天津市科协自然科学学术专著基金
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资助出版

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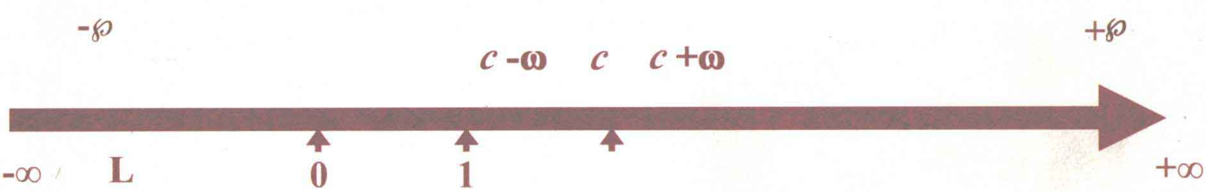
标准的欧弥伽无穷小微积分学

黄乘规 著



中英双语

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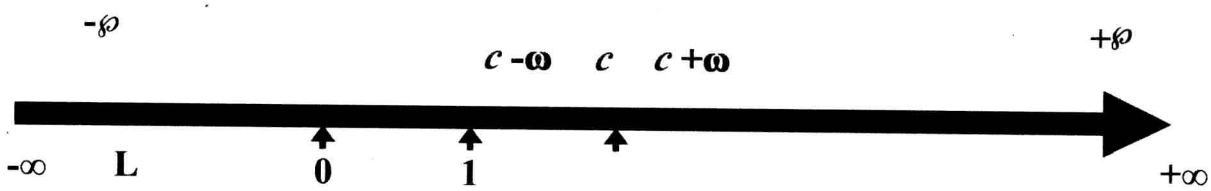

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卷首语

希腊时代，微分曾被看成既不是零又比任何数都小的无穷小量，瞬时速度看成两个无穷小量之比，曲线围成的面积看成无穷多个无穷小面积的和。这种简单看法，虽当时没有得到明确系统的论证，却有许多直观方便理解的地方。

近代，直线是一几何的形，是连续的，给了迪卡儿坐标系后，首先可以在其上建立的是有理数系，人们发现有理数系虽是密集的，但是在直线上还存在孔隙，因而引进了德氏分划，用实数来补充。通常认为，所有实数就填满了直线，线上的所有点与全体实数就能一一对应。实数可以构成直线。微积分有了系统的理论模型。但是**无穷小量却在这个模型里没有位置**。长期的努力，无穷小在实数系里找不到实体位置，只能用趋向为 0 的极限过程来表示。包含无穷小的微积分学没有建立起来。不能得到简洁的理解。但人们就认为已经解决，大部分人也就不去研究它了。

人们曾猜想 **点只表示离散性**，是位置的标志，**没有广延性**，点与点是互为外在的，相互不接触的。单个实数的测度为 0，而所有实数的测度也应该为 0，无厚是不可积的。无论如何也表示不了直线的连续性。只以点为基础，是组不成线的。所有有理数集合的测度为 0，而传统认为实数集合的测度却可以不再是 0。因为它可以覆盖全部区间或直线。是否构成测度定义的区间覆盖取法无意中有增添。延续是线的属性，以线性的区间为基础，用无穷个区间的交来定义无穷小，情况会有不同。就会得到先哲所期望的**不可分的连续统**或作为**联系纽带的单子**。也许这正可以**正确表示了延续性**，是区间测度的来源。以上都是先哲所研究过的，而没有得到真正解决的猜想。

本书的作者经过多年艰苦而坚持不懈的努力，分析了古今中外的许多前人思考过的思想观念，提出一个恰当的模型—**欧弥伽**（*Omega*）**连续统模型**，在实数系上再加进一些特殊成分，用一种**区间的无穷交**作为新成分，经过仔细验证，的确构成了扩展的**真正的连续统模型**。在这里，给无穷小，既不是零又比任何实数都小，以一个实体的位置。无厚不可积，所有全体实数的测度仍为零，而单子对实数是空集合，测度却是正的，是线性测度的真正来源。

作者对于原连续统做了恰当的扩充，测度做了恰当的定义，给无穷小量以恰当实体位置，对原来测度理论做了修正，对于各种可能的误解加以解释。并经过详细严格的论证，为新模型建立严谨的基础。

在第九章之后，在建立的新的模型基础上，重新写了微积分学，把原来的数学分析嵌入其中，而对于各种基本概念做了适当的扩展。这模型结合了古老的猜想与近代严格的理论，因而构筑了新型的严格的无穷小微积分，便于学习。这也推动人们对**数学基础问题的兴趣与注意**。长期人们的撼人心弦的追求，期望能得到满意的解决。

林建祥

2008 年 12 月 22 日于博雅西园

Foreword

In Ancient Greek Time, differential was thought of such an infinitesimal that was not zero but less than any number. Instantaneous velocity was thought of as a ratio of two infinitesimals. An area surrounded with curves was thought of as a sum of infinite infinitesimal areas. These simple views did not get any obvious systematical reasoning, but had many intuitive convenient understandings.

