Lectures in Mathematics ETH Zürich

Randall J. LeVeque

Numerical Methods for Conservation Laws

守恒定律用的数值法

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Author's address:

Randall J. LeVeque
Department of Mathematics, GN-50
University of Washington
Seattle, WA 98195
USA

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Preface

These notes developed from a course on the numerical solution of conservation laws first taught at the University of Washington in the fall of 1988 and then at ETH during the following spring.

The overall emphasis is on studying the mathematical tools that are essential in developing, analyzing, and successfully using numerical methods for nonlinear systems of conservation laws, particularly for problems involving shock waves. A reasonable understanding of the mathematical structure of these equations and their solutions is first required, and Part I of these notes deals with this theory. Part II deals more directly with numerical methods, again with the emphasis on general tools that are of broad use. I have stressed the underlying ideas used in various classes of methods rather than presenting the most sophisticated methods in great detail. My aim was to provide a sufficient background that students could then approach the current research literature with the necessary tools and understanding.

Without the wonders of TeX and LaTeX, these notes would never have been put together. The professional-looking results perhaps obscure the fact that these are indeed lecture notes. Some sections have been reworked several times by now, but others are still preliminary. I can only hope that the errors are not too blatant. Moreover, the breadth and depth of coverage was limited by the length of these courses, and some parts are rather sketchy. I do have hopes of eventually expanding these notes into a full-fledged book, going more deeply into some areas, discussing a wider variety of methods and techniques, and including discussions of more applications areas. For this reason I am particularly interested in receiving corrections, comments and suggestions. I can be reached via electronic mail at *na.rleveque@na-net.stanford.edu*.

I am indebted to Jürgen Moser and the Forschungsinstitut at ETH for the opportunity to visit and spend time developing these notes, and to Martin Gutknecht for initiating this contact. During the course of this project, I was also supported in part by a Presidential Young Investigator Award from the National Science Foundation.

Contents

1 Introduction

1.1 Conservation laws

These notes concern the solution of hyperbolic systems of conservation laws. These are time-dependent systems of partial differential equations (usually nonlinear) with a particularly simple structure. In one space dimension the equations take the form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0. \tag{1.1}$$

Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ is an m-dimensional vector of conserved quantities, or state variables, such as mass, momentum, and energy in a fluid dynamics problem. More properly, u_j is the density function for the jth state variable, with the interpretation that $\int_{x_1}^{x_2} u_j(x,t)\,dx$ is the total quantity of this state variable in the interval $[x_1, x_2]$ at time t.

The fact that these state variables are conserved means that $\int_{-\infty}^{\infty} u_j(x,t)\,dx$ should be constant with respect to t. The functions u_j themselves, representing the spatial distribution of the state variables at time t, will generally change as time evolves. The main assumption underlying (1.1) is that knowing the value of $u(x,t)$ at a given point and time allows us to determine the rate of flow, or **flux**, of each state variable at (x,t). The flux of the jth component is given by some function $f_j(u(x,t))$. The vector-valued function $f(u)$ with jth component $f_j(u)$ is called the **flux function** for the system of conservation laws, so $f : \mathbb{R}^m \to \mathbb{R}^m$. The derivation of the equation (1.1) from physical principles will be illustrated in the next chapter.

The equation (1.1) must be augmented by some initial conditions and also possibly boundary conditions on a bounded spatial domain. The simplest problem is the pure initial value problem, or **Cauchy problem**, in which (1.1) holds for $-\infty < x < \infty$ and $t \geq 0$. In this case we must specify initial conditions only,

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty. \tag{1.2}$$

We assume that the system (1.1) is **hyperbolic**. This means that the $m \times m$ Jacobian matrix $f'(u)$ of the flux function has the following property: For each value of u the eigenvalues of $f'(u)$ are real, and the matrix is diagonalizable, *i.e.*, there is a complete set of m linearly independent eigenvectors. The importance of this assumption will be seen later.

