Part I

Stochastic Differential Equations with Jumps in R^d

1

Martingale Theory and the Stochastic Integral for Point Processes

A stochastic integral is a kind of integral quite different from the usual deterministic integral. However, its theory has broad and important applications in Science, Mathematics itself, Economic, Finance, and elsewhere. A stochastic integral can be completely charaterized by martingale theory. In this chapter we will discuss the elementary martingale theory, which forms the foundation of stochastic analysis and stochastic integral. As a first step we also introduce the stochastic integral with respect to a Point process.

1.1 Concept of a Martingale.

In some sense the martingale conception can be explained by a fair game. Let us interprete it as follows:

In a game suppose that a person at the present time s has wealth x_s for the game, and at the future time t he will have the wealth x_t. The expected money for this person at the future time t is naturally expressed as $E[x_t|\mathfrak{F}_s]$, where $E[\cdot]$ means the expectation value of $\cdot$, $\mathfrak{F}_s$ means the information up to time s, which is known by the gambler, and $E[\cdot|\mathfrak{F}_s]$ is the conditional expectation value of $\cdot$ under given $\mathfrak{F}_s$. Obviously, if the game is fair, then it should be

$E[x_t|\mathfrak{F}_s] = x_s, \forall t \geq s.$

This is exactly the definition of a martingale for a random process $x_t, t \geq 0$. Let us make it more explicit for later developement.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $\{\mathfrak{F}_t\}_{t\geq 0}$ be an information family (in Mathematics, we call it a $\sigma-$algebra family or a $\sigma-$field family, see Appendix A), which satisfies the so-called "Usual Conditions":

(i) $\mathfrak{F}_s \subset \mathfrak{F}_t$, as $0 \leq s \leq t$; (ii) $\mathfrak{F}_{t+} =: \cap_{h>0}\mathfrak{F}_{t+h}$.

Here condition (i) means that the information increases with time, and condition (ii) that the information is right continuous, or say, $\mathfrak{F}_{t+h} \downarrow \mathfrak{F}_t$, as $h \downarrow 0$. In this case we call $\{\mathfrak{F}_t\}_{t\geq 0}$ a $\sigma-$field filtration.

Definition 1 *A real random process $\{x_t\}_{t\geq 0}$ is called a martingale (supermartingale, submartingale) with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$, or $\{x_t, \mathfrak{F}_t\}_{t\geq 0}$ is a martingale (supermartingale, submartingale), if*

(i) x_t is integrable for each $t \geq 0$; that is, $E|x_t| < \infty, \forall t \geq 0$;

(ii) x_t is $\mathfrak{F}_t-$adapted; that is, for each $t \geq 0$, x_t is $\mathfrak{F}_t-$measurable;

(iii) $E(x_t|\mathfrak{F}_s] = x_s$, (respectively, $\leq, \geq$), a.s. $\forall 0 \leq s \leq t$.

For the random process $\{x_t\}_{t\in[0,T]}$ and the random process $\{x_n\}_{n=1}^{\infty}$ with discrete time similar definitions can be given.

Example 2 *If $\{x_t\}_{t\geq 0}$ is a random process with independent increments; that is, $\forall 0 < t_1 < t_2 < \cdots < t_n$, the family of random variables*

$\{x_0, x_{t_1} - x_0, x_{t_2} - x_{t_1}, \cdots, x_{t_n} - x_{t_{n-1}}\}$

is independent, and the increment $x_t - x_s, \forall t > s$, is integrable and with non-negative expectation, moreover, x_0 is also integrable, then $\{x_t\}_{t\geq 0}$ is a submartingale with respect to $\{\mathfrak{F}_t^x\}_{t\geq 0}$, where $\mathfrak{F}_t^x = \sigma(x_s, s \leq t)$, which is a $\sigma-$field generated by $\{x_s, s \leq t\}$ (that is, the smallest $\sigma-$field which makes all $x_s, s \leq t$ measurable) and makes a completion.

In fact, by independent and non-negative increments,

$0 \leq E(x_t - x_s) = E[(x_t - x_s)|\mathfrak{F}_s^x], \forall t \geq s.$

Hence the conclusion is reached.

Example 3 *If $\{x_t\}_{t\geq 0}$ is a submartingale, let $y_t := x_t \vee 0 = \max(x_t, 0)$, then $\{y_t\}_{t\geq 0}$ is still a submartingale.*

In fact, since $f(x) = x \vee 0$ is a convex function, hence by Jensen's inequality for the conditional expectation

$E[x_t \vee 0|\mathfrak{F}_s] \geq E[x_t|\mathfrak{F}_s] \vee E[0|\mathfrak{F}_s] \geq x_s \vee 0, \forall t \geq s.$

So the conclusion is true.

Example 4 *If $\{x_t\}_{t\geq 0}$ is a martingale, then $\{|x_t|\}_{t\geq 0}$ is a submartingale.*

In fact, by Jensen's inequality

$E[|x_t|\,|\mathfrak{F}_s] \geq |E[x_t|\mathfrak{F}_s]| = |x_s|, \forall t \geq s.$

Thus the conclusion is deduced.

Martingales, submartingales and supermartingales have many important and useful properties, which make them become powerful tools in dealing with many theoretical and practical problems in Science, Finance and elsewhere. Among them the martingale inequalites, the limit theorems, and the

Doob-Meyer decomposition theorem for submartingales and supermartingales are most helpful and are frequently encountered in Stochastic Analysis and its Applications, and in this book. So we will discuss them in this chapter. However, to show them clearly we need to introduce the concept called a stopping time, which will be important for us later. We proceed to the next section.

1.2 Stopping Times. Predictable Process

Definition 5 *A random variable $\tau(\omega) \in [0,\infty]$ is called a $\mathfrak{F}_t-$stopping time, or simply, a stoping time, if for any $(\infty >)t \geq 0$, $\{\tau(\omega) \leq t\} \in \mathfrak{F}_t$.*

The intuitive interpletation of a stopping time is as follows: If a gambler has a right to stop his gamble at any time $\tau(\omega)$, he would of course like to choose the best time to stop. Suppose he stops his game before time t, i.e. he likes to make $\tau(\omega) \leq t$, then the maximum information he can get about his decision is only the information up to t, i.e $\{\tau(\omega) \leq t\} \in \mathfrak{F}_t$. The trivial example for a stopping time is $\tau(\omega) \equiv t, \forall\omega \in \Omega$. That is to say, any constant time t actually is a stopping time.

For a discrete random variable $\tau(\omega) \in \{0,1,2,\cdots,\infty\}$ the definition can be reduced to that $\tau(\omega)$ is a stopping time, if for any $n \in \{0,1,2,\cdots\}$, $\{\tau(\omega) = n\} \in \mathfrak{F}_n$, since $\{\tau(\omega) = n\} = \{\tau(\omega) \leq n\} - \{\tau(\omega) \leq n-1\}$, and $\{\tau(\omega) \leq n\} = \cup_{k=1}^{n} \{\tau(\omega) = k\}$. The following examples of stopping time are useful later.

Example 6 *Let B be a Borel set in R^1 and $\{x_n\}_{n=1}^{\infty}$ be a sequence of real $\mathfrak{F}_t-$adapted random variables. Define the first hitting time $\tau_B(\omega)$ to the set B (i.e. the first time that $\{x_n\}_{n=1}^{\infty}$ hits B) by*

$\tau_B(\omega) = \inf\{n : x_n(\omega) \in B\}$.

Then $\tau_B(\omega)$ is a discrete stopping time.

In fact,

$\{\tau_B(\omega) = n\} = \cap_{k=1}^{n-1} \{x_k \in B^c\} \cap \{x_n \in B\} \in \mathfrak{F}_n$.

