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PERCOLATION

SECOND EDITION

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Geoffrey Grimmett

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Second Edition
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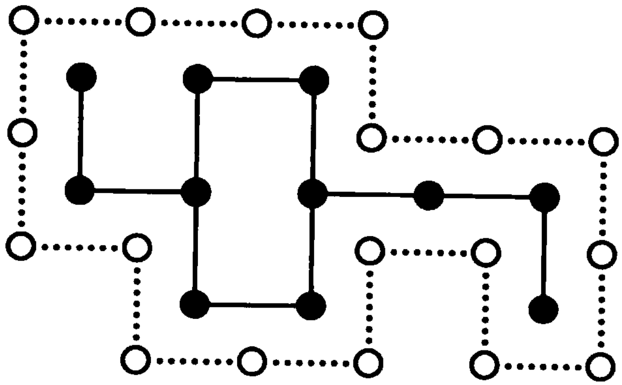


Preface to the Second Edition

Quite apart from the fact that percolation theory has its origin in an honest applied problem, it is a source of fascinating problems of the best kind for which a mathematician can wish: problems which are easy to state with a minimum of preparation, but whose solutions are apparently difficult and require new methods. At the same time, many of the problems are of interest to or proposed by statistical physicists and not dreamed up merely to demonstrate ingenuity.

As a mathematical subject, percolation is a child of the 1950s. Following the presentation by Hammersley and Morton (1954) of a paper on Monte Carlo methods to the Royal Statistical Society, Simon Broadbent contributed the following to the discussion:

> "Another problem of excluded volume, that of the random maze, may be defined as follows: A square (in two dimensions) or cubic (in three) lattice consists of "cells" at the interstices joined by "paths" which are either open or closed, the probability that a randomly-chosen path is open being p. A "liquid" which cannot flow upwards or a "gas" which flows in all directions penetrates the open paths and fills a proportion $\lambda_r(p)$ of the cells at the rth level. The problem is to determine $\lambda_r(p)$ for a large lattice. Clearly it is a non-decreasing function of p and takes the values 0 at $p = 0$ and 1 at $p = 1$. Its value in the two-dimension case is not greater than in three dimensions.
>
> It appears likely from the solution of a simplified version of the problem that as $r \to \infty$ $\lambda_r(p)$ tends strictly monotonically to $\Lambda(p)$, a unique and stable proportion of cells occupied, independent of the way the liquid or gas is introduced into the first level. No analytical solution for a general case seems to be known."

This discussion led to a fruitful partnership between Broadbent and Hammersley, and resulted in their famous paper of 1957. The subsequent publications of Hammersley initiated the mathematical study of the subject.

Much progress has been made since, and many of the open problems of the last decades have been solved. With such solutions we have seen the evolution of new techniques and questions, and the consequent knowledge has shifted the ground under percolation. The mathematics of percolation is now fairly mature, although there are major questions which remain largely unanswered. Percolation technology has emerged as a cornerstone of the theory of disordered physical systems, and the methods of this book are now being applied and extended in a variety of important settings.

The quantity of literature related to percolation seems to grow hour by hour, mostly in the physics journals. It has become difficult to get to know the subject from scratch, and one of the principal purposes of this book is to remedy this. Percolation has developed a reputation for being hard as well as important. Nevertheless, it may be interesting to note that the level of mathematical preparation required to read this book is limited to some elementary probability theory and real analysis at the undergraduate level. Readers knowing a little advanced probability theory, ergodic theory, graph theory, or mathematical physics will not be disadvantaged, but neither will their knowledge aid directly their understanding of most of the hard steps.

This book is about the mathematics of percolation theory, with the emphasis upon presenting the shortest rigorous proofs of the main facts. I have made certain sacrifices in order to maximize the accessibility of the theory, and the major one has been to restrict myself almost entirely to the special case of bond percolation on the cubic lattice $\mathbb{Z}^d$. Thus there is only little discussion of such processes as continuum, mixed, inhomogeneous, long-range, first-passage, and oriented percolation. Nor have I spent much time or space on the relationship of percolation to statistical physics, infinite particle systems, disordered media, reliability theory, and so on. With the exception of the two final chapters, I have tried to stay reasonably close to core material of the sort which most graduate students in the area might aspire to know. No critical reader will agree entirely with my selection, and physicists may sometimes feel that my intuition is crooked.