Till nowadays, a line is a geometrical form, which is continuous. After given Descartes coordinate, people might firstly set up rational number system in the line, found such a fact that although the rational number system is dense, but it exists still holes in the line. Afterwards introduced Dedekind cut to complement them by real numbers. People considered commonly that the set of real numbers could fill out the line, and that all points on the line could form a bijection with the set of real numbers. The set of real numbers could construct a line. It followed such a fact that differential and integral had a theoretical model. However ***there does not be any position for infinitesimals in the model***. Through efforts of a long time, in the set of real numbers, people could not find any position of entity for infinitesimal, while had no choice but to use such a limit process, which tends to zero, to denote. Infinitesimal Calculus had not set up yet. People were not able to obtain a simple understanding. However people thought of that the problem had solved. Most people do not think it again.

People had guessed that ***Points express only discontinuity***, they are only signs of positions, ***and have no extensibility***. Points are exterior each other, and don't touch each other. A single real number is of zero measure, since the set of all real numbers should also be of zero measure, no thickness may not gain any integration. Points can not represent any continuity of a line at all events. Only based on points people can not buildup a line. The set of all rational numbers is of zero measure. However according to the tradition, the set of all real numbers was not of zero measure, because which was able to cover all intervals or a line. Whether there were some involuntary additions in the cover method of intervals for the construction of definition of measure? Extensibility is a property of line. Based on linear intervals, and using the intersection of infinite intervals to define infinitesimal, the situation must be different. It must obtain the concept of ***indivisible continuum*** or ***monad as intimate bond*** that were ever expected by a lot of ancient wisdoms. Perhaps this just ***represents correctly the extensibility***, and is also the right source of interval measure. All these were hypotheses, which had been thought of by a lot of ancient mathematical philosophers, but never really solved

The author of the book lasts many years with hard but unremitting efforts, analyzes a lot of idea concepts which were thought of by a lot of forth goers whether in ancient times or nowadays, in China or abroad, then raises a felicitous model – ***Omega continuum model.*** By adding some special elements into real number set, and using intersections of infinite intervals to be new elements, through careful proof-tests, the author constructs certainly an extensional ***true model of continuum***. Where gives an entity position to infinitesimal which is not zero, but is smaller than every positive real number. No thickness may not gain any integration, the set of all real numbers is still of zero measure. And the monad is an empty set relative to real numbers, whose measure is certainly positive, that is the true source of linear measure.

The author makes an aptitude extension to Cantor continuum, a reasonable definition to measure, and gives a reasonable position of entity to infinitesimal, a revision to classical measure theory, and makes explanations to several eventual misconstructions. Through careful rigorous proof-tests, the author sets up a precise foundation for the new model.

After the ninth chapter, based on the new model, the author rewrites the infinitesimal calculus, in which, part contents of the classical mathematical analysis embed, and makes suitable extensions for basic conceptions. The model combines with the ancient hypotheses and rigorous theory of nowadays, and constructs a new rigorous infinitesimal calculus. It is convenient to learn. It also gives an impulse to ***the interesting and attention to the problem of mathematical foundation***. For the dramatic research of human for a long time, I expect to get a satisfactory solution.

Lin Jian Xiang

22 December, 2008, in Boya Xi Yuan

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简单译名表(A simple table of translated names)

Archimedes 阿基米德
Aristotle 亚里士多德
Borel 薄磊尔
Boyer 波耶
Cantor 康托
Carathéodory 卡拉特欧多里
Cauchy 哥西
Cavalieli 卡瓦列里
Courant 柯朗
Dedekind 狄特金
Democritus 德谟克利特
Descartes 笛卡儿
Euclid 欧几里德
Eudoxus 攸多克索
Galileo 伽利略
Gödel 哥德尔
Hegel 黑格尔
Hilbert 希尔伯脱
Iyanaga Shokichi 弥永昌吉
Jordan 若当
Lagrange 拉格朗日
Lebesgue 勒贝格
Leibniz 莱布尼茨
Newton 牛顿
Plato 柏拉图
Poincaré 彭加勒
Pythagoras 毕达哥拉斯
Pythagorean 毕达哥拉斯学派
Rao 饶
Riemann 黎曼
Robinson 鲁滨逊
Russell 罗素
Schwarz 许瓦兹
Skolem 斯科伦
Steen 史廷
Taylor 泰勒
Weierstrass 外尔斯特拉斯
Zeno 芝诺
Zhuangzhou 庄周

特殊符号表

Π^+：$-\infty$的右单子
Π^-：∞的左单子
$a\triangleright\Omega^+$：a 的右单子
$\Omega^-\triangleleft a$ ：a 的左单子
$p'(x_+)$：p 在点 x 的右导数
$p'(x_-)$：p 在点 x 的左导数
$\overline{\boldsymbol{R}}$ ：闭的实数集 $\boldsymbol{R}\cup\{\pm\infty\}$
$\int_a^b$（带箭头）：正向积分
π^+：Π^+的欧弥伽分割
π^-：Π^-的欧弥伽分割定义
$\underline{w}\rightarrow$ ：外尔斯特拉斯极限
$\underline{\Omega}\rightarrow$ ：欧弥伽极限

