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James W. Vick

Homology Theory

An Introduction to Algebraic Topology

Second Edition

同调论 第2版

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to Niki, Todd, and Stuart

Preface to the Second Edition

The 20 years since the publication of this book have been an era of continuing growth and development in the field of algebraic topology. New generations of young mathematicians have been trained, and classical problems have been solved, particularly through the application of geometry and knot theory. Diverse new resources for introductory coursework have appeared, but there is persistent interest in an intuitive treatment of the basic ideas.

This second edition has been expanded through the addition of a chapter on covering spaces. By analysis of the lifting problem it introduces the fundamental group and explores its properties, including Van Kampen's Theorem and the relationship with the first homology group. It has been inserted after the third chapter since it uses some definitions and results included prior to that point. However, much of the material is directly accessible from the same background as Chapter 1, so there would be some flexibility in how these topics are integrated into a course.

The Bibliography has been supplemented by the addition of selected books and historical articles that have appeared since 1973.

Preface to the First Edition

During the past twenty-five years the field of algebraic topology has experienced a period of phenomenal growth and development. Along with the increasing number of students and researchers in the field and the expanding areas of knowledge have come new applications of the techniques and results of algebraic topology in other branches of mathematics. As a result there has been a growing demand for an introductory course in algebraic topology for students in algebra, geometry, and analysis, as well as for those planning further work in topology.

This book is designed as a text for such a course as well as a source for individual reading and study. Its purpose is to present as clearly and concisely as possible the basic techniques and applications of homology theory. The subject matter includes singular homology theory, attaching spaces and finite CW complexes, cellular homology, the Eilenberg–Steenrod axioms, cohomology, products, and duality and fixed-point theory for topological manifolds. The treatment is highly intuitive with many figures to increase the geometric understanding. Generalities have been avoided whenever it was felt that they might obscure the essential concepts.

Although the prerequisites are limited to basic algebra (abelian groups) and general topology (compact Hausdorff spaces), a number of the classical applications of algebraic topology are given in the first chapter. Rather than devoting an initial chapter to homological algebra, these concepts have been integrated into the text so that the motivation for the constructions is more apparent. Similarly the exercises have been spread throughout in order to exploit techniques or reinforce concepts.

At the close of the book there are three bibliographical lists. The first includes all works referenced in the text. The second is an extensive list of books and notes in algebraic topology and related fields, and the third is a similar list of survey and expository articles. It was felt that these would best serve the student, teacher, and reader in offering accessible sources for further reading and study.

Acknowledgments

Acknowledgments to the First Edition

The original manuscript for this book was a set of lecture notes from Math 401–402 taught at Princeton University in 1969–1970. However, much of the technique and organization of the first four chapters may be traced to courses in algebraic topology taught by Professor E.E. Floyd at the University of Virginia in 1964–1965 and 1966–1967. The author was one of the fortunate students who have been introduced to the subject by such a masterful teacher. Any compliments that this book may merit should justifiably be directed first to Professor Floyd.

The author wishes to express his appreciation to the students and faculty of Princeton University and the University of Texas who have taken an interest in these notes, contributed to their improvement, and encouraged their publication. The typing of the manuscript by the secretarial staff of the Mathematics Department at the University of Texas was excellent, and particular thanks are due to Diane Schade, who types the majority of it. Many helpful improvements and corrections in the original manuscript were suggested by Professor Peter Landweber.

Finally, the author expresses a deep sense of gratitude to his wife and family for their boundless patience and understanding over years during which this book has evolved.

Acknowledgments to the Second Edition

When John Ewing inquired as to my interest in reissuing this book with Springer-Verlag, I was doubly pleased. First, there was interest in making it

available once more since it went out of print some years ago. Second, I would be offered the chance to include new topics that would give it broader appeal. I am grateful to John and to Springer-Verlag for their interest.

The intervening 20 years at the University of Texas have been superb. My colleagues among the faculty, staff, and students have provided much encouragement and support. In particular, I appreciate the opportunity to work jointly on research with John Alexander, Gary Hamrick, and Pierre Conner.

