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Fourier Analysis on Number Fields

数域上的傅里叶分析

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To Pat and Anand

To Brenda

Preface

This book grew out of notes from several courses that the first author has taught over the past nine years at the California Institute of Technology, and earlier at the Johns Hopkins University, Cornell University, the University of Chicago, and the University of Crete. Our general aim is to provide a modern approach to number theory through a blending of complementary algebraic and analytic perspectives, emphasizing harmonic analysis on topological groups. Our more particular goal is to cover John Tate's visionary thesis, giving virtually all of the necessary analytic details and topological preliminaries—technical prerequisites that are often foreign to the typical, more algebraically inclined number theorist. Most of the existing treatments of Tate's thesis, including Tate's own, range from terse to cryptic; our intent is to be more leisurely, more comprehensive, and more comprehensible. To this end we have assembled material that has admittedly been treated elsewhere, but not in a single volume with so much detail and not with our particular focus.

We address our text to students who have taken a year of graduate-level courses in algebra, analysis, and topology. While our choice of objects and methods is naturally guided by the specific mathematical goals of the text, our approach is by no means narrow. In fact, the subject matter at hand is germane not only to budding number theorists, but also to students of harmonic analysis or the representation theory of Lie groups. We hope, moreover, that our work will be a good reference for working mathematicians interested in any of these fields.

A brief sketch of each of the chapters follows.

(1) Topological Groups. The general discussion begins with basic notions and culminates with the proof of the existence and uniqueness of Haar (invariant) measures on locally compact groups. We next give a substantial introduction to profinite groups, which includes their characterization as compact, totally disconnected topological groups. The chapter concludes with the elementary theory of pro-p-groups, important examples of which surface later in connection with local fields.

(2) Some Representation Theory. In this chapter we introduce the fundamentals of representation theory for locally compact groups, with the ultimate

aim of proving certain key properties of unitary representations on Hilbert spaces. To reach this goal, we need some weighty analytic prerequisites, including an introduction to Gelfand theory for Banach algebras and the two spectral theorems. The first we prove completely; the second we only state, but with enough background to be thoroughly understandable. The material on Gelfand theory fortuitously appears again in the following chapter, in a somewhat different context.

(3) Duality for Locally Compact Abelian Groups. The main points here are the abstract definition of the Fourier transform, the Fourier inversion formula, and the Pontryagin duality theorem. These require many preliminaries, including the analysis of functions of positive type, their relationship to unitary representations, and Bochner's theorem. A significant theme in all of this is the interplay between two alternative descriptions of the "natural" topology on the dual group of a locally compact abelian group. The more tractable description, as the compact-open topology, is presented in the first section; the other, which arises in connection with the Fourier transform, is introduced later as part of the proof of the Fourier inversion formula.

We have been greatly influenced here by the seminal paper on abstract harmonic analysis by H. Cartan and R. Godement (1947), although we give many more details than they, some of which are not obvious—even to experts. As a subsidiary goal of the book, we certainly hope that our exposition will encourage further circulation of their beautiful and powerful ideas.

(4) The Structure of Arithmetic Fields. In the first two sections the basics of local fields, such as the p-adic rationals $\mathbf{Q}_p$, are developed from a completely topological perspective; in this the influence of Weil's *Basic Number Theory* (1974) is apparent. We also provide some connections with the algebraic construction of these objects via discrete valuation rings. The remainder of the chapter deals with global fields, which encompass the finite extensions of $\mathbf{Q}$ and function fields in one variable over a finite field. We discuss places and completions, the notions of ramification index and residual degree, and some key points on local and global bases.

(5) Adeles, Ideles, and the Class Groups. This chapter establishes the fundamental topological properties of adele and idele groups and certain of their quotients. The first two sections lay the basic groundwork of definitions and elementary results. In the third, we prove the crucial theorem that a global field embeds as a cocompact subgroup of its adele group. We conclude, in the final section, with the introduction of the idele class group, a vast generalization of the ideal class group, and explain the relationship of the former to the more traditional ray class group.

