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Martin Aigner

Combinatorial Theory

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Preface

It is now generally recognized that the field of combinatorics has, over the past years, evolved into a fully-fledged branch of discrete mathematics whose potential with respect to computers and the natural sciences is only beginning to be realized. Still, two points seem to bother most authors: The apparent difficulty in defining the scope of combinatorics and the fact that combinatorics seems to consist of a vast variety of more or less unrelated methods and results. As to the scope of the field, there appears to be a growing consensus that combinatorics should be divided into three large parts:

(a) *Enumeration*, including generating functions, inversion, and calculus of finite differences;
(b) *Order Theory*, including finite posets and lattices, matroids, and existence results such as Hall's and Ramsey's;
(c) *Configurations*, including designs, permutation groups, and coding theory.

The present book covers most aspects of parts (a) and (b), but none of (c). The reasons for excluding (c) were twofold. First, there exist several older books on the subject, such as Ryser [1] (which I still think is the most seductive introduction to combinatorics), Hall [2], and more recent ones such as Cameron–Van Lint [1] on groups and designs, and Blake–Mullin [1] on coding theory, whereas no comprehensive book exists on (a) and (b). Second, the vast diversity of types of designs, the very complicated methods usually still needed to prove existence or non-existence, and, in general, the rapid change this subject is presently undergoing do not favor a thorough treatment at this moment. I have also omitted reference to algorithms of any kind because I feel that presently nothing more can be said in book form about this subject beyond Knuth [1], Lawler [1], and Nijenhuis-Wilf [1].

As to the second point, that of systematizing the definitions, methods, and results into something resembling a theory, the present book tries to accomplish just this, admittedly at the expense of some of the spontaneity and ingenuity that makes combinatorics so appealing to mathematicians and non-mathematicians alike. To start with, mappings are grouped together into classes by placing various restrictions on them. To stick to the division outlined above, these classes of mappings are then counted, ordered, and arranged. The emphasis on ordering is well justified by the everyday experience of a combinatorist that most discrete structures, while perhaps lacking a simple algebraic structure, invariably admit a natural ordering. Following this program, the book is divided into three parts, the first part presenting the basic material on mappings and posets, in Chapters I and II, respectively, the second part dealing with enumeration in Chapters III to V, and the third part on the order-theoretical aspects in Chapters VI–VIII.

The arrangement of the material allows the reader to use the three parts almost independently and to combine several subsections into a course on special topics. For instance, Chapter II has been used as an introduction to finite lattices, Chapters VI and VII as a course on matroids, and parts of Chapter VII and Chapter VIII as a course on transversal theory and the major existence results. The exercises have been graded. Unmarked exercises can be solved without a great deal of effort; more difficult ones are marked with an asterisk (*). The symbol → indicates that the exercise is particularly helpful or interesting, but in no instance is the statement or the solution of an exercise necessary to the development of the subject. The references given at the end are, of course, by no means exhaustive; usually they have been included because they were used in one way or another in the preparation of the text. Books are indicated by an asterisk.

The German version of the present book appeared in two volumes—Kombinatorik I. Grundlagen und Zähltheorie; and II. Matroide und Transversaltheorie—as Springer Hochschultexts. Combining these two parts has been a more formidable task than I originally thought. Most of the material has been reorganized, with the major changes appearing in Chapter VIII due to many new results obtained in the last few years.

I had the opportunity of working as a research associate at the Department of Statistics of the University of North Carolina in the Combinatorial Year program 1968–1970. It was during this time that I first planned to write this book. Of the many people who have encouraged me since and furthered this work, I owe special thanks to G.-C. Rota, R. C. Bose, and T. A. Dowling for many hours of discussion; to H. Wielandt, H. Salzmann, and R. Baer for their constant support; to R. Weiss, G. Prins, R. H. Schulz, J. Schoene, and W. Mader, who read all or part of the manuscript; and finally to M. Barrett for her impeccable typing.

It is my hope that I have been able to record some of the many important changes that combinatorics has undergone in recent years while retaining its origins as an intuitively appealing mathematical pleasure.

