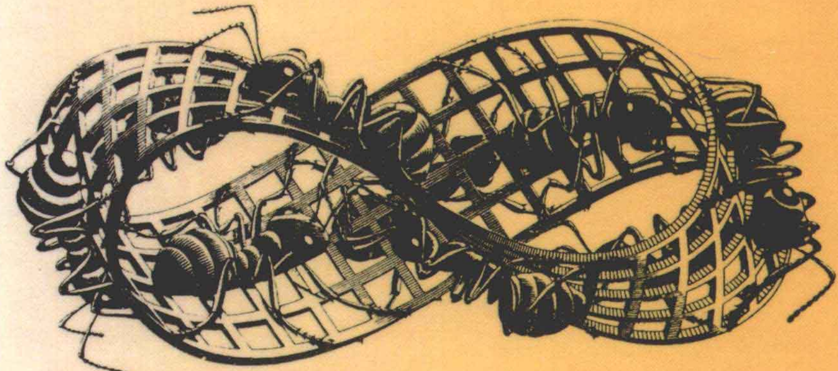
Undergraduate Texts in Mathematics

Gerard Buskes
Arnoud van Rooij

Topological Spaces

From Distance to Neighborhood

拓扑空间



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(continued after index)

Preface

This book is a text, not a reference, on Point-set Topology. It addresses itself to the student who is proficient in Calculus and has some experience with mathematical rigor, acquired, e.g., via a course in Advanced Calculus or Linear Algebra.

To most beginners, Topology offers a double challenge. In addition to the strangeness of concepts and techniques presented by any new subject, there is an abrupt rise of the level of abstraction. It is a bad idea to teach a student two things at the same moment. To mitigate the culture shock, we move from the special to the general, dividing the book into three parts:

1. The Line and the Plane
2. Metric Spaces
3. Topological Spaces.

In this way, the student has ample time to get acquainted with new ideas while still on familiar territory. Only after that, the transition to a more abstract point of view takes place.

Elementary Topology preeminently is a subject with an extensive array of technical terms indicating properties of topological spaces. In the main body of the text, we have purposely restricted our mathematical vocabulary as much as is reasonably possible. Such an enterprise is risky. Doubtlessly, many readers will find us too thrifty. To meet them halfway, in Chapter 18 we briefly introduce and discuss a number of topological properties, but even there we do not touch on paracompactness, complete normality, and extremal disconnectedness—just to mention three terms that are not really esoteric.

In a highly abstract topic like ours, it aids a student to focus on a central theme. The theme of our book is convergence. We show how, for $\mathbb{R}^n$ and for metric spaces in general, concepts such as "continuous" and "closed" can be described in terms of convergent sequences. After that, in any given set X we introduce convergence of nets relative to any given collection ω of subsets of X. This convergence leads in a natural way to the notion of a topology. The idea behind this somewhat unconventional approach is threefold.

First, it shows that the definition of "topology" is less artificial than it seems to be. Without this preparation, the definition appears to stem from an arbitrary selection of properties of the system of open sets in $\mathbb{R}^n$, and it is not clear why precisely *these* properties are the relevant ones. (The reader who finds this a digression can skip Chapter 11; in Chapter 12, the definition and some basic facts are repeated without the motivation.)

Second, it relegates the notion of "topology" to a place in the second rank. When one studies a topological space, often the topology itself is less relevant than a subbase for it (the collection ω in the situation described above). A case in point is the product topology on a Cartesian product of topological spaces: all that really matters is a subbase, and the fact that this subbase generates a topology is quite immaterial.

Third, convergent nets form a very useful tool in Topology, deserving much more attention than they generally get.

We do not assume previous knowledge of the axiomatic approach. As, however, a rigorous theory of topological spaces must have a firm base in Analysis, we start with a brief axiomatic treatment of the real-number system, explaining what axioms are and what purpose they serve.

We do not assume previous knowledge of Set Theory either. (Indeed, to be on the safe side, we have added a chapter on countability.) On the other hand, Topology unavoidably leads to nontrivial set-theoretic problems. Accordingly, in connection with the Tychonoff Theorem, we pay close attention to the Axiom of Choice and Zorn's Lemma and their role in mathematics.

