

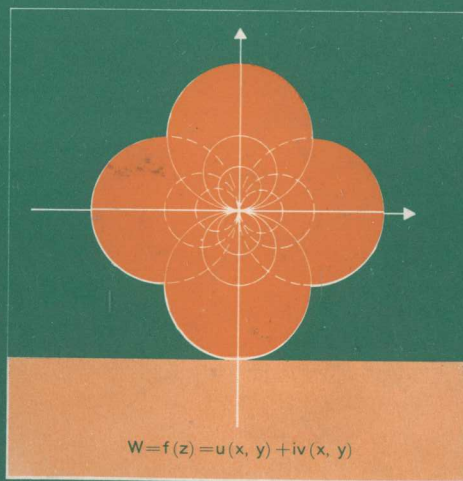
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高等工程數學詳解

(1979最新版) 第四版 第(三)冊 吳雄明譯著

ADVANCED ENGINEERING MATHEMATICS

Erwin Kreyszig Fourth Edition



久大書局

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第十一章 偏微分方程

11-1 節 基本觀念

習題 11-1

對於在波方程式(1)中，一適合的常數 c 值，證明下列各函數均爲方程式(1)的解，並畫出這些函數在空間中的曲面。

1. $u = x^2 + t^2$

解： $\frac{\partial^2 u}{\partial t^2} = 2$, $\frac{\partial^2 u}{\partial x^2} = 2$

當 $c = \pm 1$ 時

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

曲面爲旋轉拋物面。

2. $u = x^2 + 9t^2$

解： $\frac{\partial^2 u}{\partial t^2} = 18$, $\frac{\partial^2 u}{\partial x^2} = 2$

當 $c = \pm 3$ 時

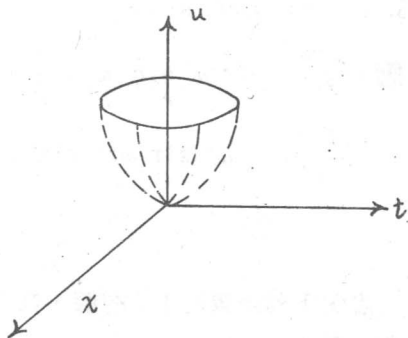
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad \text{曲面爲橢圓拋物面。}$$

3. $u = \cos t \sin x$.

解： $\frac{\partial^2 u}{\partial t^2} = -\cos t \sin x$, $\frac{\partial^2 u}{\partial x^2} = -\cos t \sin x$.

當 $c = \pm 1$ 時

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{曲面爲水波進行之波面。}$$



橢圓拋物面

(與第1題相類似)

4. $u = \sin t \sin x$.

解: $\frac{\partial^2 u}{\partial t^2} = -\sin t \sin x$, $\frac{\partial^2 u}{\partial x^2} = -\sin t \sin x$.

當 $c = \pm 1$ 時, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

其曲面與第(3)題相似。

5. $u = \cos ct \sin x$

解: $\frac{\partial^2 u}{\partial t^2} = -c^2 \cos ct \sin x$, $\frac{\partial^2 u}{\partial x^2} = -\cos ct \sin x$

$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, 其曲面與第(3)題相似。

6. $u = \sin wct \sin \omega x$

解: $\frac{\partial^2 u}{\partial t^2} = -c^2 \omega^2 \sin \omega ct \sin \omega x$,

$\frac{\partial^2 u}{\partial x^2} = -\omega^2 \sin \omega ct \sin \omega x$

$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, 其曲面與第(3)題相似。

證明下列函數爲(1)式的解, 其中 ν 與 ω 爲任意二次可微分函數, 在第7題中 $c = 1$ 。

7. $u(x, t) = \nu(x+t) + \omega(x-t)$

解: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+t)}{\partial t} + \frac{\partial \omega(x-t)}{\partial t} \right]$
 $= \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+t)}{\partial (x+t)} \frac{\partial (x+t)}{\partial t} + \frac{\partial \omega(x-t)}{\partial (x-t)} \frac{\partial (x-t)}{\partial t} \right]$
 $= \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+t)}{\partial (x+t)} - \frac{\partial \omega(x-t)}{\partial (x-t)} \right]$
 $= \frac{\partial^2 \nu(x+t)}{\partial (x+t)^2} + \frac{\partial^2 \omega(x-t)}{\partial (x-t)^2}$