The essential reason for these happy years is conveyed in the dedication: Niki, Todd, and Stuart. From kindergarten through graduate school, from little league through weddings, and from professional success back to a doctoral program, there have been enough great memories to last a lifetime.

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CHAPTER 1

Singular Homology Theory

The purpose of this chapter is to introduce the singular homology theory of an arbitrary topological space. Following the definitions and a proof of homotopy invariance, the essential computational tool (Theorem 1.14) is stated. Its proof is deferred to Appendix I so that the exposition need not be interrupted by its involved constructions. The Mayer–Vietoris sequence is noted as an immediate corollary of this theorem, and then applied to compute the homology groups of spheres. These results are applied to prove a number of classical theorems: the nonretractibility of a disk onto its boundary, the Brouwer fixed-point theorem, the nonexistence of vector fields on even-dimensional spheres, the Jordan–Brouwer separation theorem and the Brouwer theorem on the invariance of domain.

If x and y are points in $\mathbb{R}^n$, define the *segment* from x to y to be $\{(1-t)x + ty | 0 \leq t \leq 1\}$. A subset $C \subseteq \mathbb{R}^n$ is *convex* if, given x and y in C, the segment from x to y lies entirely in C. Note that an arbitrary intersection of convex sets is convex. If $A \subseteq \mathbb{R}^n$, the *convex hull* of A is the intersection of all convex sets in $\mathbb{R}^n$ which contain A.

A p-simplex s in $\mathbb{R}^n$ is the convex hull of a collection of $(p+1)$ points $\{x_0, \ldots, x_p\}$ in $\mathbb{R}^n$ in which $x_1 - x_0, \ldots, x_p - x_0$ form a linearly independent set. Note that this is independent of the designation of which point is x_0.

1.1 Proposition. *Let $\{x_0, \ldots, x_p\} \subseteq \mathbb{R}^n$. Then the following are equivalent:*

(a) *$x_1 - x_0, \ldots, x_p - x_0$ are linearly independent;*
(b) *if $\sum s_i x_i = \sum t_i x_i$ and $\sum s_i = \sum t_i$, then $s_i = t_i$ for $i = 0, \ldots, p$.*

Proof. (a) $\Rightarrow$ (b): If $\sum s_i x_i = \sum t_i x_i$ and $\sum s_i = \sum t_i$, then

$$0 = \sum_{i=0}^{p} (s_i - t_i)x_i = \sum_{i=0}^{p} (s_i - t_i)x_i - \left[\sum_{i=0}^{p} (s_i - t_i)\right] x_0$$

$$= \sum_{i=1}^{p} (s_i - t_i)(x_i - x_0).$$

By the linear independence of $x_1 - x_0, \ldots, x_p - x_0$ it follows that $s_i = t_i$ for $i = 1, \ldots, p$. Finally, this implies $s_0 = t_0$ since $\sum s_i = \sum t_i$.

(b) $\Rightarrow$ (a): If $\sum_{i=1}^{p} (t_i)(x_i - x_0) = 0$, then $\sum_{i=1}^{p} t_i x_i = (\sum_{i=1}^{p} t_i)x_0$ and by (b) the coefficients $t_1, \ldots, t_n$ must all be zero. This proves linear independence. □

Let s be a p-simplex in $\mathbb{R}^n$ and consider the set of all points of the form $t_0x_0 + t_1x_1 + \cdots + t_px_p$, where $\sum t_i = 1$ and $t_i \geq 0$ for each i. Note that this is the convex hull of the set $\{x_0, \ldots, x_p\}$ and hence from Proposition 1.1 we have the following:

1.2 Proposition. *If the p-simplex s is the convex hull of* $\{x_0, \ldots, x_p\}$, *then every point of s has a distinct unique representation in the form* $\sum t_i x_i$, *where* $t_i \geq 0$ *for all i and* $\sum t_i = 1$. □

The points x_i are the *vertices* of s. This proposition allows us to associate the points of s with $(p + 1)$-tuples $(t_0, t_1, \ldots, t_p)$ with a suitable choice of the coordinates t_i.

