Marcel Berger

A Panoramic View of Riemannian Geometry

黎曼几何概论

Springer

世界图书出版公司
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图书在版编目（CIP）数据

黎曼几何概论 = A Panoramic View of Riemannian Geometry：英文/（法）贝格（Berger，M.）著．—影印本．—北京：世界图书出版公司北京公司，2011

ISBN 978-7-5100-3306-3

Ⅰ．①黎…　Ⅱ．①贝…　Ⅲ．①黎曼几何—英文
Ⅳ．①O186.12

中国版本图书馆 CIP 数据核字（2011）第 029416 号

书　　名：	A Panoramic View of Riemannian Geometry
作　　者：	Marcel Berger
中 译 名：	黎曼几何概论
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24 开
印　　张：	35.5
版　　次：	2011 年 04 月
版权登记：	图字：01-2011-0419
书　　号：	978-7-5100-3306-3/O·883　　**定　　价：** 89.00 元

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Cataloging-in-Publication Data applied for
A catalog record for this book is available from the Library of Congress.
Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

Mathematics Subject Classification (2000): 53Axx, 53Bxx, 53Cxx, 37Cxx, 51Kxx

ISBN 3-540-65317-1 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York
a member of Springer Science+Business Media

http://www.springer.de

Heinz Götze gewidmet

This book is a tribute to the memory of Dr. Heinz Götze who dedicated his life to scientific publishing, in particular to mathematics. Mathematics publishing requires special effort and talent. Heinz Götze took up this challenge for almost 40 years with his characteristic energy and enthusiasm.

Preface

Riemannian geometry has become an important and vast subject. It deserves an encyclopedia, rather than a modest-length book. It is therefore impossible to present Riemannian geometry in a book in the standard fashion of mathematics, with complete definitions, proofs, and so on. This contrasts sharply with the situation in 1943, when Preissmann's dissertation 1943 [1041] presented all the global results of Riemannian geometry (but for the theory of symmetric spaces) including new ones, with proofs, in only forty pages.

Moreover, even at the root of the subject, the idea of a Riemannian manifold is subtle, appealing to unnatural concepts. Consequently, all recent books on Riemannian geometry, however good they may be, can only present two or three topics, having to spend quite a few pages on the foundations. Since our aim is to introduce the reader to most of the living topics of the field, we have had to follow the only possible path: to present the results without proofs.

We have two goals: first, to introduce the various concepts and tools of Riemannian geometry in the most natural way; or further, to demonstrate that one is practically forced to deal with abstract Riemannian manifolds in a host of intuitive geometrical questions. This explains why a long first chapter will deal with problems in the Euclidean plane. Second, once equipped with the concept of Riemannian manifold, we will present a panorama of current day Riemannian geometry. A panorama is never a full 360 degrees, so we will not try to be complete, but hope that our panorama will be large enough to show the reader a substantial part of today's Riemannian geometry.

In a panorama, you see the peaks, but you do not climb them. This is a way of saying that we will not prove the statements we quote. But, in a panorama, sometimes you can still see the path to a summit; analogously in many cases we will explain the main ideas or the main ingredients for the proof.

We hope that this form of presentation will leave many readers wanting to climb some peak. We will give all the needed references to the literature as the introduction and the panorama unfold. For alpinists, the equivalent of such a book will be the *refuge de haute-montagne* (the base camp) where you need to spend the night before the final climb. In the worst (we might say, the grandest) cases, like in the Himalayas, a climber has to establish as many as five base camps. The scientific analogue is that you need not only books, but also original articles.

Even without proofs or definitions, some of the peaks lie very far beyond. Distant topics will be mentioned only briefly in chapter 14. The judgement that a peak lies far away is personal; in the present case, we mean far from the author. His writing a book on Riemannian geometry does not indicate that he is an expert on every topic of it, especially the recent topics.

One may ask why we study only two objects: Euclidean domains with boundary, and Riemannian manifolds without boundary. There is a notion of Riemannian manifold with boundary, but in the Euclidean domain the interior geometry is given, flat and trivial, and the interesting phenomena come from the shape of the boundary. Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior. Asking for both at the same time risks having too much to handle (however see §§14.5.1).

