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Christian Berg
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Harmonic Analysis on Semigroups

Theory of Positive Definite and Rela[illegible]ons

半群上的调和分析

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Preface

The Fourier transform and the Laplace transform of a positive measure share, together with its moment sequence, a positive definiteness property which under certain regularity assumptions is characteristic for such expressions. This is formulated in exact terms in the famous theorems of Bochner, Bernstein–Widder and Hamburger. All three theorems can be viewed as special cases of a general theorem about functions φ on abelian semigroups with involution $(S, +, *)$ which are positive definite in the sense that the matrix $(\varphi(s_j^* + s_k))$ is positive definite for all finite choices of elements $s_1, \ldots, s_n$ from S. The three basic results mentioned above correspond to $(\mathbb{R}, +, x^* = -x)$, $([0, \infty[, +, x^* = x)$ and $(\mathbb{N}_0, +, n^* = n)$.

The purpose of this book is to provide a treatment of these positive definite functions on abelian semigroups with involution. In doing so we also discuss related topics such as negative definite functions, completely monotone functions and Hoeffding-type inequalities. We view these subjects as important ingredients of harmonic analysis on semigroups. It has been our aim, simultaneously, to write a book which can serve as a textbook for an advanced graduate course, because we feel that the notion of positive definiteness is an important and basic notion which occurs in mathematics as often as the notion of a Hilbert space. The already mentioned Laplace and Fourier transformations, as well as the generating functions for integer-valued random variables, belong to the most important analytical tools in probability theory and its applications. Only recently it turned out that positive (resp. negative) definite functions allow a probabilistic characterization in terms of so-called Hoeffding-type inequalities.

As prerequisites for the reading of this book we assume the reader to be familiar with the fundamental principles of algebra, analysis and probability, including the basic notions from vector spaces, general topology and abstract measure theory and integration. On this basis we have included Chapter 1 about locally convex topological vector spaces with the main objective of proving the Hahn–Banach theorem in different versions which will be used later, in particular, in proving the Krein–Milman theorem. We also present a short introduction to the idea of integral representations in compact convex sets, mainly without proofs because the only version of Choquet's theorem which we use later is derived directly from the Krein–Milman theorem. For later use, however, we need an integration theory for measures on Hausdorff spaces, which are not necessarily locally compact. Chapter 2 contains a treatment of Radon measures, which are inner regular with respect to the family of compact sets on which they are assumed finite. The existence of Radon product measures is based on a general theorem about Radon bimeasures on a product of two Hausdorff spaces being induced by a Radon measure on the product space. Topics like the Riesz representation theorem, adapted spaces, and weak and vague convergence of measures are likewise treated.

Many results on positive and negative definite functions are not really dependent on the semigroup structure and are, in fact, true for general positive and negative definite matrices and kernels, and such results are placed in Chapter 3.

Chapters 4–8 contain the harmonic analysis on semigroups as well as a study of many concrete examples of semigroups. We will not go into detail with the content here but refer to the Contents for a quick survey. Much work is centered around the representation of positive definite functions on an abelian semigroup $(S, +, *)$ with involution as an integral of semi-characters with respect to a positive measure. It should be emphasized that most of the theory is developed without topology on the semigroup S. The reason for this is simply that a satisfactory general representation theorem for continuous positive definite functions on topological semigroups does not seem to be known. There is, of course, the classical theory of harmonic analysis on locally compact abelian groups, but we have decided not to include this in the exposition in order to keep it within reasonable bounds and because it can be found in many books.

As described we have tried to make the book essentially self-contained. However, we have broken this principle in a few places in order to obtain special results, but have never done it if the results were essential for later development. Most of the exercises should be easy to solve, a few are more involved and sometimes require consultations in the literature referred to. At the end of each chapter is a section called Notes and Remarks. Our aim has not been to write an encyclopedia but we hope that the historical comments are fair.