L_+：代表开的射线 $(0,+\infty)$

$\boldsymbol{R}_+$：开的正实数区间 $\boldsymbol{R}(0,+\infty)$

$\text{wlim}_{n \to \infty}$：外尔斯特拉斯极限

$\text{wlim}_{x \to y}$：外尔斯特拉斯极限

$\text{wlim}_{x \to y_+}$：外尔斯特拉斯极限

$\text{wlim}_{x \to y_-}$：外尔斯特拉斯极限

⊞：欧弥伽微分加法

⊠：欧弥伽微分乘法

$\int_a^b$：逆向积分

$a \triangleright \omega^+$：$a \triangleright \Omega^+$的欧弥伽分割

$\omega^- \triangleleft a$：$\Omega^- \triangleleft a$ 的欧弥伽分割

$\overline{\Omega\Pi}$：闭的欧弥伽连续统 $\Omega\Pi \cup \{\pm\infty\}$

$\text{olim}_{n \to +P}$：欧弥伽极限

$\text{olim}_{x \to c+\omega}$：欧弥伽极限

$\text{olim}_{x \to c-\omega}$：欧弥伽极限

$\omega^\wedge$：代表以实数 c 为中心的三元素区间的并集合的微分测度

$\int_a^b$：数学分析里的积分

A Table of Special Symbols

Π^-: The right monad of $-\infty$

Π^+: The left monad of ∞

$a \triangleright \Omega^+$: The right monad of a

$\Omega^- \triangleleft a$: The left monad of a

L_+: The open ray $(0,+\infty)$

$\boldsymbol{R}_+$: The open positive real number interval

$\text{wlim}_{n \to \infty}$: Weierstrass limit

$\text{wlim}_{x \to y}$: Weierstrass limit

$\text{wlim}_{x \to y_+}$: Weierstrass limit

$\text{wlim}_{x \to y_-}$: Weierstrass limit

$p'(x_+)$: Right derivative of p at x

$p'(x_-)$: Left derivative of p at x

$\overline{\boldsymbol{R}}$: Closed real number set $\boldsymbol{R} \cup \{\pm\infty\}$

$\int_a^b$: Positive integral

⊞: Omega differential addition

⊠: Omega differential multiplication

$\int_a^b$: Inverse integral

$a \triangleright \omega^+$: Omega cut of $a \triangleright \Omega^+$

$\omega^- \triangleleft a$: Omega cut of $\Omega^- \triangleleft a$

$\overline{\Omega\Pi}$: Closed Omega continuum $\Omega\Pi \cup \{\pm\infty\}$

π^+: Omega cut of Π^+

π^-: Omega cut of Π^-

$\xrightarrow{w}$: Weierstrass limit

$\xrightarrow{\Omega}$: Omega limit

$\text{olim}_{n \to +P}$: Omega limit

$\text{olim}_{x \to c+\omega}$: Omega limit

$\text{olim}_{x \to c-\omega}$: Omega limit

$\omega^\wedge$: The differential measure of the union set of three elements interval with its center at c

$\int_a^b$: The integral of mathematical analysis

本书摘要

（注解：本书由天津市科协自然科学学术专著基金和天津师范大学学术著作出版基金资助出版）

作者在 2002 年 8 月 26 日北京举行的国际数学家大会上做了一个简短的发言，标题是：《标准的无穷小微积分学》。其摘要如下：

“此文对一条装上了标架的欧几里德直线给出了完整的微分分拆，并对以实数为标号的无穷小的积分给出了三条公理；在标准数学中证明了在实数集合之外存在正的无穷小；对若当、卡拉特欧多里和勒贝格测度论中的两条公理给出了宇观的、宏观的和微观的反例；将外尔斯特拉斯极限改进为黄氏极限，将康托连续统改进为黄氏连续统，和将牛顿和莱布尼茨公式改进为黄氏公式。”

本书是这个短的发言的发展。

本书中有两个发现和三十个成果。其中有二十五个成果是关于当代数学的，另外的五个是解决了五个公元前 550～250 年之间的或者更古老的历史难题。

第一发现了无穷小直线元素（即毕达哥拉斯学派称之为不定小的单子的对象）并系统地研究了它的性质。

第二发现了标准的无穷小微积分学的基础：一个新模型——欧弥伽连续统模型，和一个初具规模的公理系——欧弥伽微积分公理系。这个也为整个数学的发展提供了新的基础。

为了表述清楚关于当代数学的二十五个成果，令 $\boldsymbol{R}$ 代表实数集合且 $r\in\boldsymbol{R}$ 是任意的，这二十五个成果是：

1. 本书从第一章开始，逐步证明了实数集合不能填满一条装上了标架的欧几里德直线，证明了实数集合仍然是离散的，它不是一个连续统，不能填满线段。

2. 对装上了标架的欧几里德直线 L 给出了完整的微分分拆，即 L = {$-\infty$ 的右单子 ∪ r 的左单子 ∪ r ∪ r 的右单子 ∪ ∞的左单子} = 欧弥伽连续统 ΩΠ，这是无穷小微分学和积分学的基础，并对无穷小量的正向积分建立了三条公理；证明了欧几里德直线是由实数系，无穷小直线元素和无穷大直线元素所共同拼接而成，与欧几里德直线等价的这个数学结构定名为欧弥伽连续统 ΩΠ。

3. 令 ω 代表 r 的左单子和 r 的右单子（它们是一些无穷小线段元素）的共同测度，在标准数学中证明了ω是 $\boldsymbol{R}$ 之外的正无穷小，这就是证明了无穷小直线元素的存在性。这是由毕达哥拉斯开始,经过伽利略、牛顿、莱布尼茨，直到美国著名逻辑学家鲁滨逊等著名学者的两千五百多年的研究，而公开承认没有解决的。

