Undergraduate Texts in Mathematics

Robin Hartshorne

GEOMETRY: EUCLID AND BEYOND

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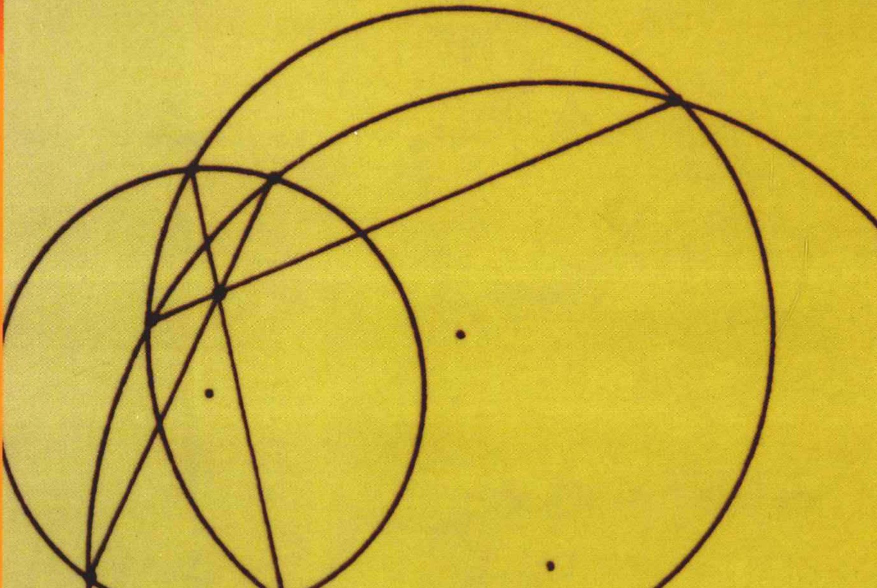

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Robin Hartshorne

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With 550 Illustrations

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Preface

In recent years, I have been teaching a junior–senior-level course on the classical geometries. This book has grown out of that teaching experience. I assume only high-school geometry and some abstract algebra. The course begins in Chapter 1 with a critical examination of Euclid's *Elements*. Students are expected to read concurrently Books I–IV of Euclid's text, which must be obtained separately. The remainder of the book is an exploration of questions that arise naturally from this reading, together with their modern answers. To shore up the foundations we use Hilbert's axioms. The Cartesian plane over a field provides an analytic model of the theory, and conversely, we see that one can introduce coordinates into an abstract geometry. The theory of area is analyzed by cutting figures into triangles. The algebra of field extensions provides a method for deciding which geometrical constructions are possible. The investigation of the parallel postulate leads to the various non-Euclidean geometries. And in the last chapter we provide what is missing from Euclid's treatment of the five Platonic solids in Book XIII of the *Elements*.

For a one-semester course such as I teach, Chapters 1 and 2 form the core material, which takes six to eight weeks. Then, depending on the taste of the instructor, one can follow a more geometric path by going directly to non-Euclidean geometry in Chapter 7, or a more algebraic one, exploring the relation between geometric constructions and field extensions, by doing Chapters 3, 4, and 6. For me, one of the most interesting topics is the introduction of coordinates into an abstractly given geometry, which is done for a Euclidean plane in Section 21, and for a hyperbolic plane in Section 41.

Throughout this book, I have attempted to choose topics that are accessible to undergraduates and that are interesting in their own right. The exercises are meant to be challenging, to stimulate a sense of curiosity and discovery in the student. I purposely do not indicate their difficulty, which varies widely.

I hope this material will become familiar to every student of mathematics, and in particular to those who will be future teachers.

I owe thanks to Marvin Greenberg for reading and commenting on large portions of the text, to Hendrik Lenstra for always having an answer to my questions, and to Victor Pambuccian for valuable references to the literature. Thanks to Faye Yeager for her patient typing and retyping of the manuscript. And special thanks to my wife, Edie, for her continual loving support.