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left[\frac{\partial \nu(x+t)}{\partial x} + \frac{\partial \omega(x-t)}{\partial t} \right] \\
 &= \frac{\partial}{\partial x} \left[\frac{\partial \nu(x+t)}{\partial (x+t)} \frac{\partial (x+t)}{\partial x} + \frac{\partial \omega(x-t)}{\partial (x-t)} \frac{\partial (x-t)}{\partial x} \right] \\
 &= \frac{\partial}{\partial x} \left[\frac{\partial \nu(x+t)}{\partial (x+t)} + \frac{\partial \omega(x-t)}{\partial (x-t)} \right] \\
 &= \frac{\partial^2 \nu(x+t)}{\partial (x+t)^2} + \frac{\partial^2 \omega(x-t)}{\partial (x-t)^2} \\
 \therefore \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}
 \end{aligned}$$

$$8. u(x, t) = \nu(x+ct) + \omega(x-ct)$$

$$\begin{aligned}
 \text{解: } \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+ct)}{\partial t} + \frac{\partial \omega(x-ct)}{\partial t} \right] \\
 &= \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+ct)}{\partial (x+ct)} \frac{\partial (x+ct)}{\partial t} + \frac{\partial \omega(x-ct)}{\partial (x-ct)} \frac{\partial (x-ct)}{\partial t} \right] \\
 &= \frac{\partial}{\partial t} \left[\frac{\partial \nu(x+ct)}{\partial (x+ct)} \cdot c + \frac{\partial \omega(x-ct)}{\partial (x-ct)} \cdot (-c) \right] \\
 &= c^2 \frac{\partial^2 \nu(x+ct)}{\partial (x+ct)^2} + c^2 \frac{\partial^2 \omega(x-ct)}{\partial (x-ct)^2} \\
 &= c^2 \left[\frac{\partial^2 \nu(x+ct)}{\partial (x+ct)^2} + \frac{\partial^2 \omega(x-ct)}{\partial (x-ct)^2} \right] \\
 \text{同理 } \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 \nu(x+ct)}{\partial (x+ct)^2} + \frac{\partial^2 \omega(x-ct)}{\partial (x-ct)^2} \\
 \therefore \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}
 \end{aligned}$$

證明下列各函數熱傳方程式(2)的解

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$9. u = e^{-t} \cos x.$$

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$$\text{解: } \frac{\partial u}{\partial t} = -e^{-t} \cos x \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = -e^{-t} \cos x.$$

$$\text{當 } c = \pm 1 \text{ 時, } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$10. u = e^{-t} \sin x.$$

$$\text{解: } \frac{\partial u}{\partial t} = -e^{-t} \sin x \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = -e^{-t} \sin x.$$

$$\text{當 } c = \pm 1 \text{ 時, } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$11. u = e^{-\omega^2 c^2 t} \sin \omega x$$

$$\text{解: } \frac{\partial u}{\partial t} = -\omega^2 c^2 e^{-\omega^2 c^2 t} \sin \omega x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = -\omega^2 c^2 e^{-\omega^2 c^2 t} \sin \omega x$$

$$\therefore \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

證明下列各函數爲拉普氏方程式(3)的解，並畫出這些函數在空間的曲面圖。

$$12. u = x^2 - y^2$$

$$\text{解: } \frac{\partial^2 u}{\partial x^2} = 2 \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{爲雙曲拋物面。}$$

$$13. u = x^3 - 3xy^2$$

$$\text{解: } \frac{\partial^2 u}{\partial x^2} = 6x \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$14. u = 3x^2 y - y^3$$

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = 6y \quad \frac{\partial^2 u}{\partial y^2} = -6y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$15. u = e^x \cos y$$

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$16. u = e^x \sin y$$

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$17. u = \ln(x^2 + y^2)$$

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{2x}{x^2 + y^2} \right) = \frac{2}{x^2 + y^2} + \frac{-4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$18. u = \arctan(y/x)$$

$$\text{解} : \frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2} \quad \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2} \quad \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ 在一固定 } u \text{ 值令为 } u_0$$

則 $\tan u_0 = y/x$

$\therefore$ 在一固定 u_0 值時，平行於 xy 平面所截之圖形為一通過原點之直線，其斜率為 $\tan u_0$ 。

19. $u = \sin x \sin hy$.