The pace of this book is relaxed with a gradual acceleration. For instance, the first three chapters and part of Chapter 4 can be relegated to home reading for a well-prepared student. However, the easy initial pace makes the first nine chapters a balanced course in metric spaces for undergraduates. The book contains more than enough material for a two-semester graduate course.

As with all mathematical learning, a substantial amount of practice is indispensable. We offer exercises of varying degrees of difficulty. Some are routine, others illustrate results of the text, and yet others go beyond the text. We have carefully crafted these exercises. Accordingly, one will find many of them, in particular the complicated ones, sectioned into more digestible pieces with hints.

Finally, in most chapters we present an "extra," a brief foray outside Topology. A beginning student is apt to consider each branch of mathematics as an autonomous unit, isolated from the rest, and also to think that mathematics is a museum piece, something created in olden times by our forefathers, that can be seen and even studied, but not touched. Our purpose of the extras is to illustrate the many connections between Topology and other subjects, such as Analysis and Set Theory. Also, in our extras we try to show that Topology was and still is built by individuals, who sometimes made mistakes. We encourage the reader to consider these extras to be part of the course. The extras are extra, not extraneous.

(continued from page ii)

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Contents

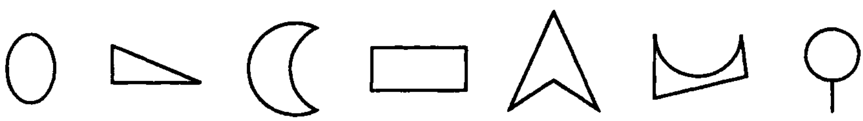
PART I

The Line and the Plane

1 CHAPTER What Topology Is About

Topological Equivalence

1.1 Question

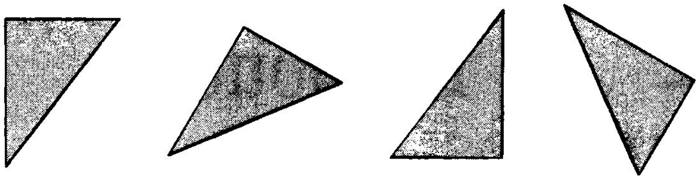


Which picture does not go with the others?

1.2

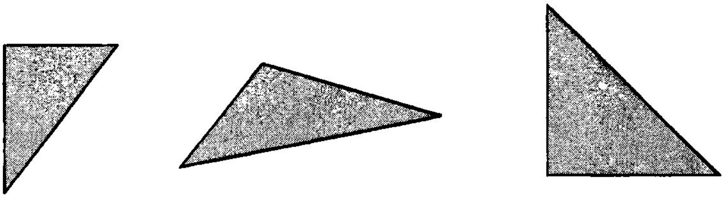
The last one, of course, but it is not so easy to describe the common feature of the first six shapes that is lacking in the seventh. A satisfactory description can be given in the language of *Topology*, the subject matter of this book.

To give you some idea of what topology is, we return for a moment to plane geometry. Suppose you have drawn a triangle with sides of 13, 14, and 15 inches and by measuring you have found that it has an angle of 54°. Then you know that *every* triangle with sides of 13, 14, and 15 inches must have a 54° angle, because all such triangles are congruent. Their positions and orientations in the plane do not matter to a geometer, as they would to a surveyor. Using an arbitrarily chosen term, we will say that the geometer's point of view is "higher" than the surveyor's. The surveyor distinguishes among the following triangles, the geometer does not:



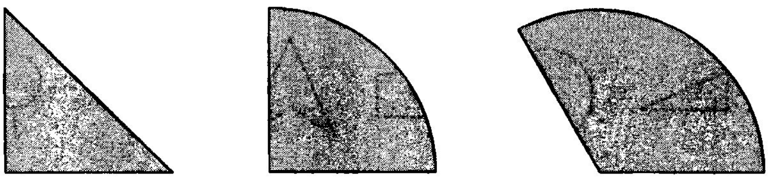
(1)

One can adopt a higher point of view than the geometer's: For certain purposes, there is no sense in distinguishing