Almost all the results and arguments of this book are valid for all bond and site percolation models, subject to minor changes only; the principal exceptions are those results of Chapter 11 which make use of the self-duality of bond percolation on the square lattice. I have no especially convincing reason for my decision to study bond percolation rather than the more general case of site percolation, but was swayed in this direction by historical reasons as well as the consequential easy access to the famous exact calculation of the critical probability of bond percolation on the square lattice. In addition, unlike the case of site models, it is easy to formulate a bond model having interactions which are long-range rather than merely nearest-neighbour. Such arguments indicate the scanty importance associated with this decision.

Here are a few words about the contents of this book. In the introductory Chapter 1 we prove the existence of a critical value p_c for the edge-probability p, marking the arrival on the scene of an infinite open cluster. The next chapter contains a general account of three basic techniques—the FKG and BK inequalities, and Russo's formula—together with certain other useful inequalities, some drawn from reliability theory. Chapter 3 contains a brief account of numerical equalities and inequalities for critical points, together with a general method for establishing strict inequalities. This is followed in Chapter 4 by material concerning the number of open clusters per vertex. Chapters 5 and 6 are devoted to subcritical percolation (with $p < p_c$). These chapters begin with the Menshikov and Aizenman–Barsky methods for identifying the critical point, and they continue with a systematic study of the subcritical phase. Chapters 7 and 8 are devoted to supercritical percolation (with $p > p_c$). They begin with an account of dynamic renormalization, the proof that percolation in slabs characterizes the supercritical phase, and a rigorous static renormalization argument; they continue with a deeper account of this phase. Chapter 9 contains a sketch of the physical approach to the critical phenomenon (when $p = p_c$), and includes an attempt to communicate to mathematicians the spirit of scaling theory and renormalization. Rigorous results are currently limited and are summarized in Chapter 10, where may be found the briefest sketch of the Hara–Slade mean field theory of critical percolation in high dimensions. Chapter 11 is devoted to percolation in two dimensions, where the technique of planar duality leads to the famous exact calculation that $p_c = \frac{1}{2}$ for bond percolation on $\mathbb{Z}^2$. The book terminates with two chapters of pencil sketches of related random processes, including continuum percolation, first-passage percolation, random electrical networks, fractal percolation, and the random-cluster model.

The first edition of this book was published in 1989. The second edition differs from the first through the reorganization of certain material, and through the inclusion of fundamental new material having substantial applications in broader contexts. In particular, the present volume includes accounts of strict inequalities between critical points, the relationship between percolation in slabs and in the whole space, the Burton–Keane proof of the uniqueness of the infinite cluster, the lace expansion and mean field theory, and numerous other results of significance. A full list of references is provided, together with pointers in the notes for each chapter.

A perennial charm of percolation is the beauty and apparent simplicity of its open problems. It has not been possible to do full justice here to work currently in progress on many such problems. The big challenge at the time of writing is to understand the proposal that critical percolation models in two dimensions are conformally invariant. Numerical experiments support this proposal, but rigorous verification is far from complete. While a full account of conformal invariance must await a later volume, at the ends of Chapters 9 and 11 may be found lists of references and a statement of Cardy's formula.

Most of the first edition of this book was written in draft form while I was visiting Cornell University for the spring semester of 1987, a visit assisted by a grant from the Fulbright Commission. It is a pleasure to acknowledge the assistance of Rick Durrett, Michael Fisher, Harry Kesten, Roberto Schonmann, and Frank Spitzer during this period. The manuscript was revised during the spring semester of 1988, which I spent at the University of Arizona at Tucson with financial support from the Center for the Study of Complex Systems and AFOSR contract no.

F49620-86-C-0130. One of the principal benefits of this visit was the opportunity for unbounded conversations with David Barsky and Chuck Newman. Rosine Bonay was responsible for the cover design and index of the first edition.