For a general random process with continuous time parameter t we have the following similar example.

Example 7 *Let x_t be a $d-$dimensional right continuous $\mathfrak{F}_t-$adapted process and let A be an open set in R^d. Denote the first hitting time $\sigma_A(\omega)$ to A by*

$\sigma_A(\omega) = \inf\{t > 0 : x_t(\omega) \in A\}$.

Then $\sigma_A(\omega)$ is a stopping time.

In fact, by the open set property and the right continuity of x_t one has that

$\{\sigma_A(\omega) \leq t\} = \cap_{n=1}^{\infty} \{\sigma_A(\omega) < t + \frac{1}{n}\}$
$= \cap_{n=1}^{\infty} \cup_{r\in Q, r<t+1/n} \{x_r(\omega) \in A\} \in \mathfrak{F}_{t+0} = \mathfrak{F}_t,$

where Q is the set of all rational numbers.

The following properties of general stopping times will be useful later.

Lemma 8 *$\tau(\omega)$ is a stopping time, if and only if $\{\tau(\omega) < t\} \in \mathfrak{F}_t, \forall t$.*

Proof. $\Longrightarrow$: $\{\tau(\omega) < t\} = \cup_{n=1}^{\infty} \{\tau(\omega) \leq t - \frac{1}{n}\} \in \mathfrak{F}_t$.
$\Longleftarrow$: $\{\tau(\omega) \leq t\} = \cap_{n=1}^{\infty} \{\tau(\omega) < t + \frac{1}{n}\} \in \mathfrak{F}_{t+0} = \mathfrak{F}_t$. ∎

Lemma 9 *Let $\sigma, \tau, \sigma_n, n = 1, 2, \cdots$ be stopping times. Then*
(i) $\sigma \wedge \tau, \sigma \vee \tau$,
(ii) $\sigma = \lim_{n \to \infty} \sigma_n$, when $\sigma_n \uparrow$ or $\sigma_n \downarrow$,
are all stopping times.

Proof. (i): $\{\sigma \wedge \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in \mathfrak{F}_t$,
$\{\sigma \vee \tau \leq t\} = \{\sigma \leq t\} \cap \{\tau \leq t\} \in \mathfrak{F}_t$.
(ii): If $\sigma_n \uparrow \sigma$, then
$\{\sigma \leq t\} = \cap_{n=1}^{\infty} \{\sigma_n \leq t\} \in \mathfrak{F}_t$.
If $\sigma_n \downarrow \sigma$, then
$\{\sigma < t\} = \cup_{n=1}^{\infty} \{\sigma_n < t\} \in \mathfrak{F}_t$.
By Lemma 8 σ is a stopping time. ∎

Now let us introduce a $\sigma-$ field which describes the information obtained up to stopping time τ. Set
$\mathfrak{F}_\tau = \{A \in \mathfrak{F}_\infty \colon \forall t \in [0, \infty), A \cap \{\tau(\omega) \leq t\} \in \mathfrak{F}_t\}$,
where we naturally define that $\mathfrak{F}_\infty = \vee_{t \geq 0} \mathfrak{F}_t$, i.e. the smallest $\sigma-$field including all $\mathfrak{F}_t, t \in [0, \infty)$. Obviously, $\mathfrak{F}_\tau$ is a $\sigma-$ algebra, and if $\tau(\omega) \equiv t$, then $\mathfrak{F}_\tau = \mathfrak{F}_t$.

Proposition 10 *Let $\sigma, \tau, \sigma_n, n = 1, 2, \cdots$ be stopping times.*
(1) If $\sigma(\omega) \leq \tau(\omega), \forall \omega$, then $\mathfrak{F}_\sigma \subset \mathfrak{F}_\tau$.
(2) If $\sigma_n(\omega) \downarrow \sigma(\omega), \forall \omega$, then $\cap_{n=1}^{\infty} \mathfrak{F}_{\sigma_n} = \mathfrak{F}_\sigma$.
(3) $\sigma \in \mathfrak{F}_\sigma$. (We use $f \in \mathfrak{F}_\sigma$ to mean that f is $\mathfrak{F}_\sigma-$measurable).

Proof. (1): $A \cap \{\tau \leq t\} = (A \cap \{\sigma \leq t\}) \cap \{\tau \leq t\} \in \mathfrak{F}_t$.
(2): By (1) $\mathfrak{F}_\sigma \subset \cap_{n=1}^{\infty} \mathfrak{F}_{\sigma_n}$. Conversely, if $A \in \cap_{n=1}^{\infty} \mathfrak{F}_{\sigma_n}$, then
$A \cap \{\sigma_n < t\} = \cup_{k=1}^{\infty} (A \cap \{\sigma_n \leq t - \frac{1}{k}\}) \in \mathfrak{F}_t, \forall t \geq 0, \forall n$.
Hence $A \cap \{\sigma < t\} = \cup_{n=1}^{\infty} (A \cap \{\sigma_n < t\}) \in \mathfrak{F}_t$, and
$A \cap \{\sigma \leq t\} = \cap_{k=1}^{\infty} (A \cap \{\sigma < t + \frac{1}{k}\}) \in \mathfrak{F}_{t+0} = \mathfrak{F}_t$, i.e. $A \in \mathfrak{F}_\sigma$.
(3): For any constant $0 \leq c < \infty$ one has that $\{\sigma \leq c\} \cap \{\sigma \leq t\} \in \mathfrak{F}_{t \wedge c} \subset \mathfrak{F}_t$, so $\{\sigma \leq c\} \in \mathfrak{F}_\sigma$. ∎

It is natural to ask that if $\{x_t\}_{t \geq 0}$ is $\mathfrak{F}_t-$adapted, and σ is a stopping time, is it true that $x_\sigma \in \mathfrak{F}_\sigma$? Generally speaking, it is not true. However, if $\{x_t\}_{t \geq 0}$ is a progressive measurable process, then it is correct. Let us introduce such a related concept.

Definition 11 *An R^d-valued process $\{x_t\}_{t \geq 0}$ is called measurable (respectively, progressive measurable), if the mapping*
$(t, \omega) \in [0, \infty) \times \Omega \to x_t(\omega) \in R^d$

(respectively, for each $t \geq 0, (s,\omega) \in [0,t] \times \Omega \to x_t(\omega) \in R^d$*)*
is $\mathfrak{B}([0,\infty)) \times \mathfrak{F} / \mathfrak{B}(R^d)$ *–measurable*
(respectively, $\mathfrak{B}([0,t]) \times \mathfrak{F}_t / \mathfrak{B}(R^d)$ *–measurable);*
that is, $\{(t,\omega) : x_t(\omega) \in B\} \in \mathfrak{B}([0,\infty)) \times \mathfrak{F}, \forall B \in \mathfrak{B}(R^d)$
(respectively, $\{(s,\omega) : s \in [0,t], x_s(\omega) \in B\} \in \mathfrak{B}([0,t]) \times \mathfrak{F}_t, \forall B \in \mathfrak{B}(R^d)$*).*

Let us introduce two useful $\sigma-$algebras as follows: Denote by $\mathcal{P}$ (respectively, $\mathcal{O}$) as the smallest $\sigma-$algebra on $[0,\infty) \times \Omega$ such that all left-continuous (respectively, right-continuous) $\mathfrak{F}_t-$adapted processes
$y_t(\omega) : [0,\infty) \times \Omega \to y_t(\omega) \in R^d$
are measurable. $\mathcal{P}$ (respectively, $\mathcal{O}$) is called the predictable (respectively, optional) $\sigma-$algebra. Thus, the following definition is natural.