Berlin
September 1979

M. Aigner

Contents

Preliminaries

It seems convenient to list at the outset a few items that will be used throughout the book.

1. Sets

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the basic number systems, and set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{N}_n = \{1, 2, \ldots, n\}$; in chapter III the notation $\underline{n}$ for $\mathbb{N}_n$ is also used. δ_{ij} is the Kronecker symbol; id_M stands for the identity mapping of a set M onto itself and 2^M for the power set of M. The cardinality of a set M is denoted by $|M|$ and we set $|M| = \infty$ whenever M is infinite. For any set M we use the symbol M^k for the cartesian product, $M^k = \{(a_1, \ldots, a_k): a_i \in M\}$, and $M^{(k)}$ for the family of k-subsets of M, $M^{(k)} = \{A \subseteq M: |A| = k\}$. A finite set M with $|M| = n$ is called an *n-set*. To define a set or a term we use $:=$ or $:\Leftrightarrow$.

The following rules are the basic tools for enumeration:

(i) *Rule of Equality*: If N and R are finite sets and if there exists a bijection between them, then $|N| = |R|$;
(ii) *Rule of Sums*: If $\{A_i: i \in I\}$ is a finite family of finite pairwise disjoint sets, then $|\bigcup_{i\in I} A_i| = \sum_{i \in I} |A_i|$;
(iii) *Rule of Products*: If $\{A_i: i \in I\}$ is a finite family of finite sets, then for the cartesian product $\prod_{i\in I} A_i$, $|\prod_{i \in I} A_i| = \prod_{i\in I} |A_i|$.

We use the symbols $A \mathbin{\dot\cup} B$ or $\dot{\bigcup}_{i\in I} A_i$ to indicate that the sets involved are disjoint.

A *multiset* on S is a set S together with a function $r: S \to \mathbb{N}_0$ (giving the multiplicity of the elements of S). A convenient notation for a multiset k on S is $k = \{a^{k_a}: a \in S\}$ with $k_a := r(a)$, $a \in S$. The usual notions for sets can be carried over to multisets. For instance, if $k = \{a^{k_a}: a \in S\}$ and $l = \{a^{l_a}: a \in S\}$ then

$$k \subseteq l :\Leftrightarrow k_a \leq l_a \quad \text{for all} \quad a \in S,$$

$$k \cap l := \{a^{\min(k_a, l_a)}: a \in S\},$$

$$k \cup l := \{a^{\max(k_a, l_a)}: a \in S\}.$$

Clearly, the family of multisets on a set S forms a lattice under inclusion; furthermore, this lattice is complete.

2. Graphs

An *undirected graph* $G(V, E)$ consists of a non-empty set V, called the *vertex-set* and a multiset E of unordered pairs $\{a, b\}$ from V, called the *edge-set*. A *simple graph* is a graph that contains no *loops* $\{a, a\}$ and no *parallel edges* $\{a, b\}, \{a, b\}$, i.e., in which $E \subseteq V^{(2)}$ is an ordinary set. A *directed graph* or *digraph* $\vec{G}(V, E)$ is a non-empty set V of vertices and a multiset E of ordered pairs (a, b) from V. The elements of E are now called *arrows* or *directed edges*. An *orientation* of an undirected graph $G(V, E)$ is a rule which designates for each edge $k = \{a, b\}$ a direction (a, b); we then write $a = k^-$, $b = k^+$. A graph is *finite* if both V and E are finite.

Except for the definition of a graph itself the terminology follows closely that of Harary [1]. (There, a graph means what we call a simple graph.) The reader is advised to consult chapter 2 in Harary's book for any term not previously defined. We shall, however, redefine most of the notions when they first appear, except for the most basic ones such as connected graph, path, circuit, etc. Whenever we simply use the term "graph" we always mean "undirected graph."

Two graphs $G(V, E)$ and $G'(V', E')$ are *isomorphic* if these exists a bijection $\phi: V \to V'$ such that $\{a, b\} \in E$ and $\{\phi(a), \phi(b)\} \in E'$ appear in E and E' with equal multiplicity. The *degree* $\gamma(v)$ of a vertex v is the number of edges incident with v where we count loops $\{v, v\}$ twice. Hence for a finite graph $G(V, E)$ we always have $\sum_{v \in V} \gamma(v) = 2|E|$.