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} (\cos x \sin hy) = -\sin x \sin hy$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} (\sin x \cos hy) = -\sin x \sin hy$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

20. $u = \sin x \cos hy$.

$$\text{解} : \frac{\partial^2 u}{\partial x^2} = -\sin x \cos hy \quad \frac{\partial^2 u}{\partial y^2} = -\sin x \cos hy$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

21. 證明函數 $u(x, y) = a \ln(x^2 + y^2) + b$ 滿足拉普拉斯方程式 (3)，並求出常數 a, b 使合於在圓周 $x^2 + y^2 = 1$ 上 $u = 0$ ，及在圓周 $x^2 + y^2 = 4$ 上 $u = 3$ 之邊界條件，並畫出 u 所代表之一種曲面圖形。

$$\text{解} : (a) \quad \frac{\partial^2 u}{\partial x^2} = \frac{a(2y^2 - 2x^2)}{(x^2 + y^2)^2} \quad \frac{\partial^2 u}{\partial y^2} = \frac{a(2x^2 - 2y^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$(b) \quad \text{若 } x^2 + y^2 = 1 \quad u = 0$$

$$\text{則 } a \ln 1 + b = 0 \Rightarrow b = 0$$

$$\text{若 } x^2 + y^2 = 4, \quad u = 3$$

$$\text{則 } a \ln 4 + b = 3 \quad \ln 4^a = 3$$

$$a = \frac{3}{2 \ln 2}, \text{ 即 } u = \frac{3}{2 \ln 2} \ln(x^2 + y^2)$$

22. 證明 $u = 1/\sqrt{x^2 + y^2 + z^2}$ 是拉普拉斯方程式 (5) 的解。

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} &= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial^2 u}{\partial y^2} &= \frac{2y^2 - z^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial^2 u}{\partial z^2} &= \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \\
 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 0
 \end{aligned}$$

與常微分方程式間之關係。

如果一個偏微分方程式僅含有其中某一自變數的導數，則我們可將其他自變數當作參數，而視為一常微分方程式求解，解下列方程式 $u = u(x, y)$ 。

23. $u_x = 0$

解： $u_x = 0 \quad \frac{\partial u(x, y)}{\partial x} = 0$

y 當作參數，則 $u_x = \frac{du(x, y)}{dx} = 0 \quad u(x, y) = f(y) + c.$

24. $u_y = 0$

解： $u_y = \frac{\partial u(x, y)}{\partial y} = 0 \Rightarrow \frac{du(x, y)}{dy} = 0$

$u(x, y) = f(x) + c$

25. $u_{xx} = 0$

解： $u_{xx} = \frac{\partial^2 u(x, y)}{\partial x^2} = 0 \Rightarrow \frac{d^2 u(x, y)}{dx^2} = 0$

