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Derivative Securities and Difference Methods

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Derivative Securities and Difference Methods

With 92 Illustrations

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Preface

In the past three decades, great progress has been made in the theory and practice of financial derivative securities. Now huge volumes of financial derivative securities are traded on the market every day. This causes a big demand for experts who know how to price financial derivative securities. This book is designed as a textbook for graduate students in a mathematical finance program and as a reference book for the people who already work in this field. We hope that a person who has studied this book and who knows how to write codes for engineering computation can handle the business of providing efficient derivative-pricing codes. In order for this book to be used by various people, the prerequisites to study the majority of this book are multivariable calculus, linear algebra, and basic probability and statistics.

In this book, the determination of the prices of financial derivative securities is reduced to solving partial differential equation problems, i.e., a PDE approach is adopted in order to find the price of a derivative security. This book is divided into two parts. In the first part, we discuss how to establish the corresponding partial differential equations and find the final and necessary boundary conditions for a specific derivative product. If possible, we derive its explicit solution and describe some properties of the solution. In many cases, no explicit solution has been found so far. In these situations, we have to use numerical methods to determine the value of financial derivative securities. Therefore, the second part is devoted to numerical methods for derivative securities. There are two styles of financial derivatives: European and American. The numerical methods for both styles of derivatives are described. The main numerical method discussed is the finite-difference method. The binomial/trinomial method is also introduced as a version of an explicit finite-difference method, and the pseudo-spectral method is discussed as a high-order finite-difference method. In this part, numerical methods for determining the market price of risk are also studied as numerical methods for inverse problems. From the viewpoint of partial differential equations, solving an inverse problem means to determine a function as a variable coefficient in a partial differential equation, according to certain values of some solutions.

During the past few years, a great number of books on financial derivative securities have been published. For example: Duffie[26], Baxter and Rennie[5], Hull[39], James and Webber[43], Jarrow[44], Kwok[49], Lamberton and Lapeyre[50], Lyuu[54], Musiela and Rutkowski[58], Pelsser[60], Tavella and Randall[69], Wilmott, Dewynne, and Howison[71], Wilmott[72], Wilmott[73], and Yan[76] have published books on this subject. However, each book has its own features and gives emphasis to some aspects of this subject. Relatively speaking, this book is similar to the books by Wilmott, Dewynne, and Howison[71], Kwok[49], and Tavella and Randall[69] because all of them deal with the partial differential equation problems in finance and their numerical methods. However, this book pays more attention to numerical methods. At least the following features of this book are unique:

1. The slopes of the payoff functions for many derivative securities are discontinuous, and American-style derivative securities usually have free boundaries. These features downgrade the efficiency of numerical methods. In this book, we will discuss how to make computation more efficient even though the solutions have such types of weak singularities.
2. Many derivative security problems are defined on an infinite domain. When a numerical method is used to solve such a problem, usually a large finite domain is taken, and some artificial boundary conditions are adopted for implicit methods. This book will discuss how to convert such a problem into a problem defined on a finite domain and without requiring any artificial boundary conditions. Also, conditions guaranteeing that a random variable is defined on a finite domain are derived. When these conditions hold, any derivative security problems will be defined on a finite domain and do not need any artificial boundary conditions in order to solve them numerically.
3. A numerical method for an inverse problem in finance, for determination of the market price of risk on the spot interest rate, has been provided. As soon as having the market price of risk on the spot interest rate, we can use partial differential equations for evaluating interest rate derivatives in practice.
4. A three-factor interest rate model has been provided. All the parameters in the model and the final values of derivatives are determined from the market data. Because of this, it can be expected that the model reflects the real market. The evaluation of interest rate derivatives is reduced to solving a final value problem of a three-dimensional partial differential equation on a finite domain. Because the correctness of the formulation of the problem is proven, the numerical method for such a problem can be designed without difficulties.

The first four chapters are related to partial differential equations in finance. Chapter 1 is an introduction, where basic features of several assets and financial derivative securities are briefly described. Chapter 2 discusses basic options. In this chapter, Itô's lemma and the Black–Scholes equation are introduced, along with the derivation of the Black–Scholes formulae. These topics are followed by a discussion on American options as both linear complementarity and free-boundary problems. Also in Chapter 2, the put–call parity relation for European options as well as the put–call symmetry relations for American options are introduced. Finally, the general equations for derivative securities are derived.

In Chapter 3, exotic options such as barrier, Asian, lookback, and multi-asset are introduced. The equations, final conditions, and necessary boundary conditions for these options are provided. In this chapter, we examine a few cases in which a two-dimensional problem may be reduced to a one-dimensional problem. Explicit solutions for some of these options are provided whenever possible. Also, the formulations as free-boundary problems have been given for several American exotic options.

In Chapter 4, one-factor interest rate models, namely, the Vasicek, Cox–Ingersoll–Ross, Ho–Lee, and Hull–White models, are carefully discussed. Then, we describe how the problem of determining the market price of risk from the market data may be formulated as an inverse problem. After that, the formulations of interest rate derivatives such as bond options and swaptions are given. Then, we discuss multi-factor models and give the details of a three-factor model that can reflect the real market and be used in practice readily. The final topics in Chapter 4 are a discussion on two-factor convertible bonds and the derivation of the equivalent free-boundary problem.