In writing the second edition, I have been aided by partial financial support from the Engineering and Physical Sciences Research Council under contract GR/L15425. I am grateful to Sarah Shea-Simonds for her help in preparing the TEXscript of this edition, and to Alexander Holroyd and Gordon Slade for reading and commenting on parts of it.

I make special acknowledgement to John Hammersley; not only did he oversee the early life of percolation, but also his unashamed love of a good problem has been an inspiration to many.

Unstinting in his help has been Harry Kesten. He read and commented in detail on much of the manuscript of the first edition, his suggestions for improvements being so numerous as to render individual acknowledgements difficult. Without his support the job would have taken much longer and been done rather worse, if at all.

G. R. G.
Cambridge
January 1999

Contents

Chapter 1

What is Percolation?

1.1 Modelling a Random Medium

Suppose we immerse a large porous stone in a bucket of water. What is the probability that the centre of the stone is wetted? In formulating a simple stochastic model for such a situation, Broadbent and Hammersley (1957) gave birth to the 'percolation model'. In two dimensions their model amounts to the following. Let $\mathbb{Z}^2$ be the plane square lattice and let p be a number satisfying $0 \leq p \leq 1$. We examine each edge of $\mathbb{Z}^2$ in turn, and declare this edge to be *open* with probability p and *closed* otherwise, independently of all other edges. The edges of $\mathbb{Z}^2$ represent the inner passageways of the stone, and the parameter p is the proportion of passages which are broad enough to allow water to pass along them. We think of the stone as being modelled by a large, finite subsection of $\mathbb{Z}^2$ (see Figure 1.1), perhaps those vertices and edges of $\mathbb{Z}^2$ contained in some specified connected subgraph of $\mathbb{Z}^2$. On immersion of the stone in water, a vertex x inside the stone is wetted if and only if there exists a path in $\mathbb{Z}^2$ from x to some vertex on the boundary of the stone, using open edges only. Percolation theory is concerned primarily with the existence of such 'open paths'.

If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^2$; we shall study the structure of this subgraph, particularly with regard to the way in which this structure depends on the numerical value of p. It is not unreasonable to postulate that the fine structure of the interior passageways of the stone is on a scale which is negligible when compared with the overall size of the stone. In such circumstances, the probability that a vertex near the centre of the stone is wetted by water permeating into the stone from its surface will behave rather similarly to the probability that this vertex is the endvertex of an infinite path of open edges in $\mathbb{Z}^2$. That is to say, the large-scale penetration of the stone by water is related to the existence of infinite connected clusters of open edges.

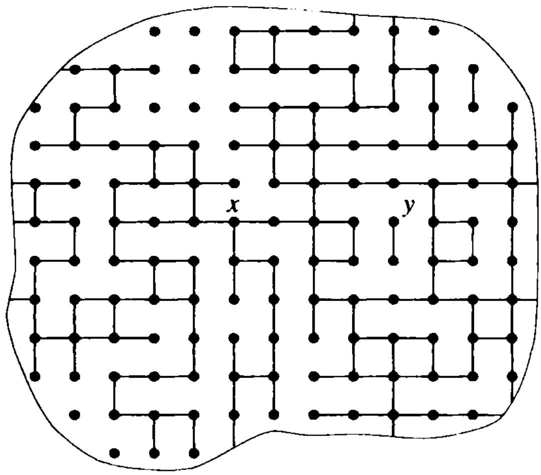


Figure 1.1. A sketch of the structure of a two-dimensional porous stone. The lines indicate the open edges; closed edges have been omitted. On immersion of the stone in water, vertex x will be wetted by the invasion of water, but vertex y will remain dry.