Two important types of graphs are the *complete graphs* K_n and the *complete bipartite graphs* $K_{m,n}$. K_n is a simple graph with n vertices with any two vertices joined by an edge. $K_{m,n}$ is a simple graph whose vertex-set is the union of two disjoint sets of cardinality m and n respectively, with two vertices being joined if and only if they are in different sets. A *bipartite graph* is any subgraph of a complete bipartite graph. We shall often denote a bipartite graph by $G(V_1 \cup V_2, E)$ to indicate the defining vertex-sets V_1, V_2, where every edge joins a vertex in V_1 with a vertex in V_2. The following rule is the single most useful tool in enumeration.

(iv) *Rule of "counting in two ways"*: Let $G(V_1 \cup V_2, E)$ be a finite bipartite graph with defining vertex-sets V_1 and V_2. Then

$$\sum_{v \in V_1} \gamma(v) = \sum_{v \in V_2} \gamma(v) \qquad (= |E|).$$

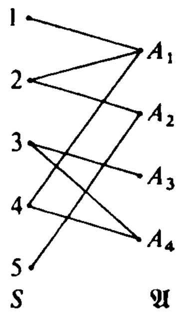
A bipartite graph $G(V_1 \cup V_2, E)$ can also be regarded as a directed graph with all edges directed from V_1 to V_2. In other words, bipartite graphs with defining vertex-sets V_1 and V_2 can be identified with *binary relations* between V_1 and V_2. For this reason, we often use the letter R for the edge-set and in $G(V_1 \cup V_2, R)$ set $R(A) := \bigcup_{a \in A} \{y \in V_2 : (a, y) \in R\}$ for $A \subseteq V_1$, and similarly $R(B) := \bigcup_{b \in B} \{x \in V_1 : (x, b) \in R\}$ for $B \subseteq V_2$. For a singleton subset $\{a\}$, we simply write $R(a)$.

Bipartite graphs have two other important equivalent interpretations. A *set system* $(S, \mathfrak{A})$ is a set S together with a family $\mathfrak{A}$ of not necessarily distinct subsets of S. Any set system $(S, \mathfrak{A})$ gives rise to its incidence graph $G(S \cup \mathfrak{A}, R)$ where

$(p, A) \in R :\Leftrightarrow p \in A$. Conversely, any bipartite graph $G(S \cup \mathfrak{A}, R)$ yields a set system $(S, \mathfrak{A})$ by identifying $A \in \mathfrak{A}$ with the set $R(A) \subseteq S$.

A set system $(S, \mathfrak{A})$ can also be described by a 0, 1-matrix $M = [m_{ij}]$ whose rows and columns are indexed by S and $\mathfrak{A}$ respectively, with $m_{ij} = 1$ or 0 depending on whether $p_i \in A_j$ or $p_i \notin A_j$. M is called the *incidence matrix* of $(S, \mathfrak{A})$. Conversely, any 0, 1-matrix gives rise to a set system by the reverse procedure.

Example.



$$S = \{1, 2, 3, 4, 5\}$$
$$\mathfrak{A} = \{\{1, 2, 4\}, \{2, 5\}, \{3\}, \{3, 4\}\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Bipartite graph — Set system — 0, 1-matrix

A graph which has no non-trivial circuits is called a *forest*. A connected forest is called a *tree*.