EXERCISE 1. Let y be a point in s. Then y is a vertex of s if and only if y is not an interior point of any segment lying in s.

If the vertices of s have been given a specific order, then s is an *ordered simplex*. So let s be an ordered simplex with vertices $x_0, x_1, \ldots, x_p$. Define σ_p to be the set of all points $(t_0, t_1, \ldots, t_p) \in \mathbb{R}^{p+1}$ with $\sum t_i = 1$ and $t_i \geq 0$ for each i. If a function

$$f: \sigma_p \to s$$

is given by $f(t_0, \ldots, t_p) = \sum t_i x_i$, then f is continuous. Moreover, from the uniqueness of representations and the fact that σ_p and s are compact Hausdorff spaces it follows that f is a homeomorphism. Thus, each ordered p-simplex is a natural homeomorphic image of σ_p. Note that σ_p is a p-simplex with vertices $x'_0 = (1, 0, \ldots, 0)$, $x'_1 = (0, 1, \ldots, 0)$, $\ldots$, $x'_p = (0, \ldots, 0, 1)$. σ_p is called the *standard* p-simplex with natural ordering.

Let X be a topological space. A *singular p-simplex* in X is a continuous function

$$\phi: \sigma_p \to X.$$

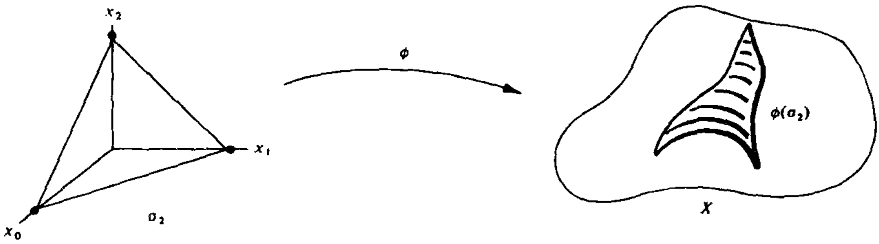

Figure 1.1

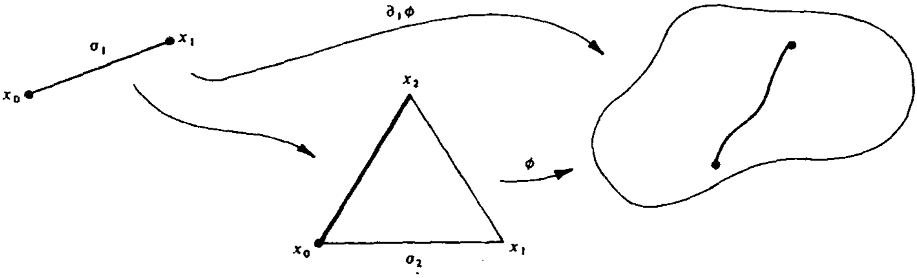

Figure 1.2

Note that the singular 0-simplices may be identified with the points of X, the singular 1-simplices with the paths in X, and so forth.

If ϕ is a singular p-simplex and i is an integer with $0 \le i \le p$, define $\partial_i(\phi)$, a singular $(p-1)$-simplex in X, by

$$\partial_i\phi(t_0,\ldots,t_{p-1}) = \phi(t_0,t_1,\ldots,t_{i-1},0,t_i,\ldots,t_{p-1}).$$

$\partial_i\phi$ is the ith *face* of ϕ.

For example, let ϕ be a singular 2-simplex in X (Figure 1.1). Then, $\partial_1\phi$ is given by the composition shown in Figure 1.2. That is, to compute $\partial_i\phi$ we embed σ_{p-1} into σ_p opposite the ith vertex, using the usual ordering of vertices, and then go into X via ϕ.