In two space dimensions a system of conservation laws takes the form

$$\frac{\partial}{\partial t}u(x,y,t) + \frac{\partial}{\partial x}f(u(x,y,t)) + \frac{\partial}{\partial y}g(u(x,y,t)) = 0 \tag{1.3}$$

where $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^m$ and there are now two flux functions $f, g : \mathbb{R}^m \to \mathbb{R}^m$. The generalization to more dimensions should be clear.

Hyperbolicity now requires that any real linear combination $\alpha f'(u) + \beta g'(u)$ of the flux Jacobians should be diagonalizable with real eigenvalues.

For brevity throughout these notes, partial derivatives will usually be denoted by subscripts. Equation (1.3), for example, will be written as

$$u_t + f(u)_x + g(u)_y = 0. \tag{1.4}$$

Typically the flux functions are nonlinear functions of u, leading to nonlinear systems of partial differential equations (PDEs). In general it is not possible to derive exact solutions to these equations, and hence the need to devise and study numerical methods for their approximate solution. Of course the same is true more generally for any nonlinear PDE, and to some extent the general theory of numerical methods for nonlinear PDEs applies in particular to systems of conservation laws. However, there are several reasons for studying this particular class of equations on their own in some depth:

- Many practical problems in science and engineering involve conserved quantities and lead to PDEs of this class.
- There are special difficulties associated with solving these systems (e.g. shock formation) that are not seen elsewhere and must be dealt with carefully in developing numerical methods. Methods based on naive finite difference approximations may work well for smooth solutions but can give disastrous results when discontinuities are present.
- Although few exact solutions are known, a great deal *is* known about the mathematical structure of these equations and their solution. This theory can be exploited to develop special methods that overcome some of the numerical difficulties encountered with a more naive approach.

1.2 Applications

One system of conservation laws of particular importance is the **Euler equations** of gas dynamics. More generally, the fundamental equations of fluid dynamics are the Navier-Stokes equations, but these include the effects of fluid viscosity and the resulting flux function depends not only on the state variables but also on their gradients, so the equations are not of the form (1.1) and are not hyperbolic. A gas, however, is sufficiently dilute that viscosity can often be ignored. Dropping these terms gives a hyperbolic system of conservation laws with $m = d + 2$ equations in d space dimensions, corresponding to the conservation of mass, energy, and the momentum in each direction. In one space dimension, these equations take the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} = 0, \tag{1.5}$$

where $\rho = \rho(x,t)$ is the density, v is the velocity, ρv is the momentum, E is the energy, and p is the pressure. The pressure p is given by a known function of the other state variables (the specific functional relation depends on the gas and is called the "equation of state"). The derivation of these equations is discussed in more detail in Chapters 2 and 5.

These equations, and some simplified versions, will be used as examples throughout these notes. Although there are many other systems of conservation laws that are important in various applications (some examples are mentioned below), the Euler equations play a special role. Much of the theory of conservation laws was developed with these equations in mind and many numerical methods were developed specifically for this system. So, although the theory and methods are applicable much more widely, a good knowledge of the Euler equations is required in order to read much of the available literature and benefit from these developments. For this reason, I urge you to familiarize yourself with these equations even if your primary interest is far from gas dynamics.

The shock tube problem. A simple example that illustrates the interesting behavior of solutions to conservation laws is the "shock tube problem" of gas dynamics. The physical set-up is a tube filled with gas, initially divided by a membrane into two sections. The gas has a higher density and pressure in one half of the tube than in the other half, with zero velocity everywhere. At time $t = 0$, the membrane is suddenly removed or broken, and the gas allowed to flow. We expect a net motion in the direction of lower pressure. Assuming the flow is uniform across the tube, there is variation in only one direction and the one-dimensional Euler equations apply.