3. Posets

We employ the usual terminology as, for instance, in Birkhoff [1]. If P is a poset then P^* denotes the *dual poset* obtained by inverting the order relation of P. If P contains a unique minimal element, then this element is called the *0-element*, denoted by 0; similarly, a unique maximal element is called the *1-element*, denoted by 1. We say, *b covers a* or *a is covered by b*, denoted by $a <\cdot\, b$, if $a < b$, and $a < x \le b$ implies $x = b$. The *atoms* are the elements covering 0 (if 0 exists); the *co-atoms* are the elements covered by 1. We most often represent a poset P by its diagram, which is the directed graph on P with an arrow from a to b if and only if b covers a. Whenever possible, we draw a diagram from the bottom up and omit the arrows. A *chain* is a poset in which any two elements are comparable. For the chain $\{a_1 < a_2 < \cdots < a_n\}$ we often use the short-hand notation $\{a_1, \ldots, a_n\}_<$. The *length* of a chain is one less than its cardinality. The length $l(a)$ of $a \in P$ is the length of the longest chain in P with a as last element. An *antichain* is a poset in which any two elements are incomparable. A chain in a poset P is called *unrefinable* if any element of the chain is covered by its successor. If L is a lattice then a non-empty subset M is called a *sublattice* if $x, y \in M$ imply $x \wedge y \in M$, $x \vee y \in M$. A subset $M \subseteq L$ may be a lattice in its own right under the induced order relation, but we reserve the term sublattice for the former situation. An *interval* of a poset P is any set $[a, b] := \{x \in P : a \leq x \leq b\}$ for $a, b \in P$. The *product* $\prod_{i \in I} P_i$ of posets P_i is the poset on the cartesian product with the coordinatewise order relation. The product of lattices is again a lattice. The *sum* $\sum_{i \in I} L_i$ of lattices, each L_i containing a 0-element, is the sublattice of the product $\prod_{i \in I} L_i$ consisting of those vectors with only a finite number of coordinates different from 0. $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{N}_n$ are assumed to be endowed with the natural ordering unless otherwise stated. A *complete lattice* is one in which any non-empty subset has an infimum and a supremum. If W is a word which contains only elements of a lattice L and the symbols $\wedge$, $\vee$, $\leq$ and (,), then we obtain the dual expression W^* by exchanging $\wedge$ with $\vee$ and $\leq$ with $\geq$. The validity of W for all variables x_i implies the validity of W^* for all variables x_i. This is called the *principle of duality* in lattices. W is called *self-dual* if $W^* = W$.

The reader is referred to Crawley–Dilworth [1, ch. 1 and 2] for all other terms not previously defined.

4. Miscellaneous Notation

(i) Let $a \in \mathbb{R}$. Then $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the largest integer $\leq a$ and the smallest integer $\geq a$, respectively.

(ii) We sometimes use the symbol $\#\{\cdots\}$ to denote the cardinality of the set $\{\cdots\}$.

(iii) By a *partition* of a set S we mean a disjoint union $S = \dot{\bigcup}_{i \in I} A_i$. We also use the notation $S = A_1 | A_2 | \cdots$ to indicate a partition. The sets A_i are assumed to be non-empty unless otherwise stated.

(iv) To facilitate the summation notation, we shall often indicate the summation index by a dot underneath. For instance, let $M = [m_{ij}]$ be an $n \times n$-matrix. Then $\sum_{1 \leq \dot{i} < j \leq n} m_{ij} = m_{1j} + m_{2j} + \cdots + m_{j-1j}$.

Chapter I

Mappings

The starting point for all our considerations is the following: We are given two sets, usually denoted by N and R, and a mapping $f: N \to R$ satisfying certain conditions. The triple (N, R, f) is called a *morphism*. Our program is to arrange mappings into *classes*, and then to *count* and *order* the resulting classes of mappings.

Accordingly, our first task will be to collect conditions of combinatorial significance which we want to impose on the mappings—and this is the content of the present chapter.

1. Classes of Mappings

Let (N, R, f) be a morphism. N and R will, in most instances, be finite sets, and we shall use the letters $n = |N|$ and $r = |R|$, respectively, for their cardinalities. A common way to describe f is by the expression

$$f = \begin{pmatrix} \cdots & a & \cdots \\ \cdots & f(a) & \cdots \end{pmatrix} \qquad (a \in N).$$

We call this the *standard representation* of $f: N \to R$. Most of the time the domain N will be totally ordered in some natural way, but, of course, any ordering is possible. For example, the three expressions

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & a & b & b \end{pmatrix}, \begin{pmatrix} 1 & 4 & 3 & 2 \\ a & b & b & a \end{pmatrix}, \begin{pmatrix} 4 & 3 & 1 & 2 \\ b & b & a & a \end{pmatrix}$$

all represent the same mapping.