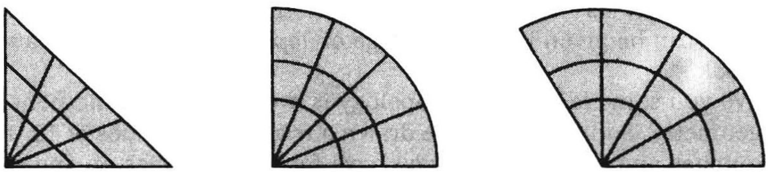
(2)

or (still higher)



(3)

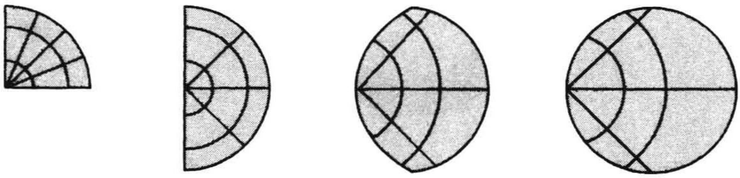
At this stage, the shapes differ considerably, but if you draw them on pieces of rubber instead of paper, you can obtain them from each other by stretching and bending:



(4)

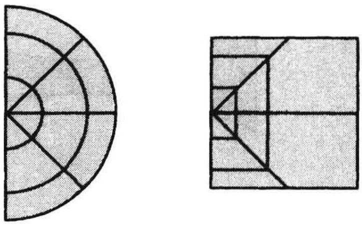
Here, we arrive at the heart of the matter. Topology is the branch of mathematics in which the differences between the shapes of (3) are irrelevant, precisely as those between the shapes of (1) are irrelevant in geometry. The topologist's point of view is "higher" than the geometer's.

Carrying the deformations a bit farther, we obtain shapes that no longer have anything triangular about them:



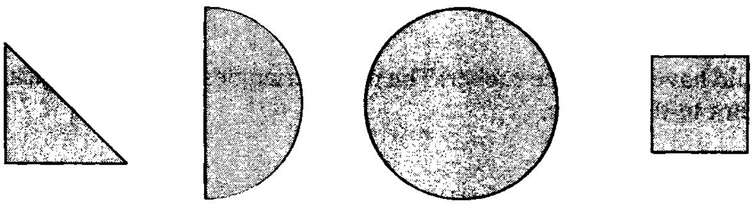
(5)

or



(6)

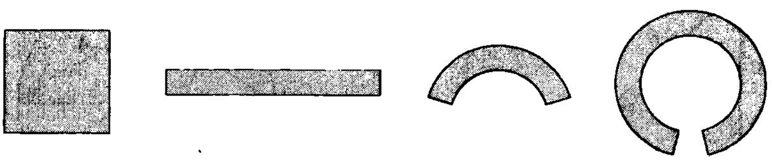
Thus, in the eyes of the topologist, the shapes



(7)

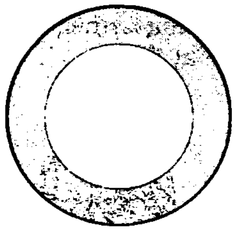
are the same; let us call them topologically equivalent [as the triangles in (1) may be called geometrically equivalent].

From the square, by stretching and bending we obtain another series of shapes:

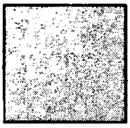


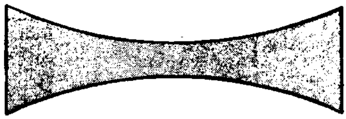
(8)

But no amount of stretching will produce a closed ring:



Here, something else is required, such as gluing two edges together. The ring and the square are not topologically equivalent; they are as different to the topologist as the square and the triangle are to the geometer. Similarly, the square may be stretched





but to obtain two triangles

one would have to tear the rubber: The pair of triangles is not topologically equivalent to the square.

1.3

What has all this to do with mathematics? Stretching a piece of rubber is hardly a mathematical operation. But the grids sketched in (4), (5), and (6) suggest how the concept of topological equivalence can be defined mathematically. Compare the first shape of (4) and the last of (5):

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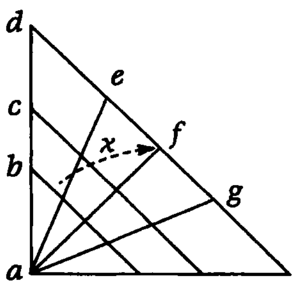

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