4. 对若当，卡拉特欧多里和勒贝格测度论中的实数的空集合具有零测度的公理和康托的实数线公理给出了宇观的、宏观的和微观的反例，并给出了欧弥伽极限协调性测度的新概念。

5. 由单个自然数的测度为零证明了自然数集合 $\boldsymbol{N}$ 的测度也等于零；并且由单个实数的测度为零证明了实数集合 $\boldsymbol{R}$ 的测度等于零，这是十分麻烦的问题，本书解决了它。

6. 在 ΩΠ 中定义了序和算术运算。这是对数学运算的重大发展。过去对算术的运算强调确定性，而 ΩΠ 中的算术运算具有明显的不确定性。

7. 把外尔斯特拉斯极限改进为欧弥伽极限。我们知道，白色光线经过三棱镜折射发散成七色光

谱。同样，外尔斯特拉斯极限经过欧弥伽连续统的透析，成为七种不同的极限。

8. 把狄特金分割改进为欧弥伽分割。狄特金分割是对有理数集合的分拆，将有理数集合分拆为上下两组，而没有中组。所以无穷小直线元素无处存身。而欧弥伽分割由上中下三组构成。而中组是此分割的中心，可能由区间套组成。 它是无穷小线素的合理的存在形式。所以在狄特金分割的导引下不能发现无穷小线素的存在。而在欧弥伽分割的帮助下发现了它。

9. 把康托连续统改进为欧弥伽连续统。康托连续统就是实数系，是当前流行数学的基础。但它是离散的，不能完整地刻画欧几里德直线，遗漏了每个实数点的左右两端的无穷小线段元素。本书把康托连续统改进为欧弥伽连续统, 完整地刻画了欧几里德直线。

10. 在 ΩΠ 中给出了欧弥伽逆向积分的定义。因为本书中证明了“实数集合 $\boldsymbol{R}$ 的测度等于零”，积分概念需要重新加以陈述。作为原始的出发点，必须对欧弥伽连续统中定义的函数一般地给出欧弥伽逆向积分的定义。

11. 然后具体化，对于一个实数的函数给出了几种欧弥伽逆向积分的定义，并给出了可积函数类。

12. 对实数函数的微分给出了定义，并给出了赋值的正向积分的公式。

13. 提出了积分等值的辅助公理，因为在各种积分等式的推导中，特别在牛顿和莱布尼茨的公式的精确陈述和严格证明中及在将牛顿和莱布尼茨的公式扩大为欧弥伽公式时，必须用到此公理。此公理是本书理论骨架形成的推理基础之一。

14. 提出了对逆向积分估值的辅助公理，因为在各种逆向积分的无穷小误差的估计中，必须用到此公理。这个有利于新理论的建立。

15. 精确陈述和严格证明了牛顿和莱布尼茨的公式，并将它改进为欧弥伽公式。 众所周知，牛顿和莱布尼茨的公式是近代数学和古代数学的分水岭，是具有划时代意义的。但是从本书的结果看来，他们的发现只是一个良好的开端。但他们所达到的水平只是一个经验公式，既没有给出严格的陈述，也没有给出严谨的证明。在原来的数学分析理论中，只是含糊地提到定积分与不定积分的联系，没有给出恰好的证明。在牛顿和莱布尼茨的公式的严格的恰当的陈述中，需要用到本书的关于实数函数的欧弥伽第一扩大的概念。在其严格的证明中，需要用到本书的积分等值的辅助公理和对逆向积分估值的辅助公理。在将牛顿和莱布尼茨的公式改进为欧弥伽公式时，除了用到上述两条公理外，还需用到关于两个实数函数的欧弥伽第二扩大的概念。对于牛顿和莱布尼茨的公式而言，欧弥伽公式是一个实质性的发展。牛顿和莱布尼茨的公式中的被积分函数只有一个。而在欧弥伽公式中，被积分函数是两个，一个代表左导数，另一个代表右导数。在牛顿和莱布尼茨的公式中导数必须存在为一个统一的函数，才能谈到积分。而在欧弥伽公式中，只要左导数和右导数存在，就可以做出关于这两个函数的积分。反映在物理问题中，只要知道左瞬时速度和右瞬时速度，就可以求得距离。所以欧弥伽公式是更深入地反映了物理问题。总之，牛顿和莱布尼茨的公式只是走了第一步，而欧弥伽公式则是它的直接发展，走了第二步。

16. 又写了《第 9.3 节　空集合∅是任意集合的子集合吗？》，在此节中论述了：在初等数学中，逻辑公理“空集合∅是任意集合的子集合”是不适合的。英国数学哲学家罗素早就指出：若承认存在一个包含着每一个集合的集合 $\mathscr{R}$， 则数学推理中会出现罗素悖论。以后在数理逻辑中就排除了这样的一个包含着每一个集合的集合 $\mathscr{R}$ 。罗素并由此进一步提出了类型论。可是，在集合论中还承认存在一个这样的空集合∅，它是每一个集合的子集合，即承认：$\varnothing \subset A$ 是一条集合论公理。承认这样的∅的存在， 与承认上述 $\mathscr{R}$ 的存在，从理论上完全是类似的。怎么这么长时间没有发现其毛病。由此可见，对于公理 $\varnothing \subset A$ 应该重新考虑，它至少对于初等数学，对于一条装上了标架的欧几