Definition 12 *A process* $\{x_t\}_{t \geq 0}$ *is called predictable (optional), if the mapping*
$(t,\omega) \in [0,\infty) \times \Omega \to x_t(\omega) \in R^d$
is $\mathcal{P} / \mathfrak{B}(R^d)$ *–measurable (respectively* $\mathcal{O} / \mathfrak{B}(R^d)$ *–measurable).*

Let us use the notation $f \in \mathcal{P}$ to mean that f is $\mathcal{P}-$ measurable; etc. It is easily seen that the following relations hold:
$f \in \mathcal{P} \Rightarrow f \in \mathcal{O} \Rightarrow f$ is progressive measurable $\Rightarrow f$ is measurable and $\mathfrak{F}_t-$adapted.

We only need to show the first two implications. The last one is obvious.

Assume that $\{x_t\}_{t \geq 0}$ is left-continuous, let $x_t^n = x_{\frac{k}{2^n}}$, as $t \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, $k = 0,1,\cdots; n = 1,2,\cdots$. Then obviously, x_t^n is right-continuous, and by the left-continuity of x_t, $x_t^n(\omega) \to x_t(\omega)$, as $n \to \infty, \forall t, \forall \omega$. So $\{x_t\}_{t \geq 0} \in \mathcal{O}$. From this one sees that $\mathcal{P} \subset \mathcal{O}$. Let us show that $\{x_t\}_{t \geq 0} \in \mathcal{O}$ implies that $\{x_t\}_{t \geq 0}$ is progressive measurable. For this for each given $t \geq 0$ we show that $\{x_s\}_{s \geq 0}$ restricted on $(s,\omega) \in [0,t] \times \Omega$ is $\mathfrak{B}([0,t]) \times \mathfrak{F}_t-$ measurable. In fact, without loss of generality we may assume that $\{x_t\}_{t \geq 0}$ is right-continuous. Now for each given $t \geq 0$, let $x_s^n = x_{\frac{k}{2^n}t}$, as $s \in [\frac{kt}{2^n}, \frac{(k+1)t}{2^n}), k = 0,1,\cdots,2^n - 1; n = 1,2,\cdots$. Then obviously, $\{x_s^n\}_{s \in [0,t]}$ is $\mathfrak{B}([0,t]) \times \mathfrak{F}_t-$measurable, so is $\{x_s\}_{s \in [0,t]}$, since by the right continuity of x_t we have that as $n \to \infty, x_s^n(\omega) \to x_s(\omega)$ $,\forall s \in [0,t], \forall \omega$.

Let us show the following

Theorem 13 *If* $\{x_t\}_{t \geq 0}$ *is a* R^d-*valued progressive measurable process, then for each stopping time* σ, $Z_\sigma I_{\sigma < \infty}$ *is* $\mathfrak{F}_\sigma-$*measurable.*

We will use the compositon of measurable maps to show this theorem. For this we need the following lemma.

Lemma 14 *If* f_i *is a measurable mapping from* $(\Omega, \mathfrak{F})$ *to* $(\Omega_i', \mathfrak{F}_i'), i = 1,2,\cdots$*; then*
$f(\omega) = (f_1(\omega), f_2(\omega), \cdots)$
is a measurable mapping from $(\Omega, \mathfrak{F})$ *to* $(\Omega_1' \times \Omega_2' \times \cdots, \mathfrak{F}_1' \times \mathfrak{F}_2' \times \cdots)$*.*

In fact, for any $B_i \in \mathfrak{F}'_i, i = 1, 2, \cdots, f^{-1}(B_1 \times B_2 \times \cdots) = \cap_{i=1}^{\infty} f_i^{-1}(B_i) \in \mathfrak{F}$. So $f^{-1}(\mathfrak{F}'_1 \times \mathfrak{F}'_2 \times \cdots) \subset \mathfrak{F}$.

Now let us prove Theorem 13.

Proof. Let $\Omega_\sigma = \{\sigma < \infty\}$. We need to show that x_σ is a measurable mapping from $(\Omega_\sigma, \mathfrak{F}_\sigma)$ to $(R^d, \mathfrak{B}(R^d))$. For any given $t \geq 0$ by Proposition 10 $\sigma \in \mathfrak{F}_\sigma$. So by the definition of $\mathfrak{F}_\sigma$, σ is a measurable mapping from $(\{\sigma \leq t\}, \mathfrak{F}_t)$ to $([0,t], \mathfrak{B}([0,t]))$. Hence by Lemma 14 $g_1(\omega) = (\sigma(\omega), \omega)$ is a measurable mapping from $(\{\sigma \leq t\}, \mathfrak{F}_t)$ to $([0,t] \times \Omega, \mathfrak{B}([0,t]) \times \mathfrak{F}_t)$. Note that by the progressive measurability of $\{x_t\}_{t \geq 0}$, $g_2(s,\omega) = x_s(\omega)$ is a measurable mapping from $([0,t] \times \Omega, \mathfrak{B}([0,t]) \times \mathfrak{F}_t)$ to $(R^d, \mathfrak{B}(R^d))$. Hence $x_{\sigma(\omega)}(\omega) I_{\sigma<\infty} = g_2 \circ g_1(\omega)$ is a measurable mapping from $(\{\sigma \leq t\}, \mathfrak{F}_t)$ to $(R^d, \mathfrak{B}(R^d))$ This shows that for any $B \in \mathfrak{B}(R^d)$, $\{x_\sigma I_{\sigma<\infty} \in B\} \cap \{\sigma \leq t\} \in \mathfrak{F}_t$.Since $t \geq 0$ is arbitrary by definition $\{x_\sigma I_{\sigma<\infty} \in B\} \in \mathfrak{F}_\sigma$. ∎

1.3 Martingales with Discrete Time

First we will show the Doob's stopping theorem (or called Doob's optional sampling theorem) for bounded stoping times.

Theorem 15 *Let $\{x_n\}_{n=0,1,2,\cdots}$ be a martingle (supermartingle, submartingale), $\sigma \leq \tau$ be two bounded stopping times. Then $\{x_n\}_{n=0,1,2,\cdots}$ is a strong martingle (respectively, strong supermartingle, strong submartingle), i.e.*

$E[x_\tau|\mathfrak{F}_\sigma] = x_\sigma$ (respectivly, $\leq$, $\geq$), a.s.

Proof. We only prove the conclusion for the case of submartingle. By assumption there exists a natural number $0 \leq n_0$ such that $\tau \leq n_0$. So $|x_\tau| \leq \max\{|x_n|, n = 0,1,2,\cdots,n_0\} \leq \sum_{n=0}^{n_0} |x_n|$. So $E\,|x_\tau| < \infty$. By the same manner $E\,|x_\sigma| < \infty$. Note that by the definition of a stopping time and $\mathfrak{F}_\sigma$ for $A \in \mathfrak{F}_\sigma$ and $0 \leq n \leq n_0$

$A \cap \{\sigma = n\} \cap \{\tau > n\} \in \mathfrak{F}_n$.

Now suppose $\tau - \sigma \leq 1$ in addition. Then by the definition of a submartingale

$\int_A (x_\sigma - x_\tau) dP = \sum_{n=0}^{n_0} \int_{A \cap \{\sigma=n\} \cap \{\tau>n\}} (x_n - x_{n+1}) dP \leq 0.$

In the general case set $T_n = \tau \wedge (\sigma + n), n = 1,2,\cdots,n_0$. Then all T_n are stopping times, and

$\sigma \leq T_1 \leq T_2 \leq \cdots \leq T_{n_0} = \tau$, $T_1 - \sigma \leq 1$, $T_{n+1} - T_n \leq 1$,
$n = 1,2,\cdots,n_0 - 1$.