> Of all the works of antiquity which have been transmitted to the present times, none are more universally and deservedly esteemed than the *Elements of Geometry* which go under the name of Euclid. In many other branches of science the moderns have far surpassed their masters; but, after a lapse of more than two thousand years, this performance still maintains its original preeminence, and has even acquired additonal celebrity from the fruitless attempts which have been made to establish a different system.
>
> – from the preface to
> Bonnycastle's Euclid
> London (1798)

Undergraduate Texts in Mathematics

For
Edie, Ben, and Joemy

and

In Loving Memory of
Jonathan Churchill Hartshorne
1972–1992

I have not found anything in Lobatchevski's work that is new to me, but the development is made in a different way from the way I had started and to be sure masterfully done by Lobatchevski in the pure spirit of geometry.

– letter from Gauss to Schumacher (1846)

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Introduction

A little after the time of Plato, but before Archimedes, in ancient Greece, a man named Euclid wrote the *Elements*, gathering and improving the work of his predecessors Pythagoras, Theaetetus, and Eudoxus into one magnificent edifice. This book soon became the standard for geometry in the classical world. With the decline of the great civilizations of Athens and Rome, it moved eastward to the center of Arabic learning in the court of the caliphs at Baghdad.

In the late Middle Ages it was translated from Arabic into Latin, and since the Renaissance it not only has been the most widely used textbook in the world, but has had an influence as a model of scientific thought that extends way beyond the confines of geometry. As Billingsley said in his preface to the first English translation (1570), "Without the diligent studie of Euclides *Elementes*, it is impossible to attaine unto the perfecte knowledge of Geometrie, and consequently of any of the other Mathematical Sciences." Even today, though few schools use the original text of Euclid, the content of a typical high-school geometry course is the same as what Euclid taught more than two thousand three hundred years ago.

In this book we will take Euclid's *Elements* as the starting point for a study of geometry from a modern mathematical perspective.

To begin, we will become familiar with the content of Euclid's work, at least those parts that deal with geometry (Books I–IV, VI, and XI–XIII). Here we find theorems that should be familiar to anyone who has had a course of high-school geometry, such as the fact (I.4) that two triangles are congruent if they have two sides and the included angle equal, or the fact (III.21) that a given arc of a circle subtends the same angle at any point of the circle from which it is seen. (Throughout this book, references such as (I.4) or (III.21) refer to the corresponding Book and Proposition number in Euclid's *Elements*.)

Many of Euclid's propositions pose construction problems, such as (I.1), to construct an equilateral triangle, or (IV.11), to construct a regular pentagon inscribed in a circle. Euclid means to construct the required figure using only the ruler, which can draw a straight line through two points, and the compass, which can draw a circle with given center and given radius. These **ruler and compass constructions** are often taught in high-school geometry. Note that Euclid casts these problems in the form of constructions, whereas a modern mathematician would be more likely to speak of proving the existence of the required figure.

At a second level, we will study the logical structure of Euclid's presentation. Euclid's *Elements* has been regarded for more than two thousand years as the prime example of the **axiomatic method**. Starting from a small number of self-evident truths, called postulates, or common notions, he deduces all the succeeding results by purely logical reasoning. Euclid thus begins with the simplest assumptions, such as Postulate 1, to draw a line through any two given points, or Postulate 3, to draw a circle with given center and radius. He then proceeds step by step to the culmination of the work in Book XIII, where he gives the construction of the five regular solids: the tetrahedron, the cube, the octahedron, the icosahedron, and the dodecahedron.

Upon closer reading, we find that Euclid does not adhere to the strict axiomatic method as closely as one might hope. Certain steps in certain proofs depend on assumptions that, however reasonable or intuitively clear they may seem, cannot be justified on the basis of the stated postulates and common notions. So, for example, the fact that the two circles in the proof of (I.1) will actually meet at some point seems obvious, but is not proved. The **method of superposition** used in the proof of (I.4), which allows one to move the triangle ABC so that it lies on top of the triangle DEF, cannot be justified from the axioms. Also, various assumptions about the relative position of figures in the plane, such as which point lies between the others, or which ray lies in the interior of a given angle, are used without any previous clarification of what such notions should mean.

These lapses in Euclid's logic lead us to the task of disengaging those implicit assumptions that are used in his arguments and providing a new set of axioms from which we can develop geometry according to modern standards of rigor. The logical foundations of geometry were widely studied in the late nineteenth century, which led to a set of axioms proposed by Hilbert in his lectures on the foundations of geometry in 1899. We will examine Hilbert's axioms, and we will see how these axioms can be used to build a solid base from which to develop Euclid's geometry pretty much according to the logical plan that he first laid out.