(6) A Quick Tour of Class Field Theory. The material in this chapter is not logically prerequisite to the development of Tate's thesis, but it is used in our

subsequent applications. We begin with the Frobenius elements (conjugacy classes) associated with unramified primes P of a global field F, first in finite Galois extensions, next in the maximal extension unramified at P. In the next three sections we state the Tchebotarev density theorem, define the transfer map for groups, and state, without proof, the Artin reciprocity law for abelian extensions of global and local fields, in terms of the more modern language of idele classes. In the fifth and final section, we explicitly describe the cyclotomic extensions of $\mathbf{Q}$ and $\mathbf{Q}_p$ and then apply the reciprocity law to prove the Kronecker-Weber theorem for these two fields.

(7) TATE'S THESIS AND APPLICATIONS. Making use of the characters and duality of locally compact abelian groups arising from consideration of local and global fields, we carefully analyze the local and global zeta functions of Tate. This brings us to the main issue: the demonstration of the functional equation and analytic continuation of the L-functions of characters of the idele class group. There follows a proof of the regulator formula for number fields, which yields the residues of the zeta function of a number field F in terms of its class number h_F and the covolume of a lattice of the group U_F of units, in a suitable Euclidean space. From this we derive the class number formula and, in consequence, Dirichlet's theorem for quadratic number fields. Further investigation of these L-functions—in fact, some rather classical analysis—next yields another fundamental property: their nonvanishing on the line $\mathrm{Re}(s)=1$. Finally, as a most remarkable application of this material, we prove the following theorem of Hecke: Suppose that χ and χ' are idele class characters of a global field K and that $\chi_P=\chi_P'$ for a set of primes of positive density. Then $\chi=\mu\chi'$ for some character μ of finite order.

One of the more parenthetical highlights of this chapter (see Section 7.2) is the explanation of the analogy between the Poisson summation formula for number fields and the Riemann-Roch theorem for curves over finite fields.

We have given a number of exercises at the end of each chapter, together with hints, wherever we felt such were advisable. The difficult problems are often broken up into several smaller parts that are correspondingly more accessible. We hope that these will promote gradual progress and that the reader will take great satisfaction in ultimately deriving a striking result. We urge doing as many problems as possible; without this effort a deep understanding of the subject cannot be cultivated.

Perhaps of particular note is the substantial array of nonstandard exercises found at the end of Chapter 7. These span almost twenty pages, and over half of them provide nontrivial complements to, and applications of, the material developed in the chapter.

The material covered in this book leads directly into the following research areas.

♢ *L-functions of Galois Representations.* Following Artin, given a finite-dimensional, continuous complex representation σ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, one associates an L-function denoted $L(\sigma,s)$. Using Tate's thesis in combination with a theorem of Brauer and abelian class field theory, one can show that this function has a meromorphic continuation and functional equation. One of the major open problems of modern number theory is to obtain analogous results for l-adic Galois representations σ_l, where l is prime. This is known to be true for σ_l arising from abelian varieties of *CM* type, and $L(\sigma_l,s)$ is in this case a product of L-functions of idele class characters, as in Tate's thesis.

♢ *Jacquet-Langlands Theory.* For any reductive algebraic group G [for instance, $\mathrm{GL}_n(F)$ for a number field F], an important generalization of the set of idele class characters is given by the irreducible *automorphic* representations π of the locally compact group $G(\mathbf{A}_F)$. The associated *L-functions* $L(\pi,s)$ are well understood in a number of cases, for example for GL_n, and by an important conjecture of Langlands, the functions $L(\sigma_l,s)$ mentioned above are all expected to be expressible in terms of suitable $L(\pi,s)$. This is often described as nonabelian class field theory.

