix Form	158
8.2	SPECIAL KINDS OF MATRICES	160
8.3	ELEMENTARY MATRICES	162
8.4	ALGEBRA OF SQUARE MATRICES	165
8.5	INPUT-OUTPUT MATRICES	174
	Proof of Theorem 8.13	178
8.6	PARTITIONED MATRICES (optional)	180
8.7	DECOMPOSING MATRICES (optional)	183
	Mathematical Induction	185
	Including Row Interchanges	185
9	Determinants: An Overview	188
9.1	THE DETERMINANT OF A MATRIX	189
	Defining the Determinant	189
	Computing the Determinant	191
	Main Property of the Determinant	192
9.2	USES OF THE DETERMINANT	194
9.3	IS-LM ANALYSIS VIA CRAMER'S RULE	197
10	Euclidean Spaces	199
10.1	POINTS AND VECTORS IN EUCLIDEAN SPACE	199
	The Real Line	199
	The Plane	199
	Three Dimensions and More	201
10.2	VECTORS	202
10.3	THE ALGEBRA OF VECTORS	205
	Addition and Subtraction	205
	Scalar Multiplication	207
10.4	LENGTH AND INNER PRODUCT IN \mathbf{R}^n	209
	Length and Distance	209
	The Inner Product	213

10.5	LINES	222
10.6	PLANES	226
	Parametric Equations	226
	Nonparametric Equations	228
	Hyperplanes	230
10.7	ECONOMIC APPLICATIONS	232
	Budget Sets in Commodity Space	232
	Input Space	233
	Probability Simplex	233
	The Investment Model	234
	IS-LM Analysis	234
11	Linear Independence	237
11.1	LINEAR INDEPENDENCE	237
	Definition	238
	Checking Linear Independence	241
11.2	SPANNING SETS	244
11.3	BASIS AND DIMENSION IN \mathbf{R}^n	247
	Dimension	249
11.4	EPILOGUE	249

P A R T I I I Calculus of Several Variables

12	Limits and Open Sets	253
12.1	SEQUENCES OF REAL NUMBERS	253
	Definition	253
	Limit of a Sequence	254
	Algebraic Properties of Limits	256
12.2	SEQUENCES IN \mathbf{R}^m	260
12.3	OPEN SETS	264
	Interior of a Set	267
12.4	CLOSED SETS	267
	Closure of a Set	268
	Boundary of a Set	269
12.5	COMPACT SETS	270
12.6	EPILOGUE	272

13	Functions of Several Variables	273
13.1	FUNCTIONS BETWEEN EUCLIDEAN SPACES	273
	Functions from \mathbf{R}^n to \mathbf{R}	274
	Functions from \mathbf{R}^k to \mathbf{R}^m	275
13.2	GEOMETRIC REPRESENTATION OF FUNCTIONS	277
	Graphs of Functions of Two Variables	277
	Level Curves	280
	Drawing Graphs from Level Sets	281
	Planar Level Sets in Economics	282
	Representing Functions from \mathbf{R}^k to \mathbf{R}^1 for $k > 2$	283
	Images of Functions from \mathbf{R}^1 to \mathbf{R}^m	285
13.3	SPECIAL KINDS OF FUNCTIONS	287
	Linear Functions on \mathbf{R}^k	287
	Quadratic Forms	289
	Matrix Representation of Quadratic Forms	290
	Polynomials	291
13.4	CONTINUOUS FUNCTIONS	293
13.5	VOCABULARY OF FUNCTIONS	295
	Onto Functions and One-to-One Functions	297
	Inverse Functions	297
	Composition of Functions	298
14	Calculus of Several Variables	300
14.1	DEFINITIONS AND EXAMPLES	300
14.2	ECONOMIC INTERPRETATION	302
	Marginal Products	302
	Elasticity	304
14.3	GEOMETRIC INTERPRETATION	305
14.4	THE TOTAL DERIVATIVE	307
	Geometric Interpretation	308
	Linear Approximation	310
	Functions of More than Two Variables	311
14.5	THE CHAIN RULE	313
	Curves	313
	Tangent Vector to a Curve	314
	Differentiating along a Curve: The Chain Rule	316
14.6	DIRECTIONAL DERIVATIVES AND GRADIENTS	319
	Directional Derivatives	319
	The Gradient Vector	320