A. Classification

With $f: N \to R$ we associate the *image* $\operatorname{im}(f)$ and the *kernel* $\ker(f)$:

$$\operatorname{im}(f) := \bigcup_{a \in N} f(a)$$

$$\ker(f) := \dot{\bigcup_{b \in \operatorname{im}(f)}} f^{-1}(b).^{(1)}$$

(1) For simplicity we write $f^{-1}(b)$ instead of the more precise $f^{-1}(\{b\})$. Similar abbreviations will be used later, e.g., $A \cup q$ for $A \cup \{q\}$.

Thus, the kernel of f is the partition of N induced by the equivalence relation

$$a \approx a' :\Leftrightarrow f(a) = f(a') \qquad (a, a' \in N).$$

It is convenient to postulate the *empty mapping* $f_\emptyset$ with $\operatorname{im}(f_\emptyset) = \emptyset$ and undefined kernel.

The mapping $f: N \to R$ is called *surjective* if $\operatorname{im}(f) = R$, and *injective* if $\ker(f) = 0$ (in the lattice of partitions of N—see section 2.B), i.e., if $a \neq a' \Rightarrow f(a) \neq f(a')$ for all $a, a' \in N$. Mappings that are both surjective and injective are called *bijective*.

With these definitions we obtain a first set of classes of mappings:

$$\begin{aligned}
\operatorname{Map}(N, R) &:= \{f: N \to R, f \text{ arbitrary}\},\\
\operatorname{Sur}(N, R) &:= \{f: N \to R, f \text{ surjective}\},\\
\operatorname{Inj}(N, R) &:= \{f: N \to R, f \text{ injective}\},\\
\operatorname{Bij}(N, R) &:= \{f: N \to R, f \text{ bijective}\}.
\end{aligned}$$

When N and R are both finite sets, we have the following obvious but important rules concerning their cardinalities:

$$\begin{aligned}
\operatorname{Sur}(N, R) \neq \emptyset &\Rightarrow |N| \geq |R|,\\
\operatorname{Inj}(N, R) \neq \emptyset &\Rightarrow |N| \leq |R|,\\
\operatorname{Bij}(N, R) \neq \emptyset &\Rightarrow |N| = |R|.
\end{aligned}$$

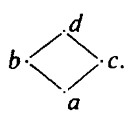
For a mapping $f: N \to N$ of a finite set N into itself the notions surjective, injective, and bijective coincide. For infinite sets this is no longer true. For example, $f: \mathbb{N} \to \mathbb{N}, f(k) = 2k$, is injective, but not surjective.

Suppose that both sets N and R are endowed with a partial order. $f \in \operatorname{Map}(N, R)$ is called *monotone* if it preserves the order relation, i.e., if $a \leq_N b \Rightarrow f(a) \leq_R f(b)$ for all $a, b \in N$, and it is called *antitone* if $a \leq_N b \Rightarrow f(a) \geq_R f(b)$ for all $a, b \in N$. The family of monotone mappings constitutes another important class:

$$\operatorname{Mon}(N, R) := \{f: N \to R, f \text{ monotone}\}.$$

Observe that if N is totally unordered we simply have $\operatorname{Mon}(N, R) = \operatorname{Map}(N, R)$, regardless of the order on R. Any monotone or antitone mapping maps chains onto chains.

Example. Let $N = \{1 < 2 < 3 < 4\}$ and

$R =$ 

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$ is monotone, whereas $\begin{pmatrix} 1 & 2 & 3 & 4 \\ d & b & c & a \end{pmatrix}$ is antitone.

Another well-known class of mappings arises in the context of algebra. Suppose there are algebraic systems of the same type defined on N and R, e.g., groups, rings, or vector spaces over the same scalar domain. A mapping $f: N \to R$ which preserves all operations is called a *homomorphism*, and we denote the class of all homomorphisms from N into R by

$$\mathrm{Hom}(N, R) := \{f: N \to R, f \text{ homomorphism}\}.$$

By combining the classes we have encountered so far it is clear what we mean by a surjective monotone mapping or an injective homomorphism, etc. We shall see that most of the combinatorial counting problems can be phrased in terms of one or more of these classes.