If $f\colon X \to Y$ is a continuous function and ϕ is a singular p-simplex in X, define a singular p-simplex $f_\#(\phi)$ in Y by $f_\#(\phi) = f \circ \phi$. Note that if $g\colon Y \to W$ is continuous and id: $X \to X$ is the identity map,

$$(g \circ f)_\#(\phi) = g_\#(f_\#(\phi)) \qquad \text{and} \qquad (\mathrm{id})_\#(\phi) = \phi.$$

An abelian group G is *free* if there exists a subset $A \subseteq G$ such that every element g in G has a unique representation

$$g = \sum_{x \in A} n_x \cdot x,$$

where n_x is an integer and equal to zero for all but finitely many x in A. The set A is a *basis* for G.

Given an arbitrary set A we may construct a free abelian group with basis A in the following manner. Let $F(A)$ be the set of all functions f from A into the integers such that $f(x) \neq 0$ for only a finite number of elements of A. Define an operation in $F(A)$ by $(f + g)(x) = f(x) + g(x)$. Then $F(A)$ is an abelian group. For any $a \in A$ define a function f_a in $F(A)$ by

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_a | a \in A\}$ is a basis for $F(A)$ as a free abelian group. Identifying a with f_a completes the construction.

For example, let $G = \{(n_1, n_2, \ldots) | n_i \text{ is an integer, eventually } 0\}$. Then G is an abelian group under coordinatewise addition, and furthermore it is free with basis

$$(1, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots.$$

For convenience we say that if $G = 0$, then G is a free abelian group with empty basis.

Note that if G is free abelian with basis A and H is an abelian group, then every function $f: A \to H$ can be uniquely extended to a homomorphism $f: G \to H$.

If X is a topological space define $S_n(X)$ to be the free abelian group whose basis is the set of all singular n-simplices of X. An element of $S_n(X)$ is called a *singular n-chain* of X and has the form

$$\sum_\phi n_\phi \cdot \phi,$$

where n_ϕ is an integer, equal to zero for all but a finite number of ϕ.

Since the ith face operator ∂_i is a function from the set of singular n-simplices to the set of singular $(n-1)$-simplices, there is a unique extension to a homomorphism

$$\partial_i: S_n(X) \to S_{n-1}(X)$$

given by $\partial_i(\sum n_\phi \cdot \phi) = \sum n_\phi \cdot \partial_i \phi$. Define the *boundary operator* to by the homomorphism

$$\partial: S_n(X) \to S_{n-1}(X)$$

given by

$$\partial = \partial_0 - \partial_1 + \partial_2 + \cdots + (-1)^n \partial_n = \sum_{i=0}^{n} (-1)^i \partial_i.$$

1.3 Proposition. *The composition* $\partial \circ \partial$ *in*

$$S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} S_{n-2}(X)$$

is zero.

EXERCISE 2. Prove Proposition 1.3. □

Geometrically this statement merely says that the boundary of any n-chain is an $(n-1)$-chain having no boundary. It is this basic property which leads to the definition of the homology groups. An element $c \in S_n(X)$ is an *n-cycle* if $\partial(c) = 0$. An element $d \in S_n(X)$ is an n-boundary if $d = \partial(e)$ for some $e \in S_{n+1}(X)$. Since ∂ is a homomorphism, its kernel, the set of all n-cycles, is a subgroup of $S_n(X)$ denoted by $Z_n(X)$. Similarly the image of ∂ in $S_n(X)$ is the subgroup $B_n(X)$ of all n-boundaries.

Note that Proposition 1.3 implies that $B_n(X) \subseteq Z_n(X)$ is a subgroup. The quotient group

$$H_n(X) = Z_n(X)/B_n(X)$$

is the nth *singular homology group* of X. The geometric motivation for this algebraic construction is evident; the objects we wish to study are cycles in topological spaces. However, in using singular cycles, the collection of all such is too vast to be effectively studied. The natural approach is then to restrict our attention to equivalence classes of cycles under the relation that two cycles are equivalent if their difference forms a boundary of a chain of one dimension higher.