When can such infinite clusters exist? Simulations are handy indicators of the likely structure of the lattice, and Figure 1.2 contains such pictures for four different values of p. When $p = 0.25$, the connected clusters of open edges are isolated and rather small. As p increases, the sizes of clusters increase also, and there is a critical value of p at which there forms a cluster which pervades the entire picture. In loose terms, as we throw in more and more open edges, there comes a moment when large-scale connections are formed across the lattice. The pictures in Figure 1.2 are of course finite. If we were able to observe the whole of the infinite lattice $\mathbb{Z}^2$, we would see that all open clusters are finite when p is small, but that there exists an infinite open cluster for large values of p. In other words, there exists a critical value p_c for the edge-density p such that all open clusters are finite when $p < p_c$, but there exists an infinite open cluster when $p > p_c$ (such remarks should be interpreted 'with probability 1'). Drinkers of Pernod are familiar with this type of phenomenon—the transparence of a glass of Pernod is undisturbed by the addition of a small amount of water, but in the process of adding the water drop by drop, there arrives an instant at which the mixture becomes opaque.

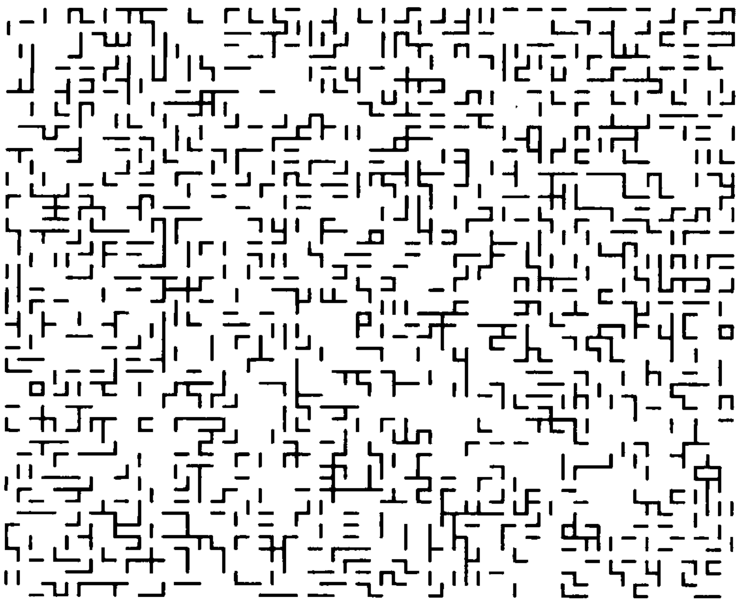
The occurrence of a 'critical phenomenon' is central to the appeal of percolation. In physical terms, we might say that the wetting of the stone is a 'surface effect' when the proportion p of open edges is small, and a 'volume effect' when p is large.

The above process is called 'bond percolation on the square lattice', and it is the most studied to date of all percolation processes. It is a very special process, largely because the square lattice has a certain property of self-duality which turns out to be extremely valuable. More generally, we begin with some periodic lattice in, say, d dimensions together with a number p satisfying $0 \le p \le 1$, and we declare each edge of the lattice to be open with probability p and closed otherwise. The resulting process is called a 'bond' model since the random blockages in the lattice are associated with the *edges*. Another type of percolation process is the 'site' percolation model, in which the *vertices* rather than the edges are declared to be open or closed at random, the closed vertices being thought of as junctions which are blocked to the passage of fluid. It is well known that every bond model may be reformulated as a site model on a different lattice, but that the converse is false (see Section 1.6). Thus site models are more general than bond models. They are illustrated in Figure 1.9.