$u(x, y) = x f(y) + g(y) + c$

26. $u_{xx} + u = 0$

解：令 $u(x, y) = c e^{mx} g(y)$ ， $u_{xx} = cm^2 e^{mx} g(y)$

$cm^2 e^{mx} g(y) + c e^{mx} g(y) = 0 \Rightarrow m^2 = -1 \quad m = \pm i$

$u(x, y) = c e^{ix} g(y) + c e^{-ix} g(y)$
 $= (A \sin x + B \cos x) g(y)$

27. 解下列偏微分方程。

(a) $u_{xx} = 0$ (b) $u_{yy} = 0$

解: $u_{xx} = 0 \Rightarrow u(x, y) = x f(y) + g(y) + bx$

$u_{yy} = 0 \Rightarrow u(x, y) = y F(x) + G(x) + cy$

$u_x = f(y) + b = y F_x(x) + G_x(x) \dots\dots\dots(1)$

$u_y = F(x) + c = x f_y(y) + g_y(y) \dots\dots\dots(2)$

由(1) $F_x(x) = a_1$ $F(x) = a_1 x + a_2$, $G_x(x) = b$ $G(x) = bx$

由(2) $f_y(y) = b_1$ $f(y) = b_1 y + b_2$, $g_y(y) = c$ $g(y) = cy$

$u(x, y) = x(b_1 y + b_2) + cy = y(a_1 x + a_2) + bx$

$a_1 = b_1 = a$ $b_2 = b$ $a_2 = c$

得 $u(x, y) = axy + bx + cy + d$

28. $u_x = 0$ $u_y = 0$

解: $u_x = 0$ $u(x, y) = f(y) + a_1$

$u_y = 0$ $u(x, y) = g(x) + a_2$

$f(y) + a_1 = g(x) + a_2$ $f(y) = c_1$ $g(x) = c_2$

$u(x, y) = c$

29. $u_{xy} = 0$ $u_{xx} = 0$ $u_{yy} = 0$

解: $u_{xx} = 0 \Rightarrow u(x, y) = x f(y) + g(y) + a_1 x$

$u_{yy} = 0 \Rightarrow u(x, y) = y F(x) + G(x) + b_1 y$

$u_{xy} = f_y(y) = F_x(x) = 0$ $f(y) = c_1$ $F(x) = c_2$

$u(x, y) = c_1 x + g(y) + a_1 x = c_2 y + G(x) + b_2 y$

得 $u(x, y) = ax + by + c$

30. $u_{xx} = 0$, $u_{xy} = 0$

解: $u_{xx} = 0 \Rightarrow u(x, y) = x f(y) + g(y) + a_1 x$

$u_{xy} = f_y(y) = 0$ $f(y) = c$

$u(x, y) = cx + g(y) + a_1 x$

即 $u(x, y) = ax + g(y) + b$

令 $u_x = P$ 解下列各偏微分方程。

31. $u_x y = u_x$

解: $u_x = p \Rightarrow p_y = p$ 得 $p = e^y f(x)$

$$u_x = f(x) e^y$$

$$u = \int f(x) e^y dx + g(y) = F(x) e^y + g(y) \quad F(x) = \int f(x) dx$$

$$32. u_{xy} + u_x = 0$$

$$\text{解: } \because u_x = p \quad p_y + p = 0 \quad \frac{dp}{dy} = -p$$

$$\text{得 } p = e^{-y} f(x)$$

$$u_x = f(x) e^{-y} \Rightarrow u(x, y) = \int f(x) e^{-y} dx + g(y) \\ = F(x) e^{-y} + g(y)$$

$$33. u_{xy} + u_x + x + y + 1 = 0$$

$$\text{解: } u_x = p \quad \therefore p_y + p + x + y + 1 = 0 \Rightarrow p = f(x) e^{-y} - x - y$$

$$\text{即 } u_x = f(x) e^{-y} - x - y$$

$$u = \int f(x) e^{-y} dx - \int x dx - \int y dx + c(y) \\ = B(x) e^{-y} - \frac{1}{2} x^2 - xy + c(y)$$

34. 證明在 $z = \text{const}$ 下, 若一平面 $z = z(x, y)$ 的水平曲線平行 x 軸, 則 z 是微分方程式 $z_x = 0$ 的解, 給予例題。

解: $\because z(x, y) = c$ 且 $z(x, y)$ 的水平曲線平行 x 軸

$$\therefore z(x, y) = f(y) = c$$

$$\text{即 } z_x = 0, \quad \text{例 } z = y$$

35. 證明方程式 $yz_x - xz_y = 0$ 的解 $z = z(x, y)$, 必代表旋轉曲面, 給予例題。提示: 令 $x = r \cos \theta$, $y = r \sin \theta$, 證明方程式 $z_\theta = 0$ 。

$$\text{解: 令 } x = r \cos \theta \quad y = r \sin \theta$$

$$z(x, y) = z(r \cos \theta, r \sin \theta) = z(r, \theta)$$

$$z_\theta = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = z_x (-r \sin \theta) + z_y (r \cos \theta)$$