This algebraic technique is a standard construction in homological algebra. A *graded* (abelian) *group* G is a collection of abelian groups $\{G_i\}$ indexed by the integers with componentwise operation. If G and G' are graded groups, a homomorphism

$$f: G \to G'$$

is a collection of homomorphisms $\{f_i\}$, where

$$f_i: G_i \to G'_{i+r}$$

for some fixed integer r. r is then called the *degree* of f. A subgroup $H \subseteq G$ of a graded group is a graded group $\{H_i\}$ where H_i is a subgroup of G_i. The quotient group G/H is the graded group $\{G_i/H_i\}$.

A *chain complex* is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

in which the composition $\partial_{n-1} \circ \partial_n = 0$ for each n. Equivalently a chain complex is a graded group $C = \{C_i\}$ together with a homomorphism $\partial: C \to C$ of degree -1 such that $\partial \circ \partial = 0$. If C and C' are chain complexes with boundary operators ∂ and ∂', a chain map from C to C' is a homomorphism

$$\Phi: C \to C'$$

of degree zero such that $\partial' \circ \Phi_n = \Phi_{n-1} \circ \partial$ for each n. (Note that the requirement that Φ have degree zero is unnecessary. It is stated here only as a convenience since all chain maps we will consider have this property.) Denoting by $Z_*(C)$ and $B_*(C)$ the kernel and image of ∂, respectively, the homology of C is the graded group

$$H_*(C) = Z_*(C)/B_*(C).$$

Note that if Φ is a chain map,

$$\Phi(Z_*(C)) \subseteq Z_*(C') \qquad \text{and} \qquad \Phi(B_*(C)) \subseteq B_*(C').$$

Therefore, Φ induces a homomorphism on homology groups

$$\Phi_*: H_*(C) \to H_*(C').$$

In this sense the graded group $S_*(X) = \{S_i(X)\}$ becomes a chain complex under the boundary operator ∂, so that the homology group of X is the homology of this chain complex. If $f: X \to Y$ is a continuous function and ϕ is a singular n-simplex in X, there is the singular n-simplex $f_\#(\phi) = f \circ \phi$ in Y. This extends uniquely to a homorphism

$$f_\#: S_n(X) \to S_n(Y) \qquad \text{for each} \quad n.$$

To show that $f_\#$ is a chain map from $S_*(X)$ to $S_*(Y)$ it must be checked that the following rectangle commutes:

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ S_{n-1}(X) & \xrightarrow{f_\#} & S_{n-1}(Y) \end{array}$$

First note that it is sufficient to check that this is true on singular n-simplices ϕ, and second, observe that it is sufficient to show $\partial_i f_\#(\phi) = f_\# \partial_i(\phi)$. Now

$$f_\# \partial_i(\phi)(t_0, \ldots, t_{n-1}) = f(\phi(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}))$$

and

$$\begin{aligned} \partial_i f_\#(\phi)(t_0, \ldots, t_{n-1}) &= f_\#(\phi)(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\ &= f(\phi(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})). \end{aligned}$$

Thus, $f_\#: S_*(X) \to S_*(Y)$ is a chain map and there is induced a homomorphism of degree zero

$$f_*: H_*(X) \to H_*(Y).$$

Note that this is suitably functorial in the sense that for $g: Y \to W$ a continuous function and id: $X \to X$ the identity, $(g \circ f)_* = g_* \circ f_*$ and id_* is the identity homomorphism.

As a first example take $X =$ point. Then for each $p \geq 0$ there exists a unique singular p-simplex $\phi_p: \sigma_p \to X$. Note further that for $p > 0$, $\partial_i \phi_p = \phi_{p-1}$. So consider the chain complex

$$\cdots \to S_2(\text{pt}) \to S_1(\text{pt}) \to S_0(\text{pt}) \to 0.$$

Each $S_n(\text{pt})$ is an infinite cyclic group generated by ϕ_n. The boundary opera-