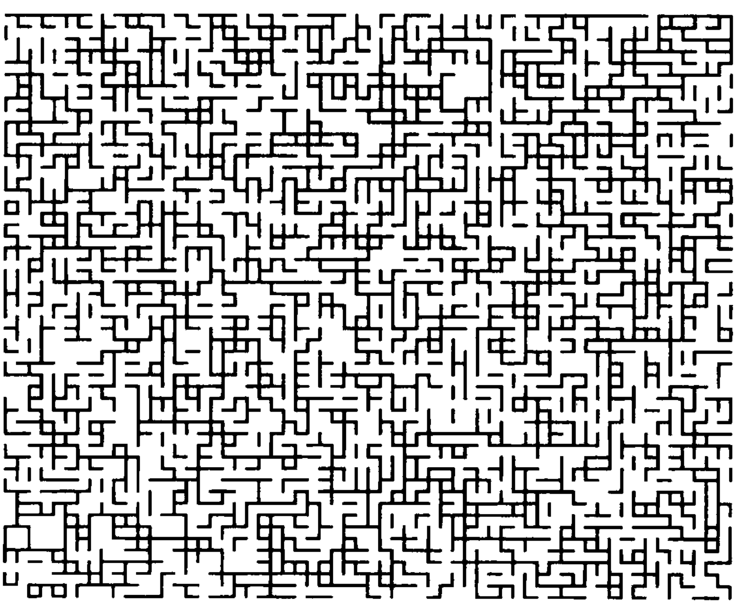
We may continue to generalize in several directions such as (i) 'mixed' models, in which both edges and vertices may be blocked, (ii) inhomogeneous models, in which different edges may have different probabilities of being open, (iii) long-range models, in which direct flow is possible between pairs of vertices which are very distant (in the above formulation, this may require a graph with large or even infinite vertex degrees), (iv) dependent percolation, in which the states of different edges are not independent, and so on. Mathematicians have a considerable talent in the art of generalization, and this has not been wasted on percolation theory. Such generalizations are often of considerable mathematical and physical interest; we shall however take the opposite route in this book. With few exceptions, we shall restrict ourselves to bond percolation on the d-dimensional cubic lattice $\mathbb{Z}^d$ where $d \ge 2$, and the main reason for this is as follows. As the level of generality rises, the accessibility of results in percolation theory is often diminished. Arguments which are relatively simple to explain in a special case can become concealed in morasses of geometrical and analytical detail when applied to some general model. This is not always the case, as illustrated by the proofs of exponential decay when $p < p_c$ (see Chapter 5) and of the uniqueness of the infinite open cluster when it exists (see Chapter 8). It is of course important to understand the limitations of an argument, but there may also be virtue in trying to describe something of the theory when stripped of peripheral detail. Bond percolation on $\mathbb{Z}^d$ is indeed a special case, but probably it exhibits the majority of properties expected of more general finite-range percolation-type models.

1.2 Why Percolation?

As a model for a disordered medium, percolation is one of the simplest, incorporating as it does a minimum of statistical dependence. Its attractions are manyfold. First, it is easy to formulate but not unrealistic in its qualitative predictions for random media. Secondly, for those with a greater interest in more complicated