Within each chapter sections, propositions, lemmas, definitions, etc. are numbered consecutively as 1.1, 1.2, 1.3, . . . in §1, as 2.1, 2.2, 2.3, . . . in §2, and so on. When making a reference to another chapter we always add the number of that chapter, e.g. 3.1.1.

We have been fascinated by the present subject since our 1976 paper and have lectured on it on various occasions. Research projects in connection with the material presented have been supported by the Danish Natural Science Research Council, die Thyssen Stiftung, den Deutschen Akademischen Austauschdienst, det Danske Undervisningsministerium, as well as our home universities. Thanks are due to Flemming Topsøe for his advice on Chapter 2. We had the good fortune to have Bettina Mann type the manuscript and thank her for the superb typing.

March 1984

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Contents

CHAPTER 1

Introduction to Locally Convex Topological Vector Spaces and Dual Pairs

§1. Locally Convex Vector Spaces

The purpose of this chapter is to provide a quick introduction to some of the basic aspects of the theory of topological vector spaces. Various versions of the Hahn–Banach theorem will be used later in the book and the exposition therefore centers around a fairly detailed treatment of these fundamental results. Other parts of the theory are only sketched, and we suggest that the reader consult one of the many books on the subject for further information, see e.g. Robertson and Robertson (1964), Rudin (1973) and Schaefer (1971).

1.1. We assume that the reader is familiar with the concept of a *vector space* E over a field $\mathbb{K}$, which is always either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and of a *topology* $\mathcal{O}$ on a set X, where $\mathcal{O}$ means the system of open subsets of X.

Generally speaking, whenever a set is equipped with both an algebraic and a topological structure, we will require that the structures match in the sense that the algebraic operations become continuous mappings.

To be precise, a vector space E equipped with a topology $\mathcal{O}$ is called a *topological vector space* if the mappings $(x, y) \mapsto x + y$ of $E \times E$ into E and $(\lambda, x) \mapsto \lambda x$ of $\mathbb{K} \times E$ into E are continuous. Here it is tacitly assumed that $E \times E$ and $\mathbb{K} \times E$ are equipped with the product topology and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with its usual topology. A topological vector space E is, in particular, a *topological group* in the sense that the mappings $(x, y) \mapsto x + y$ of $E \times E$ into E and $x \mapsto -x$ of E into E are continuous.

For each $u \in E$ the translation $\tau_u: x \mapsto x + u$ is a homeomorphism of E, so if $\mathcal{B}$ is a base for the filter $\mathcal{U}$ of neighbourhoods of zero, then $u + \mathcal{B}$ is a base for the filter of neighbourhoods of u. Therefore the whole topological structure of E is determined by a base of neighbourhoods of the origin.

A subset A of a vector space E is called *absorbing* if for each $x \in E$ there exists some $M > 0$ such that $x \in \lambda A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \geqq M$; and it is called *balanced*, if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leqq 1$. Finally, A is called *absolutely convex*, if it is convex and balanced.

1.2. Proposition. *Let E be a topological vector space and let $\mathscr{U}$ be the filter of neighbourhoods of zero. Then*:

(i) *every $U \in \mathscr{U}$ is absorbing;*
(ii) *for every $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ with $V + V \subseteq U$;*
(iii) *for every $U \in \mathscr{U}$, $b(U) = \bigcap_{|\mu| \geqq 1} \mu U$ is a balanced neighbourhood of zero contained in U.*

Proof. For $a \in E$ the mapping $\lambda \mapsto \lambda a$ of $\mathbb{K}$ into E is continuous at $\lambda = 0$ and this implies (i). Similarly the continuity at $(0, 0)$ of the mapping $(x, y) \mapsto x + y$ implies (ii). Finally, by the continuity of the mapping $(\lambda, x) \mapsto \lambda x$ at $(0, 0) \in \mathbb{K} \times E$ we can associate with a given $U \in \mathscr{U}$ a number $\varepsilon > 0$ and $V \in \mathscr{U}$ such that $\lambda V \subseteq U$ for $|\lambda| \leqq \varepsilon$. Therefore

$$\varepsilon V \subseteq b(U) \subseteq U$$

so U contains the balanced set $b(U)$ which is a neighbourhood of zero because εV is so, $x \mapsto \varepsilon x$ being a homeomorphism of E. □

From Proposition 1.2 it follows that in every topological vector space the filter $\mathscr{U}$ has a base of balanced neighbourhoods.