♢ *The p-adic L-functions.* In this volume we consider only complex-valued (smooth) functions on local and global groups. But if one fixes a prime p and replaces the target field $\mathbf{C}$ by $\mathbf{C}_p$, the completion of an algebraic closure of $\mathbf{Q}_p$, strikingly different phenomena result. Suitable p-adic measures lead to p-adic–valued L-functions, which seem to have many properties analogous to the classical complex-valued ones.

♢ *Adelic Strings.* Perhaps the most surprising application of Tate's thesis is to the study of string amplitudes in theoretical physics. This intriguing connection is not yet fully understood.

Acknowledgments

Finally, we wish to acknowledge the intellectual debt that this work owes to H. Cartan and R. Godement, J.-P. Serre (1968, 1989, and 1997), A. Weil, and, of course, to John Tate (1950). We also note the influence of other authors whose works were of particular value to the development of the analytic background in our first three chapters; most prominent among these are G. Folland (1984) and G. Pedersen (1989). (See References below for complete bibliographic data and other relevant sources.)

Index of Notation

Notation	*Section*	*Interpretation*
$\mathbf{N}, \mathbf{Z}, \mathbf{Q}$	—	natural numbers, integers, and rational numbers, respectively
$\mathbf{R}, \mathbf{C}$	—	real and complex numbers, respectively
$\mathbf{R}_+, \mathbf{R}_+^\times$	—	nonnegative reals, positive reals
1_S	—	identity map on the set S
S^c	—	complement of the set S
$\mathrm{Card}(S)$	—	cardinality of the set S
$\dot{\bigcup} S_\alpha$	—	disjoint union of sets S_α
$\mathrm{supp}(f)$	—	support of a function f
$\mathscr{C}(X)$	—	continuous (complex-valued) functions on a topological space X
$\mathscr{C}_c(X)$	—	continuous functions with compact support
$\mathscr{C}_c^+(X)$	—	positive elements of $\mathscr{C}_c(X)$ with positive sup norm
A^*, K^*	—	nonzero elements of a ring or field
$A^\times$	—	group of units of a ring A
$[K:F]$	—	degree of a finite field extension K/F
$N_{K/F}(x)$	—	norm map for a finite field extension K/F; see also Section 6.4
$\mathrm{tr}_{K/F}(x)$	—	trace map for a finite field extension K/F
KL	—	compositum of fields K and L
$\mathbf{Z}/n\mathbf{Z}$	—	integers modulo n
$\varphi(n)$	—	Euler phi function
S^1	—	the circle group
$W^\perp$	—	orthogonal complement of a subspace W
pr_W	—	orthogonal projection onto a subspace W

$k[[t]]$	—	ring of formal power series in t with coefficients in the field k
$k((t))$	—	fraction field of $k[[t]]$
$\mathrm{GL}_n(k)$	—	group of invertible $n \times n$ matrices over k
$\mathrm{SL}_n(k)$	—	$n \times n$ matrices over k of determinant 1
$B^1(X)$	A.1	unit ball in a normed linear space X
X^*	A.1	(norm) continuous dual of a normed linear space X
$l_1(\mathbf{C}^n)$	A.1	$\mathbf{C}^n$ with l_1 norm
$L(X)$	A.4	measurable functions on X modulo agreement almost everywhere
$L^p(X)$	A.4	L^p-space associated with a locally compact space X
$\lVert \cdot \rVert_p$	A.4	L^p-norm
A_S	B.1	localization of a ring A at subset S
J_K	B.2	set of fractional ideals of a global field K
P_K	B.2	set of principal fractional ideals of K
Cl_K	B.2	traditional class group of a global field K
$N(I)$	B.2	absolute norm map
$\Delta(x_1,\ldots,x_n)$	B.2	discriminant of a basis $x_1,\ldots,x_n$
$\Delta(B/A)$	B.2	discriminant ideal of a ring extension B/A
$L_h f$, $R_h f$	1.1	left and right translation operators on f
$(f:\varphi)$	1.2	Haar covering number
$\varprojlim G_i$	1.3	projective limit of a projective system $\{G_i\}$
$\hat{\mathbf{Z}}$	1.3	projective completion of $\mathbf{Z}$
$\mathbf{Z}_p$	1.3	ring of p-adic integers
G°	1.3	connected component of the identity
$\mathrm{Gal}(K/F)$	1.3	Galois group of the field extension K/F
F^S	1.3	fixed field of a set S of automorphisms of F
$\lvert G \rvert$	1.4	order of a profinite group G
$\mathrm{Aut}(V)$	2.1	algebraic automorphisms of a vector space
$\mathrm{Aut}_{\mathrm{top}}(V)$	2.1	topological automorphisms of a topological vector space
$\mathrm{Hom}(A,B)$	2.2	bounded operators between Banach spaces
$\mathrm{End}(A)$	2.2	endomorphisms on a Banach space A
$\lVert T \rVert$	2.2	norm of a bounded operator T