Let $A \in \mathfrak{F}_\sigma \subset \mathfrak{F}_{T_{n_0}}$. Then by the above conclusion

$\int_A x_\sigma dP \leq \int_A x_{T_1} dP \leq \cdots \leq \int_A x_\tau dP.$

The proof is complete. ∎

Now we have the following martingale inequality:

Theorem 16 *Let $\{x_n\}_{n=0,1,\cdots}$ be a submartingale. Then for every $\lambda > 0$ and natural number N*

$\lambda P(\max_{0\leq n\leq N} x_n \geq \lambda) \leq E(x_N I_{\max_{0\leq n\leq N} x_n \geq \lambda}) \leq E(x_N^+) \leq E|x_N|,$
and
$\lambda P(\min_{0\leq n\leq N} x_n \leq -\lambda) \leq -Ex_0 + E(x_N I_{\min_{0\leq n\leq N} x_n > -\lambda})$
$\leq Ex_0^- + E(x_N^+) \leq E|x_0| + E|x_N|.$

Proof. Let us use the first hitting time technique and strong submartingale property to show this theorem. Set
$\sigma = \min\{n \leq N : x_n \geq \lambda\}$; $\sigma = N$, if $\{\cdot\} = \phi$.
Then σ is a bounded stopping time. By Theorem 15
$Ex_N \geq Ex_\sigma = Ex_\sigma I_{\max_{0\leq n\leq N} x_n \geq \lambda} + Ex_N I_{\max_{0\leq n\leq N} x_n < \lambda}$
$\geq \lambda P(\max_{0\leq n\leq N} x_n \geq \lambda) + Ex_N I_{\max_{0\leq n\leq N} x_n < \lambda}.$
Transferring the last term to the left hand side, we obtain the first inequality. Now set
$\tau = \min\{n \leq N : x_n \leq -\lambda\}$; $\tau = N$, if $\{\cdot\} = \phi$.
Then
$Ex_0 \leq Ex_\tau = Ex_\tau I_{\min_{0\leq n\leq N} x_n \leq -\lambda} + Ex_N I_{\min_{0\leq n\leq N} x_n > -\lambda}$
$\leq -\lambda P(\min_{0\leq n\leq N} x_n \leq -\lambda) + E(x_N I_{\min_{0\leq n\leq N} x_n > -\lambda}).$
Thus the second inequality is derived. ∎

Corollary 17 *1) Assume that* $\{x_n\}_{n=0,1,\cdots}$ *is a real submartingale such that* $E((x_n^+)^p) < \infty, n = 0, 1, \cdots$, *for some* $p \geq 1$. *Then for every* N, *and* $\lambda > 0$,
$P(\max_{0\leq n\leq N} x_n^+ \geq \lambda) \leq E((x_N^+)^p)/\lambda^p,$
and if $p > 1$,
$E(\max_{0\leq n\leq N} (x_n^+)^p) \leq \left(\frac{p}{p-1}\right)^p E((x_N^+)^p).$
2) If $\{x_n\}_{n=0,1,\cdots}$ *is a real martingale such that* $E(|x_n|^p) < \infty, n = 0, 1, \cdots$, *then the conclusions in 1) hold true for* x_n^+ *and* x_N^+ *replaced by* $|x_n|$ *and* $|x_N|$, *respectively.*

Proof. 1): By Example 3 $\{x_n^+\}_{n=0,1,\cdots}$ is a non-negative submartingale. Using Jensen's inequality again one has that $\{(x_n^+)^p\}_{n=0,1,\cdots}$ is still a non-negative submartingale. Hence the first inequality is obtained from Theorem 16. Now if $p > 1$, set $\xi = \max_{0\leq n\leq N} (x_n^+)$. then by Theorem 16 again one has that
$\lambda P(\xi \geq \lambda) \leq Ex_N^+ I_{\xi \geq \lambda}.$
Hence using Fubini's theorem and Hölder's inequality one derives that
$E(\xi^p) = E\int_0^\xi p\lambda^{p-1} d\lambda = E\int_0^\infty p\lambda^{p-1} I_{\lambda\leq\xi} d\lambda = p\int_0^\infty \lambda^{p-1} P(\xi \geq \lambda) d\lambda$
$\leq p\int_0^\infty \lambda^{p-2} E(x_N^+ I_{\xi\geq\lambda}) d\lambda = \frac{p}{p-1} E[\xi^{p-1} x_N^+]$
$\leq \frac{p}{p-1} [E(x_N^+)^p]^{1/p} [E\xi^p]^{(p-1)/p}.$
Now if $E(\xi^p) = 0$, then the second inequality is trivial. If $E(\xi^p) > 0$, dividing both sides by $[E\xi^p]^{(p-1)/p}$, the second inequality is also obtained.
2): If $\{x_n\}_{n=0,1,\cdots}$ is a real martingale, then by Jensen's inequality we have that $\{|x_n|\}_{n=0,1,\cdots}$ is a submartingale, and $|x_n|^+ = |x_n|$. So by 1) the conclusions are derived in this case. ∎

In the following we will show the upcrossing inequality for a submartingale, which is the basis for proving the important limit property of a submartingale. First we introduce some notations.

For a real $\mathfrak{F}_t-$ adapted process $\{x_n\}_{n=0,1,\cdots}$ and an interval $[a, b]$, where $b > a$, let

$\tau_1 = \min\{n \geq 0 : x_n \leq a\}$,
$\tau_2 = \min\{n \geq \tau_1 : x_n \geq b\}$,
......
$\tau_{2n+1} = \min\{n \geq \tau_{2n} : x_n \leq a\}$,
$\tau_{2n+2} = \min\{n \geq \tau_{2n+1} : x_n \geq b\}$,
...... ;

where we recall that $\min \phi = +\infty$. Then $\{\tau_n\}$ is an increasing sequence of stopping times. In fact, $\forall k \geq 0$,

$\{\tau_1 = k\} = \{x_0 > a, x_1 > a, \cdots, x_{k-1} > a, x_k \leq a\} \in \mathfrak{F}_k;$
$\{\tau_2 = k\} = \cup_{j=0}^{k-1} \{\tau_1 = j, \tau_2 = k\}$
$= \cup_{j=0}^{k-1} \{\tau_1 = j, x_j \leq a, x_{j+1} < b, \cdots, x_{k-1} < b, x_k \geq b\} \in \mathfrak{F}_k;$
$\{\tau_3 = k\} = \cup_{j=1}^{k-1} \{\tau_2 = j, \tau_3 = k\}$
$= \cup_{j=1}^{k-1} \{\tau_2 = j, x_j \geq b, x_{j+1} > a, \cdots, x_{k-1} > a, x_k \leq a\} \in \mathfrak{F}_k.$

Hence τ_1, τ_2, and τ_3 are stopping times. The proofs for the rest are similar. Now set

$U_a^b[x(.), N](\omega) = \max\{k \geq 1 : \tau_{2k}(\omega) \leq N\}$,
$D_a^b[x(.), N](\omega) = \max\{k \geq 1 : \tau_{2k-1}(\omega) \leq N\}$.

Obviously the first one is the number connected to the upcrossing of $\{x_n\}_{n=0}^N$ for the interval $[a, b]$, and the second one is the number connected to the downcrossing of $\{x_n\}_{n=0}^N$ for the interval $[a, b]$.