A topological vector space need not have a base for $\mathscr{U}$ consisting of convex sets, but the spaces we will discuss always have such a base.

1.3. Definition. A topological vector space E over $\mathbb{K}$ is called *locally convex* if the filter of neighbourhoods of zero has a base of convex neighbourhoods.

1.4. Proposition. *In a locally convex topological vector space E the filter of neighbourhoods of zero has a base $\mathscr{B}$ with the following properties*:

(i) *Every $U \in \mathscr{B}$ is absorbing and absolutely convex.*
(ii) *If $U \in \mathscr{B}$ and $\lambda \neq 0$, then $\lambda U \in \mathscr{B}$.*

Conversely, given a base $\mathscr{B}$ for a filter on E with the properties (i) *and* (ii), *there is a unique topology on E such that E is a (locally convex) topological vector space with $\mathscr{B}$ as a base for the filter of neighbourhoods of zero.*

Proof. If U is a convex neighbourhood of zero then $b(U)$ is absolutely convex. If $\mathscr{B}_0$ is a base of convex neighbourhoods, then the family $\mathscr{B} = \{\lambda b(U) \mid U \in \mathscr{B}_0, \lambda \neq 0\}$ is a base satisfying (i) and (ii).

Conversely, suppose that $\mathscr{B}$ is a base for a filter $\mathscr{F}$ on E and satisfies (i) and (ii). Then every set $U \in \mathscr{F}$ contains zero. The only possible topology on E which makes E to a topological vector space, and which has $\mathscr{F}$ as the filter of neighbourhoods of zero, has the filter $a + \mathscr{F}$ as filter of neighbourhoods of $a \in E$. Calling a nonempty subset $G \subseteq E$ "open" if for every $a \in G$ there exists $U \in \mathscr{B}$ such that $a + U \subseteq G$, it is easy to see that these "open" sets form a topology with $a + \mathscr{F}$ as the filter of neighbourhoods of a, and that E is a topological vector space. □

In applications of the theory of locally convex vector spaces the topology on a given vector space E is often defined by a family of seminorms.

1.5. Definition. A function $p: E \to [0, \infty[$ is called a *seminorm* if it has the following properties:

(i) *homogeneity*: $p(\lambda x) = |\lambda| p(x)$ for $\lambda \in \mathbb{K}$, $x \in E$;
(ii) *subadditivity*: $p(x + y) \leqq p(x) + p(y)$ for $x, y \in E$.

If, in addition, $p^{-1}(\{0\}) = \{0\}$, then p is called a *norm*.

If p is a seminorm and $\alpha > 0$ then the sets $\{x \in E \mid p(x) < \alpha\}$ are absolutely convex and absorbing.

For a nonempty set $A \subseteq E$, we define a mapping $p_A: E \to [0, \infty]$ by

$$p_A(x) = \inf\{\lambda > 0 \mid x \in \lambda A\}$$

(where $p_A(x) = \infty$, if the set in question is empty).

The following lemma is easy to prove.

1.6. Lemma. *If* $A \subseteq E$ *is*

(i) *absorbing, then* $p_A(x) < \infty$ *for* $x \in E$;
(ii) *convex, then* p_A *is subadditive*;
(iii) *balanced, then* p_A *is homogeneous, and*

$$\{x \in E \mid p_A(x) < 1\} \subseteq A \subseteq \{x \in E \mid p_A(x) \leqq 1\}.$$

If A satisfies (i)–(iii) then p_A is called the *seminorm determined by* A.