$$= -yz_x + xz_y = -(yz_x - xz_y) = 0$$

$$\therefore z(r, \theta) = z(r) \text{ 與 } \theta \text{ 無關}$$

即 $z = z(r)$ 代表一繞 z 軸旋轉之旋轉面

$$\text{例 } z = x^2 + y^2 = r^2$$

11.3 習題

1. 振動繩索內基音之頻率與繩長、張力及每單位長之質量關係如何？

$$\text{解： } f_n = \frac{\lambda_n}{2\pi} = \frac{cn\pi}{2\pi\ell} = \frac{cn}{2\ell} \quad c = \sqrt{\frac{T}{\rho}}$$

$$\therefore f_n = \frac{n}{2\ell} \sqrt{\frac{T}{\rho}}$$

求振動繩索的位移函數 $u(x, t)$ (長度 $\ell = \pi$, 兩端固定, 且 $c^2 = T/\rho = 1$) 並設最初的速度爲零, 其位移如下。

2. $u(x, 0) = 0.01 \sin x$.

$$\text{解： } 0.01 \sin x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{\ell} x$$

$$B_n = \frac{2}{\ell} \int_0^{\ell} 0.01 \sin x \sin \frac{n\pi x}{\ell} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} 0.01 \sin x \sin nx dx$$

$n = 1$ 時 $B_n \neq 0$ 其均爲零。

$$B_1 = 0.01 \quad \therefore$$

$$\therefore \frac{\partial u}{\partial t} = 0 \quad \therefore B_n^* = 0$$

$$\text{得 } u(x, t) = 0.01 \cos \lambda_1 t \sin x$$

$$\lambda_n = \frac{c\pi n}{\ell} \quad \lambda_1 = \frac{\pi}{\pi} = 1$$

$$u(x, t) = 0.01 \cos t \sin x$$

3. $f(x) = k \sin 2x$.

$$\text{解： } B_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx.$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} k \sin 2x \sin nx dx$$

除 $n = 2$ 外 所有 $B_n = 0$

$$B_2 = \kappa \frac{2}{\pi} \int_0^{\pi} \sin^2 2x \, dx = \kappa$$

$$B_n^* \text{ 均爲零} \quad \lambda_n = cn\pi/\ell = cn\pi/\pi = n.$$

$$\begin{aligned} u(x, t) &= \sum B_n \cos \lambda_n t \sin \frac{n\pi x}{\ell} \\ &= \kappa \cos \lambda_2 t \sin 2x = \kappa \cos 2t \sin 2x \end{aligned}$$

4. $f(x) = \kappa (\sin x + \sin 3x)$

$$\text{解: } B_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} \kappa (\sin x + \sin 3x) \sin nx \, dx$$

除 $n = 1, n = 3$ 外 其他 $B_n = 0$

$$B_1 = \kappa \quad B_3 = \kappa$$

$$B_n^* \text{ 均爲零} \quad \lambda_n = cn\pi/\ell = cn\pi/\pi = n$$

$$\begin{aligned} u(x, t) &= \sum B_n \cos \lambda_n t \sin \frac{n\pi x}{\ell} \\ &= \kappa \cos \lambda_1 t \sin x + \kappa \cos \lambda_3 t \sin 3x \\ &= \kappa \cos t \sin x + \kappa \cos 3t \sin 3x \end{aligned}$$

$$5. \, u(x, 0) = f(x) = \begin{cases} \frac{\kappa x}{a} & 0 < x < a \\ \kappa \frac{(\pi - x)}{(\pi - a)} & a < x < \pi \end{cases}$$

$$\text{解: } B_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^a \frac{\kappa x}{a} \sin nx \, dx + \int_a^{\pi} \frac{\kappa}{a - \pi} (x - \pi) \sin nx \, dx$$

$$\begin{aligned}
 B_{n1} &= \frac{2}{\pi} \int_0^a \frac{\kappa x}{a} \sin nx \, dx \\
 &= \frac{2\kappa}{\pi a} \left[-\left(\frac{x}{n} (\cos nx)\right) \Big|_0^a + \frac{1}{n} \int_0^a \cos nx \, dx \right] \\
 &= \frac{2\kappa}{\pi a} \left[-\frac{a}{n} \cos na + \frac{1}{n^2} \sin na \right]
 \end{aligned}$$