B. Representation

There are two particularly useful and suggestive interpretations of a morphism (N, R, f). Let N be totally ordered by some fixed order. We regard the elements of N as places of a word and say the *place* $i \in N$ is occupied by the *letter* $l \in R$ if $f(i) = l$. The mapping is thus regarded as a *word* of length n with letters from the *alphabet* R, indexed by N. Now order the elements of R by some total order and regard them as *boxes*. If $f(a) = b$ we say that the object $a \in N$ has been *sorted* into the box b, or that the box b contains a. In this way we interpret $f: N \to R$ as an *occupancy pattern* of the boxes R by the objects N.
In summary:

$$f: N \to R = \begin{cases} \text{mapping from } N \text{ into } R; \\ \text{word in } R \text{ indexed by } N; \\ \text{occupancy of } R \text{ by } N. \end{cases}$$

Suppose $N = \{a_1, a_2, \ldots, a_n\}_<$ is totally ordered. Then the mapping

$$f = \begin{pmatrix} a_1 & \cdots & a_n \\ f(a_1) & \cdots & f(a_n) \end{pmatrix}$$

can be unambiguously represented by the word $f(a_1)f(a_2)\ldots f(a_n)$. We call this the *word representation* of f (relative to the given total order on N). Similarly, if $R = \{b_1, \ldots, b_r\}_<$ is totally ordered then $f^{-1}(b_1) \mathbin{\dot\cup} f^{-1}(b_2) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} f^{-1}(b_r)$ is called the *occupancy representation* of f. In most cases N or R will be the set $\{1, \ldots, n\}$ or $\{1, \ldots, r\}$ endowed with the natural order.

Example. Let $N = \{1 < 2 < 3\}$, $R = \{a < b < c\}$. We list the set Map(N, R) by giving the word representation of its members:

aaa	*acc*	*aba*	*caa*	*cbc*	*acb*
aab	*ccc*	*baa*	*cac*	*ccb*	*bac*
abb	*bbc*	*bab*	*cca*		*cab*
bbb	*bcc*	*bba*	*bcb*		*bca*
aac	*abc*	*aca*	*cbb*		*cba*.

The monotone mappings are those in the first two columns, the last column together with *abc* gives the bijective mappings. Hence we have

$$|\mathrm{Map}(N, R)| = 27, \qquad |\mathrm{Bij}(N, R)| = 6, \qquad |\mathrm{Mon}(N, R)| = 10.$$

The reader may set up a similar list using the occupancy representation.

The terms introduced in the beginning can now be interpreted as special words or occupancies, e.g., a mapping $f \in \mathrm{Inj}(N, R)$ is called a *strict word* and $f \in \mathrm{Sur}(N, R)$ a *full occupancy*. Let $N = \{1 < 2 < \cdots < n\}$ and R be an arbitrary poset. $\mathrm{Mon}(N, R)$ consists of all words $b_1 b_2 \ldots b_n$ in R with $b_1 \le b_2 \le \cdots \le b_n$. Hence we speak of $\mathrm{Mon}(N, R)$ as the class of *monotone words*. If, in particular, R is totally ordered, then the monotone words of length n are precisely the *multisets* in R of cardinality n, and it follows that there are just as many monotone words of length n in R as n-multisets in R. See the example above where the 3-multisets of R are listed in the first two columns. If we restrict ourselves to *strict monotone* words we obtain the family of all *subsets* of R of cardinality n.

Exercises I.1

1. Let $f: N \to R$. Prove:
   (i) $\mathrm{im}(f)$ minimal $\Leftrightarrow$ $\ker(f)$ maximal.
   (ii) $\mathrm{im}(f)$ maximal and $|N| < |R| \Rightarrow f$ injective.
   (iii) $\mathrm{im}(f)$ maximal and $|N| > |R| \Rightarrow f$ surjective.
   (iv) $\mathrm{im}(f)$ maximal and $|N| = |R| \Rightarrow f$ bijective.
2. Show that for $f: N \to N$ with $|N| < \infty$ the concepts injective, surjective, and bijective are equivalent, but that this is not true if N is infinite.