里德直线是不适合的。实际上，数学研究的核心是一条装上了标架的欧几里德直线及其数学等价表示：欧弥伽连续统。这是文字数学和几何学的真正的统一。如果一条数学公理，对于欧几里德直线都不适用，又何必保留它呢？

17. 在《第 19.4 节 定义在一个有穷集合和它的补集合上的斯科伦阶梯函数——在 ΩΠ 中间断函数举例》的一节中，利用对于定义在一个有穷集合和它的补集合上的斯科伦阶梯函数的讨论，说明了极限方法的局限性，并指出了进一步发展的一个重要方向是在 ΩΠ 中研究间断函数的微分和积分，各种各样的狄拉克 Delta 函数和各种各样的奇异函数及其积分的问题；在 19.5 节中陈述了广义的正向积分的公理，以此对过去和现在的工作做了简单的回顾，并对今后的工作进行展望。

18. 在 6.1.1.1-2 回中定义了*欧弥伽微观面积微分运算*：包含了欧弥伽微分加法和欧弥伽微分乘法。并对它们引进了十二条微观面积微分运算公理。这些一方面反映了*欧弥伽面积微分运算*的本质，另方面在证明无穷小直线元素的存在性时，起了重要的作用。

19. 为了研究在装上了标架的欧几里德直线上半射线的测度，引进了*有中心点的对称性公理*和*无中心点的对称性公理*。它们一方面反映了装上了标架的欧几里德直线上半射线的测度的根本性质，另方面在证明无穷大直线元素的存在性时，起了重要的作用。

20. 对欧弥伽微积分学提供了初具规模的公理体系。现在把*欧弥伽微积分公理系*的内容简介如下：它由四部分组成。第一部分是十二条微观微分运算公理，请参阅第 6.1.1 品。第二部分是两条宇观公理：*有中心点的对称性公理和无中心点的对称性公理*。请参阅 1.6 节。第三部分涉及宏观与微观的联系，分成三组。第一组是化宏观度量为微观度量的积分的三条正向积分公理，请参阅第 5 章，第 6 章，第 15 章和第 19 章。第二组是积分等值的一组公理，请参阅第 16.1 节。第三组是对逆向积分估值的一组公理，请参阅第 16.2 节。第四部分包含三种情况：①欧几里德直线 L 上的测度公理（参考 1.3.1 品）；②关于序的基本原则(参考 1.3.2 品)；③从几何上迅速判断排序的“*本源几何形式*”（参考 3.2.5-7 品）。

21. 自 18.4 至 19.3 节，将经典的数学分析中的理论初步平行地扩大到欧弥伽无穷小微积分中，为今后形成一个与数学分析理论相平行，而更加广大的无穷小微积分的理论，提供了一个出发点。今后的发展前途十分广大。例如：在 18.6.4 中推出了：黎曼可积分类可以嵌入到有界的欧弥伽可积分类中成为一个子理论。

22. 从 18.4 至 18.15 节，在欧弥伽无穷小微积分中，平行而扩大地建立了可积分函数类型，积分等式和不等式、中值公式、分部积分法、换元积分法等原则。

23. 在 18.16 节中，在欧弥伽无穷小微积分中，平行于数学分析中的连续函数的概念，扩大地建立了欧弥伽连续函数的概念以及有关的定理：如介值定理等。这存在着继续发展的空间。

24. 在 19.1-3 节中，初步平行地、扩大地和系统地建立起欧弥伽微分学的基础；定义了一阶至 n 阶欧弥伽导数、建立了四则运算公式、有关的积分表示等，建立了欧弥伽微分学的基本定理，一直到建立了欧弥伽～Taylor 公式，它是数学分析中 Taylor 公式的直接发展。

25. 在第 18.1.4 品中，证明了无穷大直线元素的存在。令 P 代表数字无穷大∞的左单子Π^+和负的数字无穷大$-\infty$的右单子Π^-的共同测度，在标准数学中证明了 P 是 $\boldsymbol{R}$ 之外的并且满足 $P<\infty$的正无穷大。本书最终的结论是：“*数学之帝座者，闭的欧弥伽连续统是也。*”在闭的欧弥伽连续统$\overline{\Omega\Pi}$ 中，微分 dx 的定义域是集合{0，ω，P，∞}。

关于古代数学的五个成果是：

① 本书系统地严格地论证了不可分量的存在和作用。将毕达哥拉斯的格言**“一切皆数”**，按照毕达哥拉斯的语言，发展为一个更加完全的格言：“*除了数，一切皆不定小单子。*”

美国著名数学史专家波耶对于整个的数学历史的评价，在他的名著《微积分概念史》中文版的313～314 页所下的结论是："现在已经看到，在分析学中，除了整数，或有穷或无穷的数系之外，就一无所需了。关于微积分学的发展，Pythagorean 格言是惊人地贴切：'万物皆数'。" 波耶的名著以 Pythagorean 格言"*万物皆数*"开始，也以这句名言结束。考虑到本书的工作，数学伟人的这个格言应当做上述发展。Robinson 在参考文献[8]和 Skolem 在参考文献[4]中的无穷大或无穷小仍然是无穷大的非标准的实数或无穷小的非标准的实数，故他们的概念仍然落在 Pythagorean 格言"**一切皆数**"的范畴之内；