A seminorm p satisfies $|p(x) - p(y)| \leqq p(x - y)$. In particular, if E is a topological vector space then p is continuous if and only if it is continuous at 0 and this is equivalent with $\{x \mid p(x) < \alpha\}$ being a neighbourhood of zero for one (and hence for all) $\alpha > 0$.

We will now see how a family $(p_i)_{i \in I}$ of seminorms on a vector space E induces a topology on E.

1.7. Proposition. *There exists a coarsest topology on E with the properties that E is a topological vector space and each* p_i *is continuous. Under this topology E is locally convex and the family of sets*

$$\{x \in E \mid p_{i_1}(x) < \varepsilon, \ldots, p_{i_n}(x) < \varepsilon\}, \qquad i_1, \ldots, i_n \in I, \quad n \in \mathbb{N}, \quad \varepsilon > 0,$$

is a base for the filter of neighbourhoods of zero.

PROOF. Let $\mathscr{B}$ denote the above family of sets. Then $\mathscr{B}$ is a base for a filter on E having the properties (i) and (ii) of Proposition 1.4, and the unique topology asserted there is the coarsest topology on E making E to a topological vector space in which each p_i is continuous. □

The above topology is called the *topology induced by the family* $(p_i)_{i \in I}$ *of seminorms.*

Note that in this topology a net (x_α) from E converges to x if and only if $\lim_\alpha p_i(x - x_\alpha) = 0$ for all $i \in I$.

The topology of an arbitrary locally convex topological vector space E is always induced by a family of seminorms, e.g. by the family of all continuous seminorms as is easily seen by 1.4 and 1.6.

1.8. Proposition. *Let E be a locally convex topological vector space, where the topology is induced by a family* $(p_i)_{i \in I}$ *of seminorms. Then E is a Hausdorff space if and only if for every* $x \in E\setminus\{0\}$ *there exists* $i \in I$ *such that* $p_i(x) \neq 0$.

PROOF. Suppose $x \neq y$ and that $(p_i)_{i \in I}$ has the above separation property. Then there exist $i \in I$ and $\varepsilon > 0$ such that $p_i(x - y) = 2\varepsilon$. The sets

$$\{u \mid p_i(x - u) < \varepsilon\}, \{u \mid p_i(y - u) < \varepsilon\}$$

are open disjoint neighbourhoods of x and y.

For the converse we prove the apparently stronger statement that the separation property of $(p_i)_{i \in I}$ is a consequence of E being a T_1-space (i.e. the one point sets are closed). In fact, if $x \neq 0$ and $\{x\}$ is closed there exists a neighbourhood U of zero such that $x \notin U$. By Proposition 1.7 there exist $\varepsilon > 0$ and finitely many indices $i_1, \ldots, i_n \in I$ such that

$$\{y \mid p_{i_1}(y) < \varepsilon, \ldots, p_{i_n}(y) < \varepsilon\} \subseteq U,$$

so for some $i \in \{i_1, \ldots, i_n\}$ we have $p_i(x) \geqq \varepsilon$. □

1.9. Finest Locally Convex Topology. Let E be a vector space over $\mathbb{K}$. Among the topologies on E, which make E into a locally convex topological vector space, there is a finest one, namely the topology induced by the family of all seminorms on E. This topology is called the *finest locally convex topology* on E. An alternative way of describing this topology is by saying that the system of all absorbing absolutely convex sets is a base for the filter of neighbourhoods of zero, cf. 1.4.

The finest locally convex topology is Hausdorff. In fact, let $e \in E\setminus\{0\}$ be given. We choose an algebraic basis for E containing e and let φ be the linear functional determined by $\varphi(e) = 1$ and φ being zero on the other vectors of the basis. Then $p = |\varphi|$ is a seminorm with $p(e) = 1$, and the result follows from 1.8.

Notice that every linear functional is continuous in the finest locally convex topology.

In Chapter 6 the finest locally convex topology will be used on the vector space of polynomials in one or more variables.