$\mathrm{sp}(a)$	2.2	spectrum of an element in a Banach algebra
$\mathrm{r}(a)$	2.2	spectral radius
$\hat{A}$	2.2	space of characters of a Banach algebra A
$\hat{a}$	2.2	Gelfand transform of a
$\mathscr{C}_0(X)$	2.3	continuous functions that vanish at infinity
T^*	2.3	adjoint of an operator T on a Hilbert space
$A_T \subseteq \mathrm{End}(H)$	2.3	the closed, self-adjoint, unital subalgebra generated by T in the ambient ring
$T^{1/2}$	2.3	square root of a positive operator
$\mathrm{Hom}_G(V, V')$	2.4	space of G-linear maps between two representation spaces
$\hat{G}$	3.1	Pontryagin dual of G
$X^{(n)} \subseteq G$	3.1	n-fold products within a group G
$W(K,V)$	3.1	local basis sets for the compact-open topology
$N(\varepsilon) \subseteq S^1$	3.1	ε-neighborhood of the identity in S^1
V_φ	3.2	Hilbert space associated with φ
$f*g$	3.2	convolution of functions
$\mathscr{P}(G)$	3.2	continuous function of positive type, bounded by 1 on G
$\mathscr{E}(G)$	3.2	elementary functions on G
$\hat{f}$	3.3	Fourier transform of a function f
$V(G)$	3.3	complex span of continuous functions of positive type
$V^1(G)$	3.3	L^1-functions in $V(G)$
$T_{\hat{\mu}}$	3.3	Fourier transform of a measure $\hat{\mu}$
$\mathrm{mod}_G(\alpha)$	4.1	module of an automorphism α on G
$B_m \subseteq k$	4.1	ball of module radius m in a topological field k
$\mathrm{ord}_k(a)$	4.2	order of an element of a local field k
$\lvert\cdot\rvert_p, \lvert\cdot\rvert_\infty$	4.2	p-norm and infinity norm on $\mathbf{Q}$ or $\mathbf{F}_q(t)$; see also Section 4.3
$\pi = \pi_k$	4.2	uniformizing parameter for a local field k
$e = e(k_1/k)$	4.3	ramification index of an extension of local fields