Theorem 18 *If $\{x_n\}_{n=0}^\infty$ is a submartingale, then for each $N \geq 1, n \geq 0$ and $a < b$*

$EU_a^b[x(.), N] \leq \frac{1}{b-a}(E[(x_N - a)^+ - (x_0 - a)^+)$,
$P(U_a^b[x(.), N] \geq n) \leq \frac{1}{b-a} E[(x_N - a)^+ I_{U_a^b[x(.),N]=n}]$,
$ED_a^b[x(.), N] \leq \frac{1}{(b-a)} E(x_N - b)^+$,
$P(D_a^b[x(.), N] \geq n+1) \leq \frac{1}{(b-a)} E(x_N - b)^+ I_{D_a^b[x(.),N]=n}.$

Proof. For a submartingale $\{x_n\}_{n=0}^\infty$ by Example 3 one sees that $\{y_n\}_{n=0}^\infty = \{(x_n - a)^+\}_{n=0}^\infty$ is a non-negative submartingale. Clearly $U_0^{b-a}[y(.), N](\omega) = U_a^b[x(.), N](\omega)$. Again define $\tau_1, \tau_2, \cdots$ as above, but with x, a, and b replaced by $y, 0$, and $b - a$ respectively. Then if $2k > N$

$E(y_N - y_0) = E\sum_{n=1}^{2k}(y_{\tau_n \wedge N} - y_{\tau_{n-1} \wedge N}) = E\sum_{n=1}^{k}(y_{\tau_{2n} \wedge N} - y_{\tau_{2n-1} \wedge N})$
$+ \sum_{n=0}^{k-1} E(y_{\tau_{2n+1} \wedge N} - y_{\tau_{2n} \wedge N}) \geq (b-a)EU_0^{b-a}[y(.), N]$,

where we have used the fact that $\{y_n\}_{n=0}^\infty$ is a submartingale, and hence a strong submartingale (Theorem 15), so $E(y_{\tau_{2n+1} \wedge N} - y_{\tau_{2n} \wedge N}) \geq 0$; and $y_n \geq 0, \forall n$. The first inequality is proved. Now observe that

$0 \geq E(y_{\tau_{2n} \wedge N} - y_{\tau_{2n+1} \wedge N})$
$= E[(y_{\tau_{2n} \wedge N} - y_{\tau_{2n+1} \wedge N})(I_{\tau_{2n} \leq N < \tau_{2n+1}} + I_{\tau_{2n+1} \leq N})]$

$= E[(b-a-y_N)I_{\tau_{2n}\le N<\tau_{2n+1}} + (b-a)I_{\tau_{2n+1}\le N}]$
$= E(b-a)I_{\tau_{2n}\le N} - Ey_N I_{\tau_{2n}\le N<\tau_{2n+1}}.$
Since $\left\{U_0^{b-a}[y(.),N] \ge n\right\} = \{N \ge \tau_{2n}\}$ and
$\{\tau_{2n} \le N < \tau_{2n}+1\} \subset \{\tau_{2n} \le N < \tau_{2n+2}\} = \left\{U_0^{b-a}[y(.),N] = n\right\}.$
Hence we find that $Ey_N I_{U_a^b[x(.),N]=n} \ge (b-a)P(U_0^{b-a}[y(.),N] \ge n)$.

For the downcrossing inequality we have to discuss $\{x_n\}_{n=0}^{\infty}$ itself directly, since $\{x_n \wedge 0\}_{n=0}^{\infty}$ is not a submartingale. Let us set $y_n = x_n - b$. Then $\{y_n\}_{n=0}^{\infty}$ is still a submartingale, and
$D^0_{-(b-a)}[y(.),N](\omega) = D_a^b[x(.),N](\omega).$

Again define $\tau_1, \tau_2, \cdots$ as above but with x, a, and b replaced by $y, -(b-a)$, and 0 respectively. We will now use another method to show the last two inequalities. First, for the fourth inequality we have that as $n \ge 1$
$0 \ge E(y_{\tau_{2n}\wedge N} - y_{\tau_{2n+1}\wedge N})$
$= E[(0-(x_N-b))I_{\tau_{2n}\le N<\tau_{2n+1}} + (b-a)I_{\tau_{2n+1}\le N}].$
Since
$\left\{D_a^b[x(.),N] \ge n+1\right\} = \left\{D^0_{-(b-a)}[y(.),N] \ge n+1\right\}$
$= \{N \ge \tau_{2n+2}\} \subset \{N \ge \tau_{2n+1}\}$
and $\{\tau_{2n} \le N < \tau_{2n+1}\} \subset \{\tau_{2n} \le N < \tau_{2n+2}\} = \left\{D_a^b[x(.),N] = n\right\}.$
Hence it follows that
$E(x_N-b)^+ I_{D_a^b[x(.),N]=n} \ge (b-a)P(D_a^b[x(.),N] \ge n+1).$
The fourth inequality holds. Now taking the summation for $n \ge 0$ it yields
$E(x_N-b)^+ \ge (b-a)\sum_{n=0}^{\infty} P(D_a^b[x(.),N] \ge n+1)$
$= (b-a)\sum_{n=0}^{\infty} nP(D_a^b[x(.),N] = n) = (b-a)ED_a^b[x(.),N].$
The third inequality is also established. ■

Corollary 19 *If $\{x_n\}_{n=0}^{\infty}$ is a supermartingale, then for each $N \ge 1, n \ge 0$ and $a < b$*
$EU_a^b[x(.),N] \le \frac{1}{b-a}E[(x_N-a)^-],$
$P(U_a^b[x(.),N] \ge n+1) \le \frac{1}{b-a}E[(x_N-a)^- I_{U_a^b[x(.),N]=n}].$
$ED_a^b[x(.),N] \le \frac{1}{(b-a)}E[(x_N-b)^- - (x_0-b)^-],$
$P(D_a^b[x(.),N] \ge n) \le \frac{1}{(b-a)}E(x_N-b)^- I_{D_a^b[x(.),N]=n}.$

Proof. Let $y_n = -x_n$. Then $\{y_n\}_{n=0}^{\infty}$ is a submartingale. Hence
$U_a^b[x(.),N] = D_{-b}^{-a}[y(.),N],$
and
$D_a^b[x(.),N] = U_{-b}^{-a}[y(.),N].$
Applying Theorem 18 we arrive at the results. ■

Theorem 18 and Corollary 19 are the classical crossing theorems on martingale. We can derive some other useful crossing results which are very useful in the mathematical finance.[16],[17],[180] Here we apply some of them to derive the important limit theorem on martingales.

Theorem 20 *If $\{x_n\}_{n=0}^{\infty}$ is a submartingale such that there exists a subsequence of $\{n\}$, denote it by $\{n_k\}$, such that*

$$\sup_k Ex_{n_k}^+ < \infty, \tag{1.1}$$

then $x_\infty = \lim_{n\to\infty} x_n$ exists a.s., and x_∞ is integrable. In particular, if $x_n \leq 0, \forall n$, then condition (1.1) is obviously satisfied, and in this case $\forall n$
$E[x_\infty|\mathfrak{F}_n] \geq x_n$, *a.s.*