1.10. Exercise. Let E be a topological vector space, and let $A, B, C, F \subseteq E$.

(a) Show that $A + B$ is open in E if A is open and B is arbitrary.
(b) Show that $F + C$ is closed in E if F is closed and C is compact.

1.11. Exercise. Let E be a topological vector space. Show that the interior of a convex set is convex. Show that if U is an absolutely convex neighbourhood of 0 in E then its interior is absolutely convex. It follows that a locally convex topological vector space has a base for the filter of neighbourhoods of 0 consisting of open absolutely convex sets.

1.12. Exercise. Show that a Hausdorff topological vector space is a regular topological space. (It is actually completely regular, but that is more difficult to prove.)

1.13. Exercise. Let E be a topological vector space and $A \subseteq E$ a nonempty and balanced subset. Then:

(i) if A is open, $A = \{x \in E \mid p_A(x) < 1\}$;
(ii) if A is closed, $A = \{x \in E \mid p_A(x) \leqq 1\}$.

1.14. Exercise. Let p, q be two seminorms on a vector space E. Then if $\{x \in E \mid p(x) \leqq 1\} = \{x \in E \mid q(x) \leqq 1\}$ it follows that $p = q$.

1.15. Exercise. Let the topology of the locally convex vector space E be induced by the family $(p_i)_{i \in I}$ of seminorms, and let f be a linear functional on E. Then f is continuous if and only if there exist $c \in \,]0, \infty[$ and some finite subset $J \subseteq I$ such that $|f(x)| \leqq c \cdot \max\{p_i(x) \mid i \in J\}$ for all $x \in E$.

§2. Hahn–Banach Theorems

One main result in the theory of locally convex topological vector spaces is the Hahn–Banach theorem about extensions of linear functionals. In the following we treat this and closely related results under the name of Hahn–Banach theorems.

We recall that a *hyperplane* H in a vector space E over $\mathbb{K}$ is a maximal proper linear subspace of E or, equivalently, a linear subspace of codimension one (i.e. $\dim E/H = 1$). Another equivalent formulation is that a hyperplane is a set of the form $\varphi^{-1}(\{0\})$ for a linear functional $\varphi: E \to \mathbb{K}$ not identically zero.

Neither local convexity nor the Hausdorff separation property is needed in our first version of the Hahn–Banach theorem. However the existence of a nonempty open convex set $A \neq E$ is a strong implicit assumption on E.

2.1. Theorem (Geometric Version). *Let E be a topological vector space over $\mathbb{K}$ and let A be a nonempty open convex subset of E. If M is a linear subspace of E with $A \cap M = \emptyset$, there exists a closed hyperplane H containing M with $A \cap H = \emptyset$.*

PROOF. We first consider the case $\mathbb{K} = \mathbb{R}$. By Zorn's lemma there exists a maximal linear subspace H of E such that $M \subseteq H$ and $A \cap H = \emptyset$. Let $C = H + \bigcup_{\lambda > 0} \lambda A$.

The sum of an open set and an arbitrary set is open, hence C is open, cf. Exercise 1.10. We now derive four properties of C and H by contradiction:

(a) $C \cap (-C) = \emptyset$.

In fact, if we assume $x \in C \cap (-C)$, we have $x = h_1 + \lambda_1 a_1 = h_2 - \lambda_2 a_2$ with $h_i \in H$, $a_i \in A$, $\lambda_i > 0$, $i = 1, 2$. By the convexity of A

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} a_2 = \frac{1}{\lambda_1 + \lambda_2} (h_2 - h_1) \in A \cap H$$

which is impossible.

(b) $H \cup C \cup (-C) = E$.

In fact, if there exists $x \in E \backslash (H \cup C \cup (-C))$ we define $\tilde{H} = H + \mathbb{R}x$, so H is a proper subspace of $\tilde{H}$. Furthermore $A \cap \tilde{H} = \emptyset$ because $y \in A \cap \tilde{H}$ can be written $y = h + \lambda x$ with $h \in H$ and $\lambda \neq 0$ ($A \cap H = \emptyset$), and then $x = (1/\lambda)y - (1/\lambda)h \in C \cup (-C)$, which is incompatible with the choice of x. Finally the existence of $\tilde{H}$ is inconsistent with the maximality of H so (b) holds.