② 破解了芝诺(Zeno of Elea, 公元前五世纪)的总格言。在《微积分概念史》英文版的 23 页，中文版 26 页，波耶写出了芝诺 (Zeno of Elea, 公元前五世纪)的总格言："最带有摧毁性的攻势来自巴门尼德的学生芝诺。他断然地反对毕达哥拉斯学派的不定小的单子之后， ……为了攻击无穷小，他加上如下的总格言：'这种东西，加到别的东西上不使其增大，从别的东西中减去又不使其减小，不过是子虚乌有而已。' 参考《非标准分析》的参考文献[8]一书的第 280 页，非标准分析的创始人 Robinson 评论道："希腊古代的数学家，攸多克索，欧几里得和阿基米德在处理面积及相关问题时所持的谨慎态度，并不反映这门新生学科的不确定性，而是由于他们一贯地不相信无穷的概念。芝诺的几个悖论就是这种思想的最著名的表现。"Robinson 显然为芝诺的各种悖论所挡住而停步不前。为什么呢？显然，芝诺设置的这个总格言是"为了攻击无穷小"。 Robinson 虽然证明了包含无穷的数学系统存在的合理性～协调性。但他思想深处并不相信无穷对象的存在，所以他不能破解芝诺的总格言。参考 Steen 的《实数线的新模型》一文[3]的 99 页，他写道："令人感到尖锐对立的是，Robinson，无穷小的再创人，不相信其真实的存在性"。在实数集合 $\boldsymbol{R}$ 之外是否存在无穷小量，这是一个历史大难题，它使得 Leibniz 终生烦恼而得不到解决。所以历史上曾经发生这样的趣味事。英国大主教贝克莱(Bishop Berkeley) 也因此嘲笑 Newton 的无穷小(或者流数)是"ghosts of departed quantities(数量死亡后变成的鬼)"。"作为哲学家的 Leibniz 抱怨数学家的 Leibniz 引入了自相矛盾的无穷小，把无穷小描述为：'虚构，但是有用的虚构'"。

作为无穷小的再创人 Robinson ，在创立了非标准分析之后，重新引用了 Leibniz 的自相矛盾的描述"虚构，但是有用的虚构"，以着重说明他自己不相信无穷小的真实的存在性。其根本原因是因为他们没有证明：在标准数学中在实数系之外存在无穷小。自 Leibniz 到 Robinson，经过很多伟大数学家三百多年的反复研究，最后不能不说无穷小是"fictions(虚构)"。至此，这种虚构论可以结束了。本书证明了无穷小量～无穷小线性元素的存在性，也就从反面破解了芝诺的总格言；

③ 对我国学者庄周的"无厚，不可积也"的猜想，做了严格论证；

④ 对亚里士多德否认数能够产生一个连续统的猜想给出了严格的论证；

⑤ 严格论证了庄子的"万世不竭"的思想。本书对庄周的"万世不竭"的原则做了两方面的发展。 第一方面是本书对无穷小线性元素存在性的证明。请参阅在 20.5 和在 20.8.2 中的陈述。第二方面是在 1.6 节中提出了*有中心点的对称性公理和无中心点的对称性公理*。这两条公理肯定了无穷大线性元素的存在性。请参考 20.8.3 品。

因此，我们更加明确地认识到： Robinson 在参考文献[8]和 Skolem 在参考文献[4]中的无穷大或无穷小仍然是无穷大的非标准的实数或无穷小的非标准的实数，故他们的概念仍然落在 Pythagorean 格言"**一切皆数**"的范畴之内。而本书则构造性地证明了无穷大和无穷小线性元素的存在。

故结合到此书的工作，对于数学伟人毕达哥拉斯的格言，按照他的语言，也可以发展成为方便

的形式：“除了数，一切皆线。” 庄周的“万世不竭”的概念也发展成新的格言：“数不能穷竭线”和“一半射线，日去一尺，万世不竭。”。

波耶称近代数学分析为“无穷的交响乐”，但自毕达哥拉斯至鲁滨逊，两千五百多年来，并没有确证无穷的实体。直至本书发现了无穷大和无穷小的直线元素，作为“无穷的交响乐”的主要演员才登上了历史的舞台。

博士，教授 黄乘规 2009 年 2 月 14 日星期六做最后修订

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Abstract of the Book

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The author had a short Communication titled "Standard Infinitesimal Calculus", in the International Congress of Mathematicians, Beijing 2002, August 20-28. The contents of the abstract of the short Communication is as follows:

"The paper gives a complete differential partition to a Euclidean straight line with a fixed frame, and three axioms for the integral of infinitesimals indexed by real numbers; proves in standard mathematics there are positive infinitesimals outside of real number set; Gives cosmic, macro and micro counterexamples to two axioms in Jordan, Carathéodory, and Lebesgue measure theory; transforms Weierstrass limit into Huang limit, Cantor continuum into Huang continuum, and Newton-Leibniz formula into Huang formula."

Obviously the book is a development of the short Communication.

There are two discoveries and thirty achievements in the book, where twenty-five achievements are of nowadays' mathematics, and five achievements of difficult problems long standing for 2250-2500 years or more.

The first discovery is to find infinitesimal linear elements, that Pythagorean called indefinitely small monad, and to study systematically their properties. The second discovery is to find a new model — Omega continuum model, that constructs the foundation of standard infinitesimal calculus: and a primary regular axiomatic system — an *axiomatic system of Omega calculus*. These provide also a new foundation for whole mathematics.