$$\begin{aligned}
 B_{n2} &= \frac{2}{\pi} \int_a^\pi \kappa \left(\frac{x-\pi}{a-\pi} \right) \sin nx \, dx \\
 &= \frac{2\kappa}{\pi(a-\pi)} \int_a^\pi (x-\pi) \sin nx \, dx \\
 &= \frac{2\kappa}{\pi(a-\pi)} \left[(x-\pi) \left(-\frac{1}{n} \cos nx \right) \Big|_a^\pi + \frac{1}{n} \int_a^\pi \cos nx \, dx \right] \\
 &= \frac{2\kappa}{\pi(a-\pi)} \left[\frac{a-\pi}{n} \cos na - \frac{1}{n^2} \sin na \right]
 \end{aligned}$$

$$\begin{aligned}
 B_n = B_{n1} + B_{n2} &= \frac{2\kappa}{\pi} \left[-\frac{\cos na}{n} + \frac{\sin na}{an^2} + \frac{\cos na}{n} \right. \\
 &\quad \left. - \frac{\sin na}{n^2(a-\pi)} \right] = \frac{2\kappa}{a(\pi-a)} \frac{\sin na}{n^2}
 \end{aligned}$$

$$B_n^* = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin nx \quad \because \lambda_n = n$$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx.$$

$$6. \quad u(x, 0) = f(x) = \begin{cases} \frac{x}{10} & 0 < x < \frac{\pi}{4} \\ \frac{1}{20}(\pi - 2x) & \frac{\pi}{4} < x < \frac{3}{4}\pi \\ \frac{1}{10}(x - \pi) & \frac{3}{4}\pi < x < \pi \end{cases}$$

$$\begin{aligned} \text{解: } B_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \\ &= \frac{2}{\pi} \left[\int_0^{\pi/4} \frac{x}{10} \sin nx dx + \int_{\pi/4}^{3\pi/4} \frac{1}{20} (\pi - 2x) \sin nx dx \right. \\ &\quad \left. + \int_{3\pi/4}^{\pi} \frac{1}{10} (x - \pi) \sin nx dx \right] \end{aligned}$$

$$\begin{aligned} B_{n1} &= \frac{2}{\pi} \int_0^{\pi/4} \frac{x}{10} \sin nx dx \\ &= \frac{1}{5\pi} \left[-\frac{\pi}{4n} \cos \frac{n\pi}{4} + \frac{1}{n^2} \sin \frac{n\pi}{4} \right] \end{aligned}$$

$$\begin{aligned} B_{n2} &= \frac{2}{\pi} \int_{\pi/4}^{3\pi/4} \frac{1}{20} (\pi - 2x) \sin nx dx \\ &= \frac{1}{10\pi} \left[\frac{\pi}{2n} \cos \frac{3n\pi}{4} + \frac{\pi}{2n} \cos \frac{n\pi}{4} - \frac{2}{n^2} \sin \frac{3n\pi}{4} \right. \\ &\quad \left. + \frac{2}{n^2} \sin \frac{n\pi}{4} \right] \end{aligned}$$

$$\begin{aligned} B_{n3} &= \frac{2}{\pi} \int_{3\pi/4}^{\pi} \frac{1}{10} (x - \pi) \sin nx dx \\ &= \frac{1}{5\pi} \left[-\frac{\pi}{4n} \cos \frac{3n\pi}{4} - \frac{1}{n^2} \sin \frac{3n\pi}{4} \right] \end{aligned}$$

$$B_n = B_{n1} + B_{n2} + B_{n3} = \frac{-4}{5\pi} \frac{1}{n^2} \left[\cos \frac{n\pi}{2} \sin \frac{n\pi}{4} \right]$$

$$B_n^* = 0$$

$$\begin{aligned} \therefore u(x, t) &= \sum_{n=1}^{\infty} B_n \cos \lambda_n + \sin nx \\ &= \sum_{n=1}^{\infty} -\frac{4}{5\pi} \frac{1}{n^2} \left[\cos \frac{n\pi}{2} \sin \frac{n\pi}{4} \right] \cos nt \sin nx \end{aligned}$$

$$7. u(x, 0) = f(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{4} \\ \frac{4\kappa}{\pi} \left(x - \frac{\pi}{4}\right) & \frac{\pi}{4} < x < \frac{\pi}{2} \\ \frac{4\kappa}{\pi} \left(\frac{3\pi}{4} - x\right) & \frac{\pi}{2} < x < \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < x < \pi \end{cases}$$