(c) $H \cap (C \cup (-C)) = \emptyset$.

In fact, if $x \in H \cap C$ then $x = h + \lambda a$ with $h \in H$, $a \in A$ and $\lambda > 0$, but then $a = (1/\lambda)(x - h) \in A \cap H$, which is a contradiction.

From (b) and (c) follows that H is the complement of the open set $C \cup (-C)$, hence closed.

(d) H is a hyperplane.

If H is not a hyperplane there exists $x \in E \backslash H$ such that $\tilde{H} = H + \mathbb{R}x \neq E$. Without loss of generality we may assume $x \in C$ and we can choose $y \in (-C) \backslash \tilde{H}$. The function $f: [0, 1] \to E$ defined by $f(\lambda) = (1 - \lambda)x + \lambda y$ is continuous, so $f^{-1}(C)$ and $f^{-1}(-C)$ are disjoint open subsets of $[0, 1]$ containing, respectively, 0 and 1. Since $[0, 1]$ is connected there exists $\alpha \in]0, 1[$ such that $f(\alpha) \in H$. But this implies $y = (1/\alpha)(f(\alpha) - (1 - \alpha)x) \in \tilde{H}$, which is a contradiction.

This finishes the proof of the real case.

A complex vector space can be considered as a real vector space, and if H denotes a real closed hyperplane containing M and such that $A \cap H = \emptyset$, then $H \cap (iH)$ is a complex hyperplane with the desired properties. □

The following important criterion for continuity of a linear functional will be used several times.

2.2. Proposition. *Let E be a topological vector space over $\mathbb{K}$, let $\varphi: E \to \mathbb{K}$ be a nonzero linear functional and let $H = \varphi^{-1}(\{0\})$ be the corresponding hyperplane. Then precisely one of the following two statements is true:*

(i) *φ is continuous and H is closed;*
(ii) *φ is discontinuous and H is dense.*

PROOF. The closure $\bar{H}$ is a linear subspace of E. By the maximality of H we therefore have either $H = \bar{H}$ or $\bar{H} = E$. If φ is continuous then $H = \varphi^{-1}(\{0\})$ is closed. Suppose next that H is closed. Let $a \in E \backslash H$ be chosen such that $\varphi(a) = 1$. By Proposition 1.2 there exists a balanced neighbourhood V of zero such that $(a + V) \cap H = \emptyset$, and therefore $\varphi(V)$ is a balanced subset of $\mathbb{K}$ such that $0 \notin 1 + \varphi(V)$, hence $\varphi(V) \subseteq \{x \in \mathbb{K} \mid |x| < 1\}$. It follows that $|\varphi(x)| < \varepsilon$ for all $x \in \varepsilon V$, $\varepsilon > 0$, so φ is continuous at zero, and hence continuous. □

2.3. Theorem of Separation. *Let E be a locally convex topological vector space over $\mathbb{K}$. Suppose F and C are disjoint nonempty convex subsets of E such that F is closed and C is compact. Then there exists a continuous linear functional $\varphi: E \to \mathbb{K}$ such that*

$$\sup_{x \in C} \operatorname{Re} \varphi(x) < \inf_{x \in F} \operatorname{Re} \varphi(x).$$