14.7	EXPLICIT FUNCTIONS FROM \mathbf{R}^n TO \mathbf{R}^m	323
	Approximation by Differentials	324
	The Chain Rule	326
14.8	HIGHER-ORDER DERIVATIVES	328
	Continuously Differentiable Functions	328
	Second Order Derivatives and Hessians	329
	Young's Theorem	330
	Higher-Order Derivatives	331
	An Economic Application	331
14.9	Epilogue	333
15	Implicit Functions and Their Derivatives	334
15.1	IMPLICIT FUNCTIONS	334
	Examples	334
	The Implicit Function Theorem for \mathbf{R}^2	337
	Several Exogenous Variables in an Implicit Function	341
15.2	LEVEL CURVES AND THEIR TANGENTS	342
	Geometric Interpretation of the Implicit Function Theorem	342
	Proof Sketch	344
	Relationship to the Gradient	345
	Tangent to the Level Set Using Differentials	347
	Level Sets of Functions of Several Variables	348
15.3	SYSTEMS OF IMPLICIT FUNCTIONS	350
	Linear Systems	351
	Nonlinear Systems	353
15.4	APPLICATION: COMPARATIVE STATICS	360
15.5	THE INVERSE FUNCTION THEOREM (optional)	364
15.6	APPLICATION: SIMPSON'S PARADOX	368

P A R T I V Optimization

16	Quadratic Forms and Definite Matrices	375
16.1	QUADRATIC FORMS	375
16.2	DEFINITENESS OF QUADRATIC FORMS	376
	Definite Symmetric Matrices	379

	Application: Second Order Conditions and Convexity	379	
	Application: Conic Sections	380	
	Principal Minors of a Matrix	381	
	The Definiteness of Diagonal Matrices	383	
	The Definiteness of 2×2 Matrices	384	
16.3	LINEAR CONSTRAINTS AND BORDERED MATRICES	386	
	Definiteness and Optimality	386	
	One Constraint	390	
	Other Approaches	391	
16.4	APPENDIX	393	
17	Unconstrained Optimization	396	
17.1	DEFINITIONS	396	
17.2	FIRST ORDER CONDITIONS	397	
17.3	SECOND ORDER CONDITIONS	398	
	Sufficient Conditions	398	
	Necessary Conditions	401	
17.4	GLOBAL MAXIMA AND MINIMA	402	
	Global Maxima of Concave Functions	403	
17.5	ECONOMIC APPLICATIONS	404	
	Profit-Maximizing Firm	405	
	Discriminating Monopolist	405	
	Least Squares Analysis	407	
18	Constrained Optimization I: First Order Conditions		411
18.1	EXAMPLES	412	
18.2	EQUALITY CONSTRAINTS	413	
	Two Variables and One Equality Constraint	413	
	Several Equality Constraints	420	
18.3	INEQUALITY CONSTRAINTS	424	
	One Inequality Constraint	424	
	Several Inequality Constraints	430	
18.4	MIXED CONSTRAINTS	434	
18.5	CONSTRAINED MINIMIZATION PROBLEMS		436
18.6	KUHN-TUCKER FORMULATION	439	

18.7	EXAMPLES AND APPLICATIONS	442	
	Application: A Sales-Maximizing Firm with Advertising	442	
	Application: The Averch-Johnson Effect	443	
	One More Worked Example	445	
19	Constrained Optimization II	448	
19.1	THE MEANING OF THE MULTIPLIER	448	
	One Equality Constraint	449	
	Several Equality Constraints	450	
	Inequality Constraints	451	
	Interpreting the Multiplier	452	
19.2	ENVELOPE THEOREMS	453	
	Unconstrained Problems	453	
	Constrained Problems	455	
19.3	SECOND ORDER CONDITIONS	457	
	Constrained Maximization Problems	459	
	Minimization Problems	463	
	Inequality Constraints	466	
	Alternative Approaches to the Bordered Hessian Condition	467	
	Necessary Second Order Conditions	468	
19.4	SMOOTH DEPENDENCE ON THE PARAMETERS	469	
19.5	CONSTRAINT QUALIFICATIONS	472	
19.6	PROOFS OF FIRST ORDER CONDITIONS	478	
	Proof of Theorems 18.1 and 18.2: Equality Constraints	478	
	Proof of Theorems 18.3 and 18.4: Inequality Constraints	480	
20	Homogeneous and Homothetic Functions	483	
20.1	HOMOGENEOUS FUNCTIONS	483	
	Definition and Examples	483	
	Homogeneous Functions in Economics	485	
	Properties of Homogeneous Functions	487	
	A Calculus Criterion for Homogeneity	491	
	Economic Applications of Euler's Theorem	492	
20.2	HOMOGENIZING A FUNCTION	493	
	Economic Applications of Homogenization	495	
20.3	CARDINAL VERSUS ORDINAL UTILITY	496	