Proof. First, clearly
condition (1.1) $\Longleftrightarrow \sup_n Ex_n^+ < \infty \Longleftrightarrow \sup_n E|x_n| < \infty$.
In fact, by the properties of submartingales one has that
$Ex_k^+ \leq Ex_{n_k}^+, \ \forall k.$
and
$E|x_n| = 2Ex_n^+ - Ex_n \leq 2Ex_n^+ - Ex_0.$
Hence the equivalent relations hold. Now let $U_a^b(x(.)) = \lim_{N\to\infty} U_a^b(x(.), N)$.
Then by Theorem 18
$EU_a^b(x(.)) \leq \frac{1}{b-a}\sup_N E(x_N - a)^+ < \infty.$
Hence $U_a^b(x(.)) < \infty$, $a.s.$ Let
$W = \cup_{a,b\in Q,a<b} W_{a,b} = \cup_{a,b\in Q,a<b}\{\underline{\lim}_n x_n < a < b < \overline{\lim}_n x_n\}.$
Then
$P(W) \leq \sum_{a,b\in Q,a<b} P(W_{a,b}) \leq \sum_{a,b\in Q,a<b} P(U_a^b(x(.)) = \infty) = 0.$
Now we can let $x_\infty(\omega) = \lim_{n\to\infty} x_n(\omega)$, as $\omega \notin W$; and $x_\infty(\omega) = 0$, as $\omega \in W$. By Fatou's lemma
$E|x_\infty| \leq \sup_n E|x_n| < \infty.$
Hence x_∞ is integrable. In the case $x_n \leq 0, \forall n$, by the definition of a submartingale
$0 \geq E[x_m|\mathfrak{F}_n] \geq x_n$, a.s. $\forall m$.
Again by Fatou's lemma letting $m \to \infty$ one reaches the final conclusion.
∎

1.4 Uniform Integrability and Martingales

It is well known in the theory of real analysis that if a sequence of measurable functions is dominated by an integrable function, then one can take the limit under the integral sign for the function sequence. That is the famous Lebesgue's dominated convergence theorem. However, sometimes it is difficult to find such a dominated function. In this case the uniform integrability of that function sequence can be a great help. Actually, in many cases it is a powerful tool .

Definition 21 *A family of functions $A \subset L^1(\Omega, \mathfrak{F}, P)$ is called uniformly integrable, if $\lim_{\lambda\to\infty}\sup_{f\in A} E(fI_{|f|>\lambda}) = 0$, where $L^1(\Omega, \mathfrak{F}, P)$ is the totality of random variables ξ, (that is, all ξ are $\mathfrak{F}-$measurable) such that $E|\xi| < \infty$.*

Lemma 22 *Suppose that* $\{x_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathfrak{F}, P)$ *is uniformly integrable, and as* $n \to \infty$,

$x_n \to x$, *in probability,*

i.e. $\forall \varepsilon > 0$, $\lim_{n\to\infty} P(|x_n - x| > \varepsilon) = 0$, *then*

$\lim_{n\to\infty} E|x_n - x| = 0$. *(i.e.* $x_n \to x$, *in* $L^1(\Omega, \mathfrak{F}, P)$*).*

In particular, $\lim_{n\to\infty} Ex_n = Ex$

Proof. In fact, $\forall \varepsilon > 0$,

$E|x_n - x| \le E(|x_n - x| I_{|x_n - x| > \lambda}) + E(|x_n - x| I_{|x_n - x| \le \lambda}) = I_1^{n,\lambda} + I_2^{n,\lambda}$.

Hence one can take a λ large enough such that $I_1^{n,\lambda} < \varepsilon/2$, since clearly $\{x_n - x\}_{n=1}^{\infty}$ is uniformly integrable. Then for this fixed λ by using Lebesgue's dominated convergence theorem one can have a sufficiently large N such that as $n \ge N$, $I_2^{n,\lambda} < \varepsilon/2$. ■

For the sufficient conditions of uniform integrability of a family A we have

Lemma 23 *Suppose that* $A \subset L^1(\Omega, \mathfrak{F}, P)$. *Any one of the following conditions makes A uniformly integrable:*

1) There exists an integrable $g \in L^1(\Omega, \mathfrak{F}, P)$ *such that*

$|x| \le g$, $\forall x \in A$.

2) There exists a $p > 1$ *such that* $\sup_{x\in A} E|x(\omega)|^p < \infty$.

Proof. 1): Since as $\lambda \to \infty$

$\sup_{x\in A} P(|x(\omega)| > \lambda) \le \frac{1}{\lambda}\sup_{x\in A} E|x(\omega)| \le \frac{1}{\lambda} E|g(\omega)| \to 0$.

So by the integrability of g one has that as $\lambda \to \infty$

$E|x(\omega)| I_{|x(\omega)|>\lambda} \le Eg(\omega) I_{|x(\omega)|>\lambda} \to 0$, uniformly w.r.t. $x \in A$.

2): Since $\sup_{x\in A} P(|x(\omega)| > \lambda) \le \frac{1}{\lambda}\sup_{x\in A} E|x(\omega)| \to 0$, as $\lambda \to \infty$. So as $\lambda \to \infty$

$\sup_{x\in A} E|x(\omega)| I_{|x(\omega)|>\lambda}$

$\le \sup_{x\in A}(E|x(\omega)|^p)^{1/p} \sup_{x\in A}[P(|x(\omega)| > \lambda)]^{(p-1)/p} \to 0$. ■

Now we know that the uniform integrability condition is weaker than the domination condition. Actually, it is also the neccessary condition for the L^1-convergence of the sequence of integrable random variables or, say, integrable functions.

Theorem 24 *Suppose that* $\{x_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathfrak{F}, P)$. *Then the following two statements are equivalent:*

1) $\{x_n\}_{n=1}^{\infty}$ *is uniformly integrable.*

2) $\sup_{n=1,2,\cdots} E|x_n| < \infty$; *and* $\forall \varepsilon > 0, \exists \delta > 0$ *such that* $\forall B \in \mathfrak{F}$, *as* $P(B) < \delta$

$\sup_{n=1,2,\cdots} E|x_n| I_B < \varepsilon$.

Furthermore, if there exists an $x \in L^1(\Omega, \mathfrak{F}, P)$ *such that as* $n \to \infty$, $x_n \to x$, *in probability; then the following statement is also equivalent to 1):*

3) $x_n \to x$, *in* $L^1(\Omega, \mathfrak{F}, P)$.

Proof. Since 1) $\Longrightarrow$ 3) is already proved in Lemma 22, we will show that 3) $\Longrightarrow$ 2) $\Longrightarrow$ 1) $\Longrightarrow$2).

1) $\Longrightarrow$ 2): Take a λ_0 large enough such that $\sup_{n=1,2,\cdots} E|x_n| I_{|x_n|>\lambda_0} < 1$. Then

$\sup_{n=1,2,\cdots} E|x_n| \leq \lambda_0 + 1.$

On the other hand, for any $B \in \mathfrak{F}$ since

$E|x_n| I_B = E|x_n| I_{\{|x_n|>\lambda\}\cap B} + E|x_n| I_{\{|x_n|\leq\lambda\}\cap B}$

$\leq \sup_{n=1,2,\cdots} E|x_n| I_{\{|x_n|>\lambda\}} + \lambda P(B) = I_1^\lambda + I_2^{\lambda,B}.$

Hence $\forall\varepsilon > 0$ one can take a $\lambda_\varepsilon > 0$ large enough such that $I_1^{\lambda_\varepsilon} < \frac{\varepsilon}{2}$, then let $\delta_\varepsilon = \frac{\varepsilon}{2\lambda_\varepsilon}$. For this $\delta_\varepsilon > 0$ one has that $\forall B \in \mathfrak{F}, P(B) < \delta_\varepsilon \Longrightarrow \sup_{n=1,2,\cdots} E|x_n| I_B < \varepsilon$.