PROOF. Let us first suppose $\mathbb{K} = \mathbb{R}$, and consider the set $B = F - C$. Obviously B is convex, and using the compactness of C it may be seen that B is closed, cf. Exercise 1.10. Since $F \cap C = \emptyset$ we have $0 \notin B$, so by 1.4 there exists an absolutely convex neighbourhood U of 0 such that $U \cap B = \emptyset$. The interior V of U is an open absolutely convex neighbourhood (cf. Exercise 1.11) so $A = B + V = B - V$ is a nonempty open convex set (1.10) such that $0 \notin A$. Since $\{0\}$ is a linear subspace not intersecting A, there exists by Theorem 2.1 a closed hyperplane H with $A \cap H = \emptyset$. Let φ be a linear functional on E with $H = \varphi^{-1}(\{0\})$. By 2.2, φ is continuous. Now $\varphi(A)$ is a convex subset of $\mathbb{R}$, hence an interval, and since $0 \notin \varphi(A)$ we may assume $\varphi(A) \subseteq]0, \infty[$. (If this is not the case we replace φ by $-\varphi$). We next claim

$$\inf_{x \in B} \varphi(x) > 0,$$

which is equivalent to the assertion. If the contrary was true there exists a sequence (x_n) from B such that $\varphi(x_n) \to 0$. Since V is absorbing there exists $u \in V$ with $\varphi(u) < 0$, but $x_n + u \in A$ so that $\varphi(x_n) + \varphi(u) > 0$ for all n, which is in contradiction with $\varphi(x_n) \to 0$.

In the case $\mathbb{K} = \mathbb{C}$ we consider E as a real vector space and find a $\mathbb{R}$-linear functional $\varphi: E \to \mathbb{R}$ as above. To finish the proof we notice that there exists precisely one $\mathbb{C}$-linear functional $\psi: E \to \mathbb{C}$ with $\operatorname{Re} \psi = \varphi$ namely $\psi(x) = \varphi(x) - i\varphi(ix)$, which is continuous since φ is so. □

Applying the theorem to two one-point sets we find

2.4. Corollary. *Let E be a locally convex Hausdorff topological vector space. For $a, b \in E$, $a \neq b$, there exists a continuous linear functional f on E such that $f(a) \neq f(b)$.*

We shall now treat the versions of the Hahn–Banach theorem which are called extension theorems. Although they may be derived from the geometric version, we give a direct proof using Zorn's lemma.

The first extension theorem is purely algebraic and very useful in the theory of integral representations. It uses the following weakened form of the concept of a seminorm.

2.5. Definition. Let E be a vector space. A function $p: E \to \mathbb{R}$ is called *sublinear* if it has the following properties:

(i) *positive homogeneity*: $p(\lambda x) = \lambda p(x)$ for $\lambda \geqq 0$, $x \in E$;
(ii) *subadditivity*: $p(x + y) \leqq p(x) + p(y)$ for $x, y \in E$.

A function $f: E \to \mathbb{R}$ is called *dominated* by p if $f(x) \leqq p(x)$ for all $x \in E$.

2.6. Theorem (Extension Version). *Let M be a linear subspace of a real vector space E and let $p: E \to \mathbb{R}$ be a sublinear function. If $f: M \to \mathbb{R}$ is linear and dominated by p on M, there exists a linear extension $\tilde{f}: E \to \mathbb{R}$ of f, which is dominated by p.*

Proof. We first show that it is always possible to perform one-dimensional extensions assuming $M \neq E$.

Let $e \in E \backslash M$ and define $M' = \operatorname{span}(M \cup \{e\})$. Every element $x' \in M'$ has a unique representation as $x' = x + te$ with $x \in M$, $t \in \mathbb{R}$. For every $\alpha \in \mathbb{R}$ the functional $f'_\alpha: M' \to \mathbb{R}$ defined by $f'_\alpha(x + te) = f(x) + t\alpha$ is a linear extension of f. We shall see that α may be chosen such that f'_α is dominated by p.

By the subadditivity of p we get for all $x, y \in M$

$$f(x) + f(y) = f(x + y) \leqq p(x + y) \leqq p(x - e) + p(e + y),$$

or

$$f(x) - p(x - e) \leqq p(e + y) - f(y).$$

Defining

$$k = \sup\{f(x) - p(x - e) \mid x \in M\},$$
$$K = \inf\{p(e + y) - f(y) \mid y \in M\},$$

we have

$$-\infty < k \leqq K < \infty.$$