The present text is an introduction, so we have to refrain from saying too much. For example, we will mainly consider compact Riemannian manifolds. But noncompact ones are also a very important subject; they are more challenging and more difficult to study.

We will conform to the following principles:

- This book is not a handbook of Riemannian geometry, nor a systematic awarding of prizes. We give only the best recent results, not all of the intermediate ones. However, we mention when the desired type of results started to appear, this being of historical interest and at the same time helping the reader to realize the difficulty of the problem. We hope that those whose results are not mentioned will pardon us.
- We present open problems as soon as they can be stated. This encourages the reader to appreciate the difficulty and the current state of each problem.

Since this text is unusual, it is natural to expect unusual features of presentation. First, references are especially important in a book about mathematical culture. But there should not be too many. Generally, we will only give a few of the recent references. From these, the interested reader will be able to trace back to most of the standard sources. When we are considering very basic notions (like that of manifold or billiard) we will typically give many references. The reader might prefer to work with one more than another. Second, since we will not give formal definitions in the text, we thought the reader might find it useful to have most of them collected in the final chapter.

Some words about organization: first, the immensity of the field poses a problem of classification; in our division into chapters, necessarily arbitrary, we did not follow any logical or historical order. We have tried to follow a certain naturalness and simplicity. This explains why many recent discoveries, like those concerning the isoperimetric profile, the systolic inequalities, the spectrum, the geodesic flow and periodic geodesics come before a host of discoveries relating the topology of the underlying manifolds with various assumptions on curvature, although the latter results came to light much earlier than the former.

Second, our treatment of topics is certainly uneven, but this reflects the tastes and knowledge of the author. Disparities appear in the choice of results presented and in what we will offer as ideas behind the proofs. We apologize for that. For example, everything concerning bundles over Riemannian manifolds, especially spin bundles and spin geometry, will be very sketchy.

We hope that despite these weaknesses, the present book will bring pleasure and be of help to professional Riemannian geometers as well as those who want to enter into the realm of Riemannian geometry, which is an amazingly beautiful, active and natural field of research today. The reader who finds this book worthwhile will be interested in reading Dillen & Verstraelen 2000 [449].

Acknowledgements

I was able to write this book with enthusiasm thanks to the Università di Roma "La Sapienza", the Indian Institute of Technology at Powai–Bombay, the University of Pennsylvania and the Zürich Polytechnicum, where I was invited to give lectures, in Rome in 1992, in Bombay in 1993, in Philadelphia in 1994, and in Zürich in the winter semester of 1995–1996. These four departments permitted me to give lectures entitled "Topics in Riemannian geometry" where I covered a lot but with almost no proofs, only sketching ideas and ingredients. I want to thank them for having allowed me to give lectures which were not set in a classical frame. Many thanks also to the people who greatly helped me to write the surveys Berger 1998 [171], 2000 [172], 2003 [173]. They are too many to be thanked individually (their names are listed in Berger 1998 [171]), although I make an exception for Shanta Shrinivasan who wrote a first draft of my Bombay lectures.

I am deeply indebted to Dr. Benjamin McKay for taking on and carrying out the difficult task of language editing and typesetting the manuscript, inserting all the figures as he did so. Finally my special thanks go to Springer's mathematical editorial for agreeing to embark on this extraordinary project. Personal thanks go to my old friends Dr. Joachim Heinze and Dr. Catriona Byrne. Finally Mss. Susanne Denskus and Leonie Kunz had the hard time to completely put the manuscript in its final form.

Conventions of This Book

- We will generally assume (with the notable exception of all of chapter 1) that all manifolds are **compact and without boundary**.
- Einstein summation convention will be used.
- The lemmas, theorems, propositions, questions, corollaries, etc. are all numbered with a single sequence, to make it easier to find them; most mathematics books use separate sequences for the lemmas, theorems, etc.

Bures-Sur-Yvette, France,
June 2003 *Marcel Berger*

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