In order to saying clearly the twenty-five achievements of nowadays' mathematics, let $\boldsymbol{R}$ denote real number set and $r \in \boldsymbol{R}$ be arbitrary. The twenty-five achievements of nowadays mathematics are as follows:

1. The book begins with the first chapter to gradually prove that the real number set cannot fill up a Euclidean straight line with a fixed frame, and that the real number set is still discrete and not a continuum, and cannot fill up any linear segment.

2. Gives a complete differential partition to a Euclidean straight line with a fixed frame L, i.e. L = {the right monad of $-\infty$ $\cup$ the left monad of r $\cup$ r $\cup$ the right monad of r $\cup$ the left monad of ∞} = Omega continuum $\Omega\Pi$, that is the foundation of infinitesimal calculus, sets up three axioms for the positive integral of infinitesimals; and proves that the Euclidean straight line with a fixed frame L consists of real number set $\boldsymbol{R}$, and simultaneously of infinitesimal and infinitelarge linear elements, the mathematical model which is equal to the Euclidean straight line with a fixed frame, is named Omega continuum $\Omega\Pi$.

3. Let ω denote the common measure of the left monad and right monad (which are infinitesimal

linear elements) of r, proves in standard mathematics that ω is a positive infinitesimal outside of $\boldsymbol{R}$. This proves the existence of infinitesimal linear elements. This study began with Pythagorean, through Galileo, Newton, Leibniz, till USA celebrated logician Robinson. They declared in public that the problem was unsolved for two thousand and five hundred years or more.

4. Gives cosmic, macro and micro counterexamples for two axioms which are axioms claiming real number empty set being of zero measure in Jordan, Carathéodory, and Lebesgue measure theory, and Cantor's real number line axiom, and sets up a new concept of Omega limit consistence measure.

5. Proves that the natural number set N is of zero measure from a single natural number with zero measure; And that the real number set $\boldsymbol{R}$ is of zero measure from a single real number with zero measure. These were also difficult problems; however they are solved in the book.

6. Defines order and arithmetic operators in $\Omega\Pi$. This is an important development for mathematical operators. In the past for arithmetic operators people insisted on definitude. However in $\Omega\Pi$, arithmetic operators have obviously indefinitude.

7. Transforms Weierstrass limit into Omega limit. We know that by refraction though a triangular prism a bar of white light diverges to be seven color lights. Similarly by the refraction of Omega continuum $\Omega\Pi$ Weierstrass limit diverges to be seven different limits.

8. Transforms Dedekind cut into Omega cut. Dedekind cut is a partition of rational number set, to partition it into two subsets, upper and lower sets. It has no middle set. Therefore there is no place for the live of infinitesimal linear elements. However Omega cut, it consists of upper, middle and lower sets. Its middle set is the center of the cut, may be an interval nest. It is a reasonable place to live infinitesimal linear elements. This is why under Dedekind cut people could not introduce infinitesimal linear element, and by using Omega cut we find infinitesimal linear elements.

9. Transforms Cantor continuum into Omega continuum. Cantor continuum is namely real number set. It is the foundation of current popular mathematics. However it is discrete, can't describe completely a Euclidean straight line with a fixed frame and misses the infinitesimal linear elements on the left and right sides of every real number point. The book transforms Cantor continuum into Omega continuum, and completely describes a Euclidean straight line with a fixed frame.

10. Gives the definition of Omega inverse integral in $\Omega\Pi$. Because in the book we prove that the measure of the real number set $\boldsymbol{R}$ is equal to zero, the concept of integral needs a new statement. As an original start, we must generally give definition of Omega inverse integral for a function defined in Omega continuum.

11. Then concretely, gives several definitions for Omega inverse integral for a real number function and several integrable function classes.

12. Gives definition of differential of a real number function, and the definition of valuation formula of positive integral.

13. Brings forward auxiliary axioms for integral equality. They are necessary for the deduction of integral equality. Especially in the accurate statement and rigorous proof of Newton-Leibniz formula, and in its transformation to Omega formula, they are necessary. The axioms are deductive foundations of forming the theoretical frame of the book.

14. Puts forward auxiliary estimate axioms for inverse integrals. They are necessary for the estimates

of infinitesimal error for inverse integral. These have advantage over the constitution of the new theory.

15. Accurately states and thoroughly proves the so-called Newton-Leibniz formula, and transforms it into Omega formula. It is well known that Newton-Leibniz formula is the watershed between the modern mathematics and ancient mathematics, and has the epoch-making sense. However from the results of the book, we see that their discovery is only a good dayspring. And the level that they arrived is only an experiential formula. They had not given rigorous statement and justification. In the popular mathematical analysis, it is only equivocally to make mention the link theorem between definite and indefinite integrals and gives not any justification proof. In the just statement of Newton-Leibniz formula, it need use the concept of the first Omega enlargement of a real function in the book. In the rigorous proof of Newton-Leibniz formula, it need use the auxiliary axioms of integral equality and the auxiliary axioms for the estimate of inverse integral in the book. In the transformation of Newton-Leibniz formula to Omega formula, except for the just announcement two axioms, it need use the concept of the second Omega enlargement of two real functions in the book. For Newton-Leibniz formula, Omega formula is an essential development. In Newton-Leibniz formula, the integrand is only one function. However in Omega formula, the integrand has two functions, in which one stands for the left derivative, and the other stands for the right derivative. In Newton-Leibniz formula, people talk about integral only when there is a united derivative. However in Omega formula, only if there are left derivative and right derivative, one can make out the integral of the two functions. Correspondingly for problems in physics, one can solve the distance if knowing the left and right instantaneous velocities. Omega formula images deeply problems in physics. In a word, Newton-Leibniz formula only walks the first step, Omega formula is its direct development, and walks the second step.