2) $\Longrightarrow$ 1): $\forall\varepsilon > 0$ Take $\delta > 0$ such that 2) holds. Since

$P(|x_n| > \lambda) \leq \frac{1}{\lambda} \sup_{n=1,2,\cdots} E|x_n|.$

Hence one can take an N large enough such that as $\lambda > N$,

$P(|x_n| > \lambda) < \delta, \forall n = 1, 2, \cdots.$

Thus by 2) as $\lambda > N$,

$E|x_n| I_{|x_n|>\lambda} < \varepsilon, \forall n = 1, 2, \cdots.$

3) $\Longrightarrow$ 2): Take an N_0 large enough such that as $n > N_0$,

$E|x - x_n| < 1.$

Thus

$\sup_{n=1,2,\cdots} E|x_n| \leq \max\{1 + E|x|, E|x_1|, \cdots, E|x_{N_0}|\} < \infty.$

On the other hand, observe that

$E|x_n| I_B \leq E|x_n - x| + E|x| I_B.$

Hence $\forall\varepsilon > 0$, one can take an N_ε large enough such that as $n > N_\varepsilon$,

$E|x - x_n| < \frac{\varepsilon}{2}.$

Then take a $\delta > 0$ small enough such that $\forall B \in \mathfrak{F}$, as $P(B) < \delta$,

$\max_{n=1,\cdots,N_\varepsilon} \{E|x_n| I_B\} < \varepsilon$, and $E|x| I_B < \varepsilon/2$.

Thus as $P(B) < \delta, E|x_n| I_B < \varepsilon, \forall n = 1, 2, \cdots.$ ∎

Now let us use uniform integrability as a tool to study the martingales.

Theorem 25 *If $\{x_n\}_{n=0}^\infty$ is a submartingale such that $\{x_n^+\}_{n=0}^\infty$ is uniformly integrable, then $x_\infty = \lim_{n\to\infty} x_n$ exists, a.s., and*

$E[x_\infty|\mathfrak{F}_n] \geq x_n, \forall n,$

i.e. $\{x_n\}_{n=0,1,2,\cdots,\infty}$ is also a submartingale, and we call it a right-closed submartingale.

This theorem actually tells us that a uniformly integrable submartingale is a right-closed submartingale.

Proof. By uniform integrability one has that

$\sup_{n=0,1,2,\cdots} Ex_n^+ < \infty.$

Hence applying Theorem 20 one has that $x_\infty = \lim_{n\to\infty} x_n$ exists, a.s. Now by the submartingale property $\{x_n^+\}_{n=0}^\infty$ is also a submartingale (Example 3). Hence for any $a > 0$, and $B \in \mathfrak{F}_n$, as $m \geq n$,

$\int_B[(-a) \vee x_n]dP \leq \int_B[(-a) \vee x_m]dP.$

Letting $m \to \infty$ by the uniform integrability of $\{x_n^+\}_{n=0}^\infty$ one has that

$\int_B[(-a)\vee x_n]dP \le \int_B[(-a)\vee x_\infty]dP.$
Now letting $a \uparrow \infty$ by Fatou's lemma one obtains that
$\int_B x_n dP \le \int_B x_\infty dP = \int_B E[x_\infty|\mathfrak{F}_n]dP, \forall B \in \mathfrak{F}_n.$
The conclusion is established. ∎

We also have the following inverse theorem.

Theorem 26 *If $\{x_n\}_{n=0,1,2,\cdots,\infty}$ is a submartingale, where*
$x_\infty = \lim_{n\to\infty} x_n$ *exists, a.s.,*
then $\{x_n^+\}_{n=0,1,2,\cdots}$ is uniformly integrable.

Proof. *By Jensen's inequality $\{x_n^+\}_{n=0,1,2,\cdots,\infty}$ is also a submartingale. Now $\forall\lambda > 0$, denote $B_\lambda^n = \{x_n^+ > \lambda\}$, then by the submartingale definition as $\lambda \to \infty$*
$P(B_\lambda^n) \le \frac{1}{\lambda}Ex_n^+ \le \frac{1}{\lambda}Ex_\infty^+ \to 0,$ *uniformly w.r.t. n.*
Therefore as $\lambda \to \infty$
$\int_{B_\lambda^n} x_n^+ dP \le \int_{B_\lambda^n} x_\infty^+ dP \to 0,$ *uniformly w.r.t. n.* ∎

Corollary 27 *1) If $\{x_n\}_{n=0,1,2,\cdots,\infty}$ is a right-closed martingale, then one has that $\{x_n\}_{n=0,1,2,\cdots}$ is uniformly integrable.*

2) For a sequence of random variables $\{y_n\}_{n=0,1,2,\cdots}$ if there exists a $z \in L^1(\Omega, \mathfrak{F}, P)$ such that $|y_n| \le E(|z|\,|\mathfrak{F}_n)$, where $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$ are $\sigma-$fields, then $\{y_n\}_{n=0,1,2,\cdots}$ is uniformly integrable.

Proof. 1): It can be derived from Theorem 26, since $\{x_n\}_{n=0,1,2,\cdots,\infty}$ is both a submartingale and a supermartingale.

2): In fact, let $z_n = E(z|\mathfrak{F}_n)$. Then, obviously $\{z_n\}_{n=0,1,2,\cdots}$ is a martingale, and $\sup_{n=0,1,2,\cdots} E|z_n| \le E|z| < \infty$. Hence by Theorem 20 $z_\infty = \lim_{n\to\infty} z_n$ exists, *a.s.*, $z_\infty \in L^1(\Omega, \mathfrak{F}, P)$, and $z_n = E(z_\infty|\mathfrak{F}_n), \forall n$. Therefore $\{z_n\}_{n=0,1,2,\cdots,\infty}$ is a right-closed martingale. Now by 1) it is uniformly integrable, so is $\{y_n\}_{n=0,1,2,\cdots}$. ∎

Condition in 2) for the uniform integrability is weaker than the usual Lebesgue's dominated condition. Moreover, from the proof of 2) in Corollary 27 one also can obtain the following thoerem

Theorem 28 *(Levi's theorem) If $z \in L^1(\Omega, \mathfrak{F}, P)$, and $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$ are $\sigma-$fields, then as $n \uparrow \infty$, $E[z|\mathfrak{F}_n] \to E[z|\mathfrak{F}_\infty]$, where $\mathfrak{F}_\infty = \vee_{n=1,2,\cdots}\mathfrak{F}_n$, i.e. $\mathfrak{F}_\infty$ is the smallest $\sigma-$field including all $\mathfrak{F}_n, n = 1, 2, \cdots$.*

Proof. By the proof of 2) in Corollary 27 one already has that $z_\infty = \lim_{n\to\infty} z_n = \lim_{n\to\infty} E[z|\mathfrak{F}_n]$ exists, *a.s.*, $z_\infty \in L^1(\Omega, \mathfrak{F}, P)$, and $z_n = E(z_\infty|\mathfrak{F}_n), \forall n$. Let us show that $z_\infty = E[z|\mathfrak{F}_\infty], a.s.$ In fact, by limit one has $z_\infty \in \mathfrak{F}_\infty$, i.e. it is $\mathfrak{F}_\infty-$measurable. Moreover, $\forall n, \forall B \in \mathfrak{F}_n$,
$Ez_\infty I_B = Ez_n I_B = E(E[z|\mathfrak{F}_n]I_B) = E(E[zI_B|\mathfrak{F}_n]) = EzI_B$
$= EE[z|\mathfrak{F}_\infty]I_B.$
From this one also has that $\forall B \in \mathfrak{F}_\infty = \vee_{n=1,2,\cdots}\mathfrak{F}_n$,
$Ez_\infty I_B = EE[z|\mathfrak{F}_\infty]I_B.$

Since z_∞ and $E[z|\mathfrak{F}_\infty]$ both are $\mathfrak{F}_\infty$−measurable. Hence $z_\infty = E[z|\mathfrak{F}_\infty], a.s.$
■

Now let us consider the discrete time $\{\cdots, -k, -k+1, \cdots, -2, -1, 0\}$ with right-end point 0 but without the initial left–starting-time. We still call $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ a martingale (supermartingale, submartingale), if
(i) x_n is integrable for each $n = 0, -1, -2, \cdots$;
(ii) x_n is $\mathfrak{F}_n$−adapted, i.e. for each $n = 0, -1, -2, \cdots$, x_n is $\mathfrak{F}_n$−measurable, where $\mathfrak{F}_n, n = 0, -1, -2, \cdots$, are σ−fields still with an increasing property, i.e. $\mathfrak{F}_n \subset \mathfrak{F}_m$, as $n \le m; \forall n, m \in \{\cdots, -k, -k+1, \cdots, -2, -1, 0\}$;
(iii) $E[x_m|\mathfrak{F}_s] = x_n$, (respectively, $\le$, $\ge$), $a.s.$ $\forall 0 \le n \le m; \forall n, m \in \{\cdots, -k, -k+1, \cdots, -2, -1, 0\}$.
We have the following limit thoerem.