→ 3. Let $N_<$ and $R_<$ be posets. Describe $\mathrm{Mon}(N, R)$ when $N_<$ is an antichain and when $R_<$ is an antichain.

→ 4. Show that there are precisely 5 non-isomorphic posets with 3 elements and 16 with 4 elements. How many are isomorphic to their dual? How many are lattices?

→ 5. Show that a directed graph $\vec{G}(V, E)$ is the diagram of some poset on V if and only if for any directed path $a_0 \to a_1 \cdots \to a_t$ of length $t \ge 2$ we always have $(a_0, a_t) \notin E$ and $(a_t, a_0) \notin E$.

6. Show that a poset is a chain if and only if all subposets are lattices.

7. Find a bijection from $[\mathscr{C}(1)]^2$ to $\mathscr{C}(3)$ which is monotone but preserves neither infima or suprema, where $\mathscr{C}(n)$ is the chain of length n.

→ 8. Let $N = \{1, 2, 3\}_<$ and $R = \{1, 2, 3, 4\}_<$. Compute $|\mathrm{Map}(N, R)|$, $|\mathrm{Inj}(N, R)|$, $|\mathrm{Sur}(N, R)|$, and $|\mathrm{Mon}(N, R)|$.

9. Let G and H be groups. Prove that a partition π of G is the kernel of a homomorphism $f: G \to H$ if and only if the following holds: If a and a'

lie in a common block of π and similarly for b and b' then $a \cdot b$ and $a' \cdot b'$ also lie in a common block of π. Deduce from this the homomorphism theorem for groups.

10. Let N and R both be cyclic groups of order 9. What is $|\mathrm{Hom}(N, R)|$?

→11. Suppose V and W are vector spaces over $GF(2)$ of dimension m and n, respectively. Compute $|\mathrm{Hom}(V, W)|$.

→12. In how many ways can we sort the elements 1, 2, 3 into the boxes A, B if, in addition, we require that the boxes are linearly ordered? For example,

$$\underset{A}{1, 2, 3} \,|\, \underset{B}{\varnothing}$$

is different from

$$\underset{A}{1, 3, 2} \,|\, \underset{B}{\varnothing};$$

similarly 2|1, 3 is different from 2|3, 1. (Answer: 24)

→13. Suppose $v = \sum_{i=1}^{b} l_i - b + 1$ objects are sorted into b boxes $B_1, \ldots, B_b$. Show that for some i, box B_i contains at least l_i objects. This is called the "pigeon hole principle."

14. By using ex. 13 prove that a sequence of $mn + 1$ distinct integers contains either an increasing subsequence of length greater than m or a decreasing subsequence of length greater than n.

15. Prove that there are two people in New York City who have precisely the same number of hairs on their head. (Hint: Use the pigeon hole principle.)

2. Fundamental Orders

We mentioned at the outset that our main object was to count and order classes of mappings. Later on we shall see that a good many of the counting problems consist in evaluating certain coefficients of the underlying order structure. So, let us first find out what order relations arise in connection with our general set-up.

A. Inclusion

First we may compare mappings by looking at their images. Let f, g be mappings with $\mathrm{im}(f) \subseteq R$, $\mathrm{im}(g) \subseteq R$. Regarding $\mathrm{im}(f)$, $\mathrm{im}(g)$ as multisets $\{b^{|f^{-1}(b)|} : b \in R\}$, $\{b^{|g^{-1}(b)|} : b \in R\}$, we obtain a natural relation

$$f \subseteq g :\Leftrightarrow |f^{-1}(b)| \le |g^{-1}(b)| \quad \text{for all } b \in R.$$

This inclusion relation $\subseteq$ is obviously reflexive and transitive but, in general, not antisymmetric since it disregards the nature of the elements which are mapped into