$$\begin{aligned} \text{解: } B_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \\ &= \frac{2}{\pi} \left[\int_{\pi/4}^{\pi/2} \frac{4\kappa}{\pi} \left(x - \frac{\pi}{4}\right) \sin nx dx \right. \\ &\quad \left. + \int_{\pi/2}^{3\pi/4} \frac{4\kappa}{\pi} \left(\frac{3\pi}{4} - x\right) \sin nx dx \right] \\ B_{n1} &= \frac{2}{\pi} \int_{\pi/4}^{\pi/2} \frac{4\kappa}{\pi} \left(x - \frac{\pi}{4}\right) \sin nx dx \\ &= \frac{8\kappa}{\pi^2} \left[-\frac{\pi}{4n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{n\pi}{4} \right) \right] \\ B_{n2} &= \frac{2}{\pi} \int_{\pi/2}^{3\pi/4} \frac{4\kappa}{\pi} \left(\frac{3\pi}{4} - x\right) \sin nx dx \\ &= \frac{8\kappa}{\pi^2} \left[\frac{\pi}{4n} \cos \frac{n\pi}{2} - \frac{1}{n^2} \sin \frac{3}{4} n\pi + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \\ B_n &= B_{n1} + B_{n2} = \frac{8\kappa}{\pi^2} \left[\frac{2}{n^2} \sin \frac{n\pi}{2} - \frac{1}{n^2} \sin \frac{n\pi}{4} - \frac{1}{n^2} \sin \frac{3n\pi}{4} \right] \\ B_1 &= \frac{8\kappa}{\pi^2} \left[2 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right] = \frac{8\kappa}{\pi^2} (2 - \sqrt{2}) \\ B_2 &= \frac{8\kappa}{\pi^2} \left[-\frac{1}{4} + \frac{1}{4} \right] = 0 \\ B_3 &= -\frac{8\kappa}{\pi^2} \left(\frac{2}{9} - \frac{\sqrt{2}}{9} \right) \end{aligned}$$

$$B_4 = 0$$

$$\vdots$$

$$\begin{aligned}\therefore u(x, t) &= \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin nx \\ &= \frac{8\kappa}{\pi^2} [(2 - \sqrt{2}) \cos t \sin x \\ &\quad - \frac{1}{9} (2 + \sqrt{2}) \cos 3t \sin 3x + \cdots]\end{aligned}$$

$$8. u(x, 0) = f(x) = 0.01x(\pi - x)$$

$$\begin{aligned}\text{解: } B_n &= \frac{2}{\pi} \int_0^{\pi} 0.01x(\pi - x) \sin nx dx \\ &= \frac{0.02}{\pi} \left[-\frac{x(\pi - x)}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} (\pi - 2x) \cos nx dx \right] \\ &= \frac{0.02}{\pi} \cdot \frac{1}{n} \left[\frac{\pi - 2x}{n} \sin nx \Big|_0^{\pi} + \frac{2}{n} \int_0^{\pi} \sin nx dx \right] \\ &= \frac{-0.04}{n^3 \pi} [(-1)^n - 1]\end{aligned}$$

$$\begin{aligned}\therefore u(x, t) &= \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin nx \\ &= \frac{0.08}{\pi} \left[\cos t \sin x + \frac{1}{3^3} \cos 3t \sin 3x \right. \\ &\quad \left. + \frac{1}{5^3} \cos 5t \sin 5x + \cdots \right]\end{aligned}$$

$$9. u(x, 0) = f(x) = 0.01x(\pi^2 - x^2)$$

$$\begin{aligned}\text{解: } B_n &= \frac{2}{\pi} \int_0^{\pi} 0.01x(\pi^2 - x^2) \sin nx dx \\ &= \frac{2}{\pi} \times 0.01 \left[\int_0^{\pi} x(\pi^2 - x^2) \sin nx dx \right] \\ &= \frac{0.02}{\pi} \left[-\frac{x\pi^2 - x^3}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} (\pi^2 - 3x^2) \cos nx dx \right]\end{aligned}$$