Most of basic materials in these four chapters can be found from many books, for example, from the books listed above. Readers who need to know more about these subjects are referred to those books. Some of the materials are the authors' research results. For more details, see those corresponding papers given in the references.

As is well-known, exact solutions to the vanilla American option problems are not known, and the problems need to be solved numerically. For vanilla European options, if σ depends on S or the dividend is paid discretely, then explicit solutions may not exist. Therefore, in order to evaluate their prices, we often rely on numerical methods. For pricing exotic options and interest rate derivatives, we rely on numerical methods even more due to the complexity of these problems.

The next four chapters are devoted to numerical methods for partial differential equations in finance. In Chapter 5, we provide the basic numerical methods that will be used for solving partial differential equation problems and discuss the basic theory on finite-difference methods – stability, convergence and the extrapolation technique of numerical solutions. Most of these concepts can be found in many books. In the next chapter, Initial-Boundary Value and LC (Linear Complementarity) Problems, we discuss the numerical methods for European-style derivative securities and for American-style derivative securities formulated as an LC problem. In Chapter 7, Free-Boundary Problems, we carefully discuss how to solve one-factor and two-factor American option problems as free-boundary problems by implicit finite-difference methods. We also describe how to solve a two-factor convertible bond problem as a free-boundary problem by the pseudo-spectral method. In this chapter, we provide a comparison among these methods given in this chapter and in Chapter 6 as well. In the last chapter, Interest Rate Modeling, we begin with another formulation of the inverse problem and some numerical examples on the market price of risk. Then, we discuss how to price interest rate derivatives, such as swaptions, using one-factor models with numerical market prices of risk and show some numerical results. Finally, how to use the three-factor model to price interest rate derivatives in practice is discussed. Most of the materials presented in the last three chapters are from research results, especially from the authors' research.

This book can be used as a text book for two courses as a sequence. In the first course, the subject "Partial Differential Equations in Finance" is taught by using the materials in Part I. The second one is a course on "Numerical Methods for Derivative Securities" based on Part II of this book. In order to help students to understand the materials and check whether or not students have understood them, a number of problems are given at the end of each chapter. Also, at the ends of Chapters 5–8, some projects are given in order for students to be trained in evaluating derivative securities. This book is considered as a book between a textbook for graduate students and a monograph. If time is not enough, some portions can be omitted and left to students as reference materials. We have used it as a textbook in our mathematical finance program and almost all the materials can be taught in class. The following materials are basic and more important:

- Sections 1.1–2;
- Sections 2.1–4, Subsections 2.5.1–2, 2.6.1–3, 2.9.1–4, 2.10.1–2;
- Section 3.1, Subsections 3.2.1, 3.3.1–4, 3.4.1–2;
- Sections 4.1–2, 4.6–7;
- Subsections 5.1.1–2, 5.2.1–2, Section 5.3, Subsection 5.4.1, Section 5.5;
- Subsections 6.1.1–3, 6.1.5, 6.2.1–3, 6.3.2–3, 6.3.6;
- Section 7.1, Subsections 7.2.1, 7.2.3, 7.2.5–6, Section 7.3;
- Section 8.3.

These materials can be taught in one semester. Thus, if only one course is offered, this book can also be used.

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Part I

Partial Differential Equations in Finance

1

Introduction

1.1 Assets

We first introduce some basic knowledge on stocks, bonds, foreign currencies, commodities, and indices, all of which are called **assets** in this book.

Huge volumes of stocks are traded on the stock market every day, and the price of a stock changes all the time. Such a price is a typical random variable. As examples, the prices of the stocks issued by IBM and GE during the period 1990–2000 are plotted in Figs. 1.1 and 1.2. Stocks are issued by corporations. A corporation like IBM, for example, is a business unit, which gets its capital through issuing stocks. A holder of a share of stock owns a fixed portion of the corporation. For example, if a corporation issues 10 million shares of stock, then the holder of a share of stock owns 10^{-7} portion of the corporation. Stock prices, especially those of high technology stocks, have large volatilities. However, stocks usually have higher returns than bonds, which attracts people to buy them. Many corporations distribute a small amount of cash to its stockholders in proportion to the number of shares of stock held periodically. The amount is not fixed and is determined by the corporation after the stocks have been issued. This payment is commonly known as the dividend. A corporation sometimes splits its stock. When a stock split occurs, the value of the stock changes. If one share splits into two shares, the value of a new share of stock is one half of the value of an old share of stock because the value of the corporation does not change when the stock split occurs.

Bonds and other debt instruments are other types of securities that are traded on the market frequently. Besides issuing stocks, a corporation can also get its capital through issuing bonds. Governments at various levels issue bonds for some special purposes, too. The holder of a bond will get the face value (the par value) at the maturity as long as the issuer has the ability to pay. Therefore, the price of a bond usually goes to the face value as the maturity approaches, which is a feature any price of stock does not have and is called the pull-to-par phenomenon. Periodically, a bondholder will receive a fixed amount of cash, usually a few percent of the face value. This percentage is