The structure of this flow turns out to be very interesting, involving three distinct waves separating regions in which the state variables are constant. Across two of these waves there are discontinuities in some of the state variables. A **shock wave** propagates into the region of lower pressure, across which the density and pressure jump to higher values and all of the state variables are discontinuous. This is followed by a **contact discontinuity**, across which the density is again discontinuous but the velocity and pressure are constant. The third wave moves in the opposite direction and has a very different

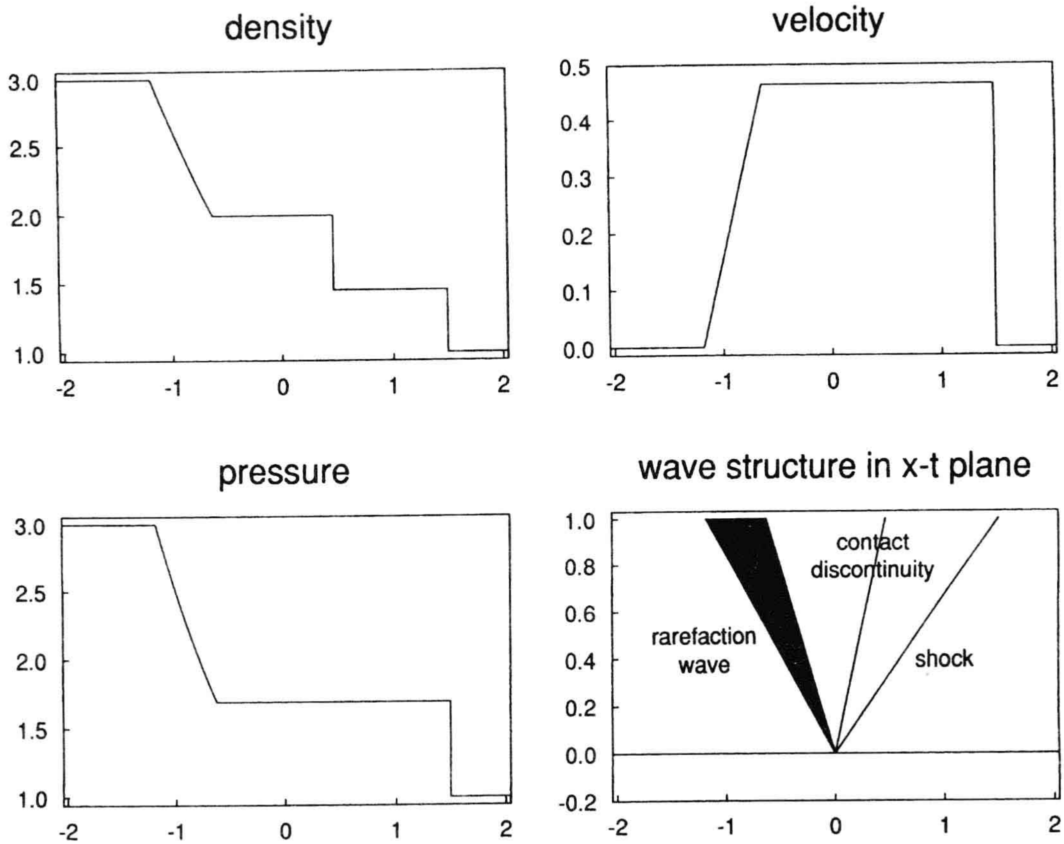

Figure 1.1. Solution to a shock tube problem for the one-dimensional Euler equations.

structure: all of the state variables are continuous and there is a smooth transition. This wave is called a **rarefaction wave** since the density of the gas decreases (the gas is rarefied) as this wave passes through.

If we put the initial discontinuity at $x = 0$, then the resulting solution $u(x,t)$ is a "similarity solution" in the variable x/t, meaning that $u(x,t)$ can be expressed as a function of x/t alone, say $u(x,t) = w(x/t)$. It follows that $u(x,t) = u(\alpha x, \alpha t)$ for any $\alpha > 0$, so the solution at two different times t and αt look the same if we rescale the x-axis. This also means that the waves move at constant speed and the solution $u(x,t)$ is constant along any ray $x/t =$ constant in the x-t plane.

Figure 1.1 shows a typical solution as a function of x/t. We can view this as a plot of the solution as a function of x at time $t = 1$, for example. The structure of the solution in the x-t plane is also shown.