$f = f(k_1/k)$	4.3	residual degree of an extension of local fields
K_v	4.4	completion of a field K at a place v
K_Q	4.4	completion of global field K at the place corresponding to a prime Q
$\mathcal{P}_K$	4.4	set of places of K
$\mathcal{P}_{K,\infty}$	4.4	set of Archimedean places of K
$\mathcal{P}_{K,f}$	4.4	set of ultrametric places of K
$r_{K/F}: \mathcal{P}_K \to \mathcal{P}_F$	4.4	restriction map for places of a field extension K/F
$v \mid u$	4.4	place v restricts to place u
$\mathfrak{o}_v$	4.4	local ring of integers with respect to a place v
$\mathfrak{o}_K$	4.4	ring of integers of a global field K
D_Q	4.5	decomposition group of a prime Q
ρ_Q	4.5	canonical map from D_Q to $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F})$
j_Q	4.5	induced isomorphism from D_Q onto $\mathrm{Gal}(K_Q/F_P)$ where Q lies over P
$\mathrm{Hom}_k(L,M)$	4.5	embedding of L into M over k
$\Pi' G_v$	5.1	restricted direct product
G_S	5.1	S-version of the restricted direct product
$\prod_v dg_v$	5.1	induced Haar measure on a restricted direct product of locally compact groups
$\mathbf{A}_K$	5.2	adele group of a global field K
$\mathbf{I}_K$	5.2	idele group of a global field K
S_ω	5.2	set of infinite places of a global field
$\mathbf{A}_\omega$	5.2	the open subgroup $\mathbf{A}_{S_\omega}$ of the adele group
C_K	5.4	idele class group of global field K; see also Section 6.4
$\lvert x \rvert_{\mathbf{A}_K}$	5.4	standard absolute value on the adele group
$C_K^1 = \mathbf{I}_K^1 / K^*$	5.4	norm-one idele class group
S_∞	5.4	set of Archimedean places of a global field
$\mathbf{I}_{K,S}$	5.4	S-ideles of the global field K
$\mathbf{I}_{K,S}^1$	5.4	S-ideles of norm one
R_S	5.4	S-integers of a global field
$\mathbf{A}_{K,S}$	5.4	S-adeles of the global field K

$C_{K,S}$	5.4	S-class group of a global field K
v_P	5.4	discrete valuation associated with a prime P in a Dedekind domain
$K_{M,1}$	5.4	elements of K congruent to 1 modulo the integral ideal M
$J_K(M)$	5.4	fractional ideals relatively prime to M
$Cl_K(M)$	5.4	wide ray class group of K relative to M
$K_{\widetilde{M},1}$	5.4	elements of K congruent to 1 modulo the ideal M extended by a set of real places
$Cl_K(\widetilde{M})$	5.4	narrow ray class group of K relative to $\widetilde{M}$
$\varphi_{Q/P}$	6.1	Frobenius element associated with primes Q and P, where Q lies over P
$(P,K/F)$	6.1	Artin symbol (or Frobenius class)
$F^{\mathrm{ur}}(P)$	6.1	maximal unramified extension of F at P
Σ_F	6.2	set of places of a global field F
(G,G)	6.3	commutator subgroup of a group G
G^{ab}	6.3	abelianization of a group G
$V: G^{\mathrm{ab}} \to H^{\mathrm{ab}}$	6.3	transfer map
C_F	6.4	idele class group for F global, F^* for F local
$N_{K/F}: C_K \to C_F$	6.4	norm homomorphism
$j_{K/F}: C_F \to C_K$	6.4	map induced by inclusion
$\Gamma_K = \mathrm{Gal}\ \overline{F}/K)$	6.4	Galois group of the separable closure of F over a finite extension K of F
$i_{K/F}: \Gamma_K \to \Gamma_F$	6.4	inclusion map of Galois groups
$V: \Gamma_F^{\mathrm{ab}} \to \Gamma_K^{\mathrm{ab}}$	6.4	transfer map on Galois groups
$\theta_F: C_F \to \Gamma_F^{\mathrm{ab}}$	6.4	Artin map
$\theta_{K/F}$	6.4	Artin map with projection onto $\mathrm{Gal}(K/F)$
F^{ab}	6.5	maximal abelian extension of a field F
F_n	6.5	extension of F by all nth roots of unity
F_∞	6.5	extension of F by *all* roots of unity
$\theta(z)$	7.0	theta function
$d^*x = dx/\lvert x\rvert$	7.1	Haar measure on F^* as given by the Haar measure dx on a local field F
U_F	7.1	elements of F^* of unit absolute value
$\mathcal{G}_F$	7.1	valuation group of a local field F

Contents