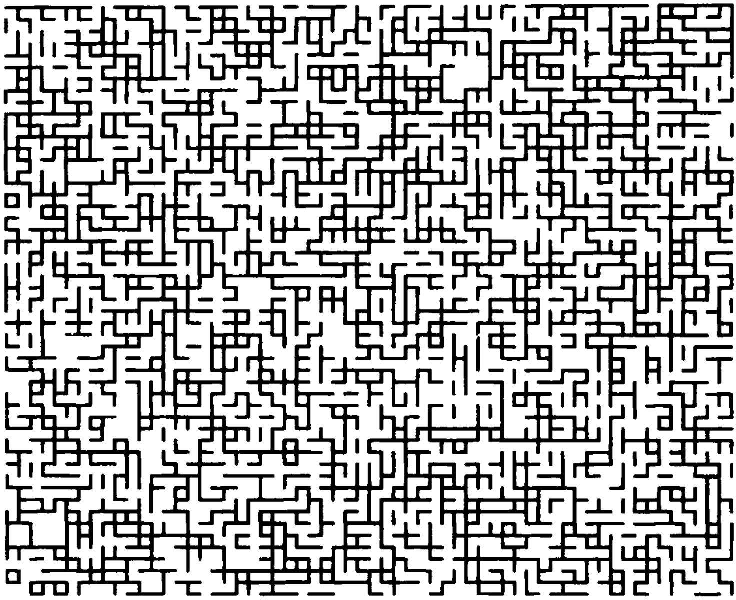


(a) $p = 0.25$

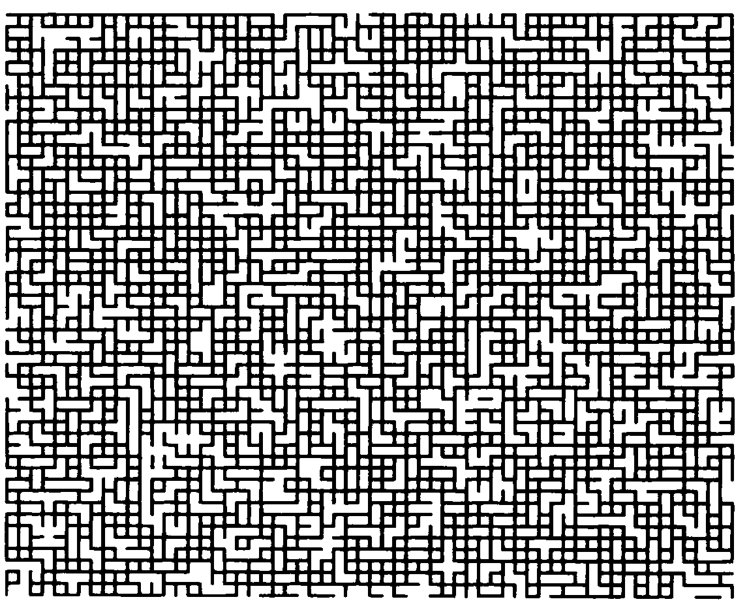


(b) $p = 0.49$

Figure 1.2. Realizations of bond percolation on a 50×60 section of the square lattice for four different values of p. The pictures have been created using the same sequence of pseudorandom numbers, with the result that each graph is a subgraph of the next. Readers with good



(c) $p = 0.51$



(d) $p = 0.75$

eyesight may care to check that there exist open paths joining the left to the right side when $p = 0.51$ but not when $p = 0.49$. The (random) value of p at which such paths appear for this realization is $0.5059 \ldots$.

processes, it is a playground for developing mathematical techniques and insight. Thirdly, it is well endowed with beautiful conjectures which are easy to state but apparently rather hard to settle.

There is a fourth reason of significance. A great amount of effort has been invested in recent years towards an understanding of complex interacting random systems, including disordered media and other physical models. Such processes typically involve families of dependent random variables which are indexed by $\mathbb{Z}^d$ for some $d \geq 2$. To develop a full theory of such a system is often beyond the current methodology. Instead, one may sometimes obtain partial results by making a comparison with another process which is better understood. It is sometimes possible to make such a comparison with a percolation model. In this way, one may derive valuable results for the more complex system; these results may not be the best possible, but they may be compelling indicators of the directions to be pursued.

Here is an example. Consider a physical model having a parameter T called 'temperature'. It may be suspected that there exists a critical value T_c marking a phase transition. While this fact may itself be unproven, it may be possible to prove by comparison that the behaviour of the process for small T is qualitatively different from that for large T.

It has been claimed that percolation theory is a cornerstone of the theory of disordered media. As evidence to support this claim, we make brief reference to four types of disordered physical systems, emphasizing the role of percolation for each.

A. *Disordered electrical networks.* It may not be too difficult to calculate the effective electrical resistance of a block of either material A or of material B, but what is the effective resistance of a mixture of these two materials? If the mixture is disordered, it may be reasonable to assume that each component of the block is chosen at random to be of type A or of type B, independently of the types of all other components. The resulting effective resistance is a random variable whose distribution depends on the proportion p of components of type A. It seems to be difficult to say much of interest about the way in which this distribution depends on the numerical value of p. An extreme example arises when material B is a perfect insulator, and this is a case for which percolation comes to the fore. We illustrate this in a special example.

Let U_n be the square section $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$ of the square lattice, and let S_n and T_n be the bottom and top sides of U_n,

$$S_n = \{(m, 0) : 0 \leq m \leq n\}, \qquad T_n = \{(m, n) : 0 \leq m \leq n\}.$$

We turn U_n into an electrical network as follows. We examine each edge of U_n in turn, and replace it by a wire of resistance 1 ohm with probability p, otherwise removing the connection entirely; this is done independently of all other edges. We now replace S_n and T_n by silver bars and we apply a potential difference between