16. Writes the section 9.3: 《Is the empty set $\varnothing$ a subset of every set?》. In the section states that in elementary mathematics, the logical axiom "For every set A, $\varnothing \subset A$." is not consistent. Russell, an English celebrated mathematical philosopher, pointed early out: If accepting a set $\mathscr{R}$ that contains every set, then we might see Russell paradox in the mathematical deduction. Afterwards it debars such a set $\mathscr{R}$ containing every set in mathematical logic. Furthermore Russell put forward his type theory. However in set theory people still accept such an empty set, denoted by $\varnothing$, which is a subset of every set, i.e. accept that $\varnothing \subset A$ is an axiom in set theory. Accepting the existence of the set $\varnothing$, and accepting the existence of the set $\mathscr{R}$, are theoretically similar. Why people were not conscious of the fault of the former. From this we see that we should newly consider over the axiom $\varnothing \subset A$. At least it is not suit elementary mathematics and the Euclidean straight line with a fixed frame. In fact the hardcore of the study of mathematics is to study the Euclidean straight line with a fixed frame and its mathematical equivalent: Omega continuum which is the true union of mathematics and geometry. If an axiom is not suit for the Euclidean straight line with a fixed frame, why should we hold it?

17. In the section 19.4: Skolem step function defined on a finite set and its supplementary set — Examples of discontinuous function in $\Omega\Pi$, by using the discussion for Skolem step function defined on a finite set and its supplementary set, we explain the localization of the limit method. Besides we point out that the future development is to study the differential and integral calculus for discontinuous function in $\Omega\Pi$, various Dirac Delta functions and various problems of singular function and its integral; In the section

19.5 states the general positive integral axiom, and by using the axiom to look back the works of the past and present, and to view the future development.

18. In the Article 6.1.1.1-2 defines *Omega area differential operators*, in which contain Omega differential addition and Omega differential multiplication in ΩΠ. And in which introduces twelve micro Omega area differential operational axioms. These reflect the radical properties of Omega area differential operators on one hand, and play an important role for the justification of existence of infinitesimal linear elements on the other hand.

19. For the study of the measure of a ray in a Euclidean straight line with a fixed frame, introduces *a cosmic symmetric axiom with a central point* and *a cosmic symmetric axiom without a central point*. These reflect the radical properties of a ray in a Euclidean straight line with a fixed frame on one hand, and play an important role for the justification of existence of infinitelarge linear elements on the other hand.

20. Puts forward a primary regular axiomatic system for *Omega calculus*. Now we introduce simply the system as follows: That consists of four parts. The first part consists of twelve axioms of differential operators; please refer to Article 6.1.1. The second part consists of two cosmic axioms: one is the *cosmic symmetric axiom with a central point* and another is the *cosmic symmetric axiom without a central point*; please refer to Section 1.6. The third part is about the link between micro and macro, which consists of three groups. The first group consists of three axioms of positive integral. That transfer macro measure into an integral of micro measures; please refer to Chapters 5, 6, 15, and 19. The second group consists of auxiliary axioms of integral equality; please refer to Section 16.1. The third group consists of auxiliary axioms for inverse integral estimate; please refer to Section 16.2. The fourth part consists of three cases: ①Basic measure axiom in straight line L（refer to Article 1.3.1）; ②Basic principle of the order (refer to Article 1.3.2); and ③ "*original geometrical form*" to quickly determine order between measures geometrically（refer to Articles 3.2.5-7）.

21. From the section 18.4 to section 19.3, sets up an introductory theory which is parallel to the theory of mathematical analysis and enlarges it into Omega Infinitesimal Calculus so that will come into being an extended Omega Infinitesimal Calculus theory which is parallel to and wider than the classical mathematical analytical theory. For example, from 18.6.4 it follows a fact that Riemann integrable class may be embedded into Omega bounded integrable function class to be a sub-theory.

22. From the section 18.4 to section 18.15, parallel and enlargement to analysis, in Omega Infinitesimal Calculus sets up the classes of integrable functions, integral equalities and inequalities, integral middle value formulas, integration formula by substitution, and integration formula by parts.

23. In the section 18.16, parallel and enlargement to the notion of continuity in analysis, in Omega Infinitesimal Calculus sets up the notion of Omega continuous function and corresponding theorems, for example, Intermediate value theorem of a continuous function and etc, so that will be an expansive foreground.

24. In the sections 19.1-3, parallel and enlargement to analysis, in Omega Infinitesimal Calculus sets up systematically an expansive introductory theory of differential calculus, defines Omega derivative from one order to *n* order, sets up arithmetic operations for them, and sets up corresponding integral formulae, sets up the basic theorems in Omega differential calculus, until Omega Taylor expansion, which is a development of Taylor formula.