In a real experimental shock tube, the state variables would not be discontinuous

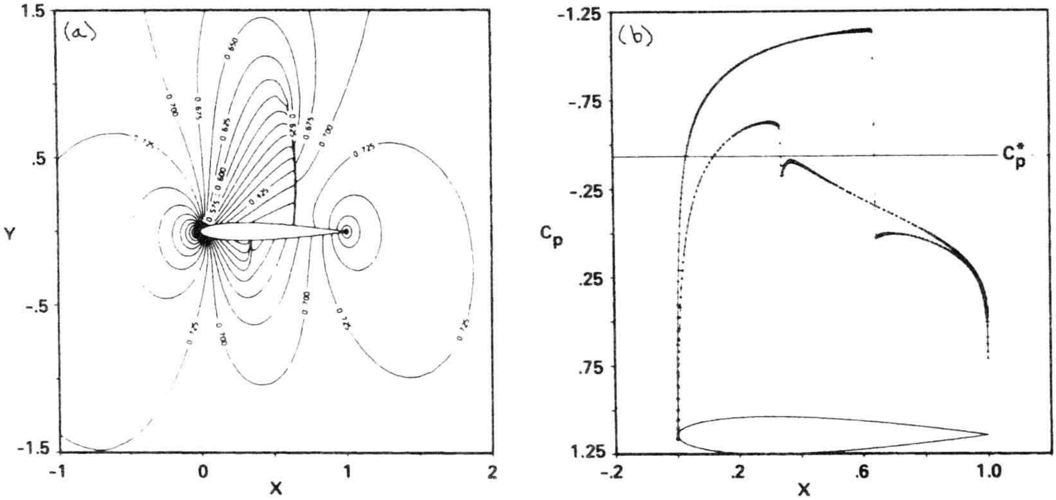

Figure 1.2. (a) Pressure contours for flow around an airfoil. (b) Pressure coefficient plotted along the top and bottom surface. Figure taken from Yee and Harten[100]. (Reprinted with permission.)

across the shock wave or contact discontinuity because of effects such as viscosity and heat conduction. These are ignored in the Euler equations. If we include these effects, using the full Navier-Stokes equations, then the solution of the partial differential equations would also be smooth. However, these smooth solutions would be nearly discontinuous, in the sense that the rise in density would occur over a distance that is microscopic compared to the natural length scale of the shock tube. If we plotted the smooth solutions they would look indistinguishable from the discontinuous plots shown in Figure 1.1. For this reason we would like to ignore these viscous terms altogether and work with the simpler Euler equations.

The Euler equations are used extensively in aerodynamics, for example in modeling the flow of air around an aircraft or other vehicle. These are typically three dimensional problems, although 2D and even 1D problems are sometimes of interest. A typical 2D problem is the flow of air over an airfoil, which is simply the cross section of a wing. Figure 1.2a (taken from Yee and Harten[100]) shows the contours of pressure in a steady state solution for a particular airfoil shape when the freestream velocity is Mach 0.8. Note the region above the upper surface of the airfoil where many contour lines coincide. This is again a shock wave, visible as a discontinuity in the pressure. A weaker shock is visible on the lower surface of the airfoil as well.

Small changes in the shape of an airfoil can lead to very different flow patterns, and so the ability to experiment by performing calculations with a wide variety of shapes is required. Of particular interest to the aerodynamical engineer is the pressure distribution along the airfoil surface. From this she can calculate the lift and drag (the vertical and horizontal components of the net force on the wing) which are crucial in evaluating its performance. Figure 1.2b shows the "pressure coefficient" along the upper and lower surfaces. Again the shocks can be observed as discontinuities in pressure.

The location and strength of shock waves has a significant impact on the overall solution, and so an accurate computation of discontinuities in the flow field is of great importance.