Theorem 29 *If* $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ *is a submartingale such that* $\inf_{n=0,-1,-2,\cdots} Ex_n > -\infty$,
then $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ *is uniformly integrable,*
$x_{-\infty} = \lim_{n\to-\infty} x_n$ *exists a.s.,*
and as $n \to -\infty$, $x_n \to x_{-\infty}$, *in* $L^1(\Omega, \mathfrak{F}, P)$.

Proof. For each N consider the finite sequence of random variables $\{x_n\}_{n=-N,-N+1,\cdots,-1,0}$. Denot by $U_a^b[x(.), -N]$ the number of upcrossing of $\{x_n\}_{n=-N,-N+1,\cdots,-1,0}$ for the interval $[a,b]$. Then by Theorem 18 $EU_a^b[x(.), -N] \le \frac{1}{b-a}E(x_0 - a)^+$, and
$EU_a^b[x(.)] \le \frac{1}{b-a}E(x_0 - a)^+ < \infty$,
where $U_a^b[x(.)] = \lim_{N\to\infty} U_a^b[x(.), -N]$. By the proof of Theorem 18 one has that $x_{-\infty} = \lim_{n\to-\infty} x_n$ exists $a.s.$ However, it still remains to be proved that $x_{-\infty}$ is finite, $a.s.$. Let us show that $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ is uniformly integrable. If this can be done, then all conclusions will be derived immediately by the property of uniform integrability. Observe that $Ex_n \downarrow$, as $n \downarrow$, since $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ is a submartingale. Hence by assumption a finite limit exists:
$\lim_{n\to-\infty} Ex_n \ge \inf_{n=0,-1,-2,\cdots} Ex_n > -\infty$.
Note that $\{E[x_0|\mathfrak{F}_n]\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ is a martingale and uniformly integrable. Hence $\{x_n - E[x_0|\mathfrak{F}_n]\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ is a non-positive submartingale, and the uniform integrability of it is the same as that of $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$. So we may assume that $x_n \le 0$. Now $\forall\varepsilon > 0$, take a $-k$ large enough such that $Ex_k - \lim_{n\to-\infty} Ex_n < \varepsilon$. Then by the property of submartingales and the property that $Ex_n \downarrow$ as $n \downarrow$,
$P(x_n < -\lambda) \le \frac{1}{\lambda}E|x_n| = \frac{1}{\lambda}(2Ex_n^+ - Ex_n)$
$\le \frac{1}{\lambda}(2Ex_0^+ - \lim_{n\to-\infty} Ex_n) \to 0$, uniformly w.r.t. n, as $\lambda \to \infty$.
and if $n \le k \le 0$,
$0 \le E[(-x_n)I_{x_n<-\lambda}] \le -Ex_n - E[(-x_n)I_{x_n\ge-\lambda}]$
$\le -\lim_{n\to-\infty} Ex_n + E[x_k I_{x_n\ge-\lambda}]$
$\le -Ex_k + E[x_k I_{x_n\ge-\lambda}] + \varepsilon = E[(-x_k)I_{x_n<-\lambda}] + \varepsilon$.

From this one easily derives that $\{x_n\}_{n\in\{\cdots,-k,-k+1,\cdots,-2,-1,0\}}$ with $x_n \le 0$ is uniformly integrable. ∎

1.5 Martingales with Continuous Time

Now let us consider the martingale (submartingale, and supermartingale) $\{x_t\}_{t\in[0,\infty)}$ with continuous time. First, we will still introduce the upcrossing numbers of a random process for an interval. Let $\{x_t\}_{t\ge 0} = \{x_t\}_{t\in[0,\infty)}$ be an adapted process, and $U = \{t_1, t_2, \cdots, t_n\}$ be a finite subset of $R_+ = [0,\infty)$. Denote its rearrangement to the natural order by $\{s_1, s_2, \cdots, s_n\}$, i.e. $s_1 < s_2 < \cdots < s_n$. Let $U_a^b[x(.), U]$ be the number of upcrossings of $\{x_{s_k}\}_{k=1}^n$ for interval $[a,b]$, and we also call it the number of upcrossings of $\{x_t\}_{t\in U}$ for interval $[a,b]$. For any subset D of R_+, define

$U_a^b[x(.), D] = \sup\left\{U_a^b[x(.), U] : U \text{ is a finite subset of } D\right\}.$

In case $D = \{t_1, t_2, \cdots, t_n, \cdots\}$, obviously

$U_a^b[x(.), D] = \lim_{n\to\infty} U_a^b[x(.), U_n],$

where $U_n = \{t_1, t_2, \cdots, t_n\}$. By using the results on discrete time we have the following theorem.

Theorem 30 *If $\{x_t\}_{t\ge 0}$ is a submartingale, $D = \{t_1, t_2, \cdots, t_n, \cdots\}$, then for any $0 \le r < s, a < b$ and $\lambda > 0$ one has that*

$\lambda P(\sup_{t\in D\cap[r,s]} |x_t| > -\lambda) \le Ex_r^- + 2E(x_s^+),$

$EU_a^b[x(.), D\cap[r,s]] \le \frac{1}{b-a}(E[(x_s - a)^+ - (x_r - a)^+).$

Proof. Set $\widehat{D} = \{r, s, t_1, t_2, \cdots, t_n, \cdots\}$ Notice that

$U_n = (\{r, s, t_1, t_2, \cdots, t_n\} \cap [r,s]) \uparrow (\widehat{D} \cap [r,s]).$

So the conlusions for the set $\widehat{D} \cup [r,s]$ are derived by applying Theorem 16 and 18 and taking the limit for $n \to \infty$. However, $(D \cap [r,s]) \subset (\widehat{D} \cap [r,s])$. So the two conclusions for $(D \cap [r,s])$ hold true. ∎

Now we can generalize the limit theorem to submartingales with continuous time.

Theorem 31 *Let $\{x_t\}_{t\ge 0}$ be a submartingale. Then $\widehat{x}_t = \lim_{r\downarrow t, r\in Q} x_r$ exists and finite a.s. and $\{\widehat{x}_t\}_{t\ge 0}$ is still a submartingale such that $\widehat{x}_t$ is right continuous with left-hand limits a.s. Furthermore, $x_t \le \widehat{x}_t$ a.s. for $\forall t \ge 0$.*

(Recall that $Q =$ the totality of real rational numbers). To establish the above theorem we divide it into two steps. The first step can be written as the following lemma.

Lemma 32 *If $\{x_t\}_{t\ge 0}$ is a submartingale, then $\forall t \ge 0$*

$\lim_{r\uparrow t, r\in Q} x_r$ *and* $\lim_{r\downarrow t, r\in Q} x_r$

exist and are finite a.s. (Here, we define $x_{0-} = x_0$, and $\mathfrak{F}_{0-} = \mathfrak{F}_0$).