The flow field shown in Figure 1.2 is a **steady state** solution, meaning the state variables $u(x,y,t)$ are independent of t. This simplifies the equations since the time derivative terms drop out and (1.3) becomes

$$f(u)_x + g(u)_y = 0. \tag{1.6}$$

In these notes we are concerned primarily with time-dependent problems. One way to solve the steady state equation (1.6) is to choose some initial conditions (e.g. uniform flow) and solve the time-dependent equations until a steady state is reached. This can be viewed as an iterative method for solving the steady state equation. Unfortunately, it is typically a very inefficient method since it may take thousands of time steps to reach steady state. A wide variety of techniques have been developed to accelerate this convergence to steady state by giving up time accuracy. The study of such acceleration techniques is a whole subject in its own right and will not be presented here. However, the discrete difference equations modeling (1.6) that are solved by such an iterative method must again be designed to accurately capture discontinuities in the flow, and are often identical to the spatial terms in a time-accurate method. Hence much of the theory developed here is also directly applicable in solving steady state equations.

Unsteady problems also arise in aerodynamics, for example in modeling wing flutter, or the flow patterns around rotating helicopter blades or the blades of a turbine. At high speeds these problems involve the generation of shock waves, and their propagation and interaction with other shocks or objects is of interest.

Meteorology and weather prediction is another area of fluid dynamics where conservation laws apply. Weather fronts are essentially shock waves — "discontinuities" in pressure and temperature. However, the scales involved are vastly greater than in the shock tube or airfoil problems discussed above, and the viscous and dissipative effects cause these fronts to have a width of several miles rather than the fractions of an inch common in aerodynamics.

Astrophysical modeling leads to systems of conservation laws similar to the Euler equations for the density of matter in space. A spiral galaxy, for example, may consist of alternating arms of high density and low density, separated by "discontinuities" that are again propagating shock waves. In this context the shock width may be two or three light years! However, since the diameter of a galaxy is on the order of 10^5 light years, this is

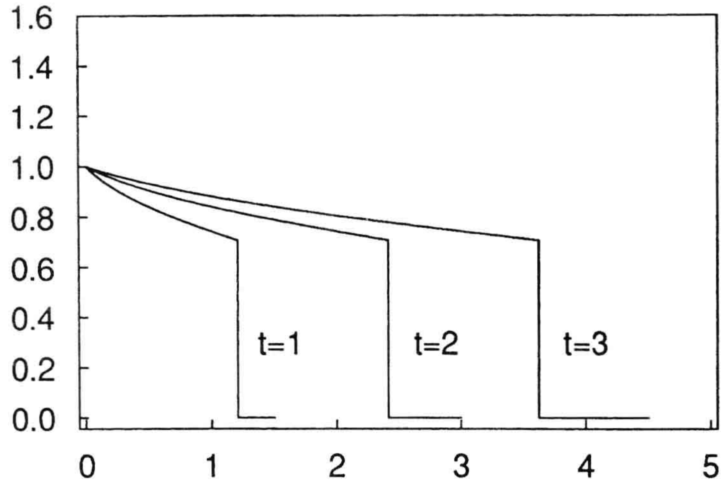

Figure 1.3. Solution of the Buckley-Leverett equation at three different times.

still a small distance in relative terms. In particular, in a practical numerical calculation the shock width may well be less than the mesh width.

Modeling the dynamics of a single star, or the plasma in a fusion reactor, also requires conservation laws. These now involve electromagnetic effects as well as fluid dynamics. The magnetohydrodynamics (MHD) equations are one system of this type.

Multiphase flow problems in porous materials give rise to somewhat different systems of conservation laws. One important application area is secondary oil recovery, in which water (with some additives, perhaps) is pumped down one well in an effort to force more oil out of other wells. One particularly simple model is the Buckley-Leverett equation, a scalar conservation law for a single variable u representing the saturation of water in the rock or sand ($u = 0$ corresponds to pure oil, $u = 1$ to pure water). Figure 1.3 shows the solution to this 1D problem at three different times. Here the initial condition is $u(x,0) = 0$ and a boundary condition $u(0,t) = 1$ is applied, corresponding to pure water being pumped in at the left boundary. Note that the advancing front again has a discontinuity, or propagating shock wave. The Buckley-Leverett equation is discussed further in Chapter 4. More realistic models naturally involve systems of conservation laws in two or three dimensions.

Systems of conservation laws naturally arise in a wide variety of other applications as well. Examples include the study of explosions and blast waves, the propagation of waves in elastic solids, the flow of glaciers, and the separation of chemical species by chromatography, to name but a few.