高等学校试用

英语理工科教材选

第十分册 数学(二)

王传法 吴修珉 主编

戴鸣钟 谢卓杰 柯秉衡 主审

机械工业部部属院校选编

高等学校试用

英语理工科教材选

Book X

Mathematics (2)

王传法　　吴修琨　主编
戴鸣钟　谢卓杰　柯秉衡　主审

机械工业部部属院校选编

本分册内容选自 MURRAY H. PROTTER
与 CHARLES B. MORREY, JR. 所著
College Calculus with Analytic Geometry
英文版书（第三版）中第16，18二章。

高等学校试用

英语理工科教材选

（第十分册 数学(二)）

王传法 吴修珉 主编

戴鸣钟 谢卓杰 柯秉衡 主审

上海机械学院教务处出版发行

上海机械学院印刷厂印刷

编者的话

为了提高机械工业部部属院校学生的外语水平，培养学生阅读英语科技书刊的能力，我们选编了这套“英语理工科教材选”。整套“教材选”共分九个分册，内容包括数学、物理、理论力学、材料力学(与理论力学合为一个分册)、电工学、工业电子学、金属工艺学、机械原理、机械另件(与机械原理合为一个分册)、计算机算法语言、管理工程等十一门业务课程。各业务课都选了三章英语原版教材（个别也有选四章），供机械工业部部属院校试用。

在业务课中使用部分外语原版教材，这是我们的一次尝试，也是业务课教材改革、汲收国外先进科学技术的探索。在选材时，我们考虑了我国现行各课程的体系、内容以及学生的外语程度，尽可能选用适合我国实际的外国材料。

本“教材选”的选编工作，是在机械工业部教育局的直接领导下，由部属院校的有关教研室做了大量调查研究后选定的，并进行注释和词汇整理工作。由马泰来、卢思源、李国瑞、柯秉衡、谢卓杰、戴炜华、戴鸣钟等同志(以姓氏笔划为序)组成的审编小组，对选材的文字、注释、词汇作了审校。戴鸣钟教授担任整套“教材选”的总审。

由于时间仓促，选材、注释和编辑必有不尽完善之处，希广大读者提出宝贵意见，以利改进。

1983年4月

CONTENTS

Unit One

(Chapter 16 *College Calculus with Analytic Geometry*)

INFINITE SERIES

Unit Two

(Chapter 18 *College Calculus with Analytic Geometry*)

MULTIPLE INTEGRATION

Notes

16

INFINITE SERIES

1. INDETERMINATE FORMS

If $f(x)$ and $F(x)$ both approach 0 as x tends to a value a, the quotient

$$\frac{f(x)}{F(x)}$$

may approach a limit, may become infinite, or may fail to have any limit. We saw in the definition of derivative that it is the evaluation of just such expressions that leads to the usual differentiation formulas. We are aware that the expression

$$\frac{f(a)}{F(a)} = \frac{0}{0}$$

is in itself a meaningless one, and we use the term **indeterminate form** for the ratio 0/0.

If $f(x)$ and $F(x)$ both tend to infinity as x tends to a, the ratio $f(x)/F(x)$ may or may not tend to a limit. We use the same term, **indeterminate form,** for the expression ∞/∞, obtained by direct substitution of $x = a$ into the quotient $f(x)/F(x)$.

We recall the Theorem of the Mean, which we established (Chapter 6, p. 143).

Theorem 1 (Theorem of the Mean). *Suppose that f is continuous for*

$$a \leq x \leq b$$

and that $f'(x)$ exists for each x between a and b. Then there is an x_0 between a and b (that is, $a < x_0 < b$) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

Remark. Rolle's Theorem (p. 141) is the special case $f(a) = f(b) = 0$. ◀

The evaluation of indeterminate forms requires an extension of the Theorem of the Mean which we now prove.

Theorem 2 (Generalized Theorem of the Mean). *Suppose that f and F are continuous for $a \le x \le b$, and $f'(x)$ and $F'(x)$ exist for $a < x < b$ with $F'(x) \neq 0$ there. Then $F(b) - F(a) \neq 0$ and there is a number ξ with*

$$a < \xi < b$$

such that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)} \tag{1}$$

Proof. The fact that $F(b) - F(a) \neq 0$ is obtained by applying the Theorem of the
① Mean (Theorem 1) to F. For then, $F(b) - F(a) = F'(x_0)(b - a)$ for some x_0 such that $a < x_0 < b$. By hypothesis, the right side is different from zero.

For the proof of the main part of the theorem, we define the function $\phi(x)$ by the formula

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)}[F(x) - F(a)].$$

We compute $\phi(a)$, $\phi(b)$, and $\phi'(x)$, getting

$$\phi(a) = f(a) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)}[F(a) - F(a)] = 0,$$

$$\phi(b) = f(b) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)}[F(b) - F(a)] = 0,$$

$$\phi'(x) = f'(x) - \frac{f(b) - f(a)}{F(b) - F(a)}F'(x).$$

Applying the Theorem of the Mean (i.e., in the special form of Rolle's Theorem) to $\phi(x)$ in the interval $[a, b]$, we find

$$\frac{\phi(b) - \phi(a)}{b - a} = 0 = \phi'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{F(b) - F(a)}F'(\xi)$$

for some ξ between a and b. Dividing by $F'(\xi)$, we obtain formula (1). ◀

The next theorem, known as **l'Hôpital's Rule,** is useful in the evaluation of
② indeterminate forms.

Theorem 3 (l'Hôpital's Rule). *Suppose that*

$$\lim_{x\to a} f(x) = 0, \qquad \lim_{x\to a} F(x) = 0, \qquad \text{and} \qquad \lim_{x\to a}\frac{f'(x)}{F'(x)} = L,$$

and that the hypotheses of Theorem 2 hold in some deleted interval about a. Then*

$$\lim_{x\to a}\frac{f(x)}{F(x)} = \lim_{x\to a}\frac{f'(x)}{F'(x)} = L.$$

* Let I be an interval which has a as an interior point. The interval I with a removed from it is called a **deleted interval** about a.

Proof. For some h we apply Theorem 2 in the interval $a < x < a + h$. Then

$$\frac{f(a+h) - f(a)}{F(a+h) - F(a)} = \frac{f(a+h)}{F(a+h)} = \frac{f'(\xi)}{F'(\xi)}, \qquad a < \xi < a + h,$$

where we have taken $f(a) = F(a) = 0$. As h tends to 0, ξ tends to a, and so

$$\lim_{h\to 0} \frac{f(a+h)}{F(a+h)} = \lim_{\xi\to a} \frac{f'(\xi)}{F'(\xi)} = L.$$

A similar proof is valid for x in the interval $a - h < x < a$.

Example 1. Evaluate

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x - 3}{x^2 - 9}.$$

Solution. We set $f(x) = x^3 - 2x^2 - 2x - 3$ and $F(x) = x^2 - 9$. We see at once that $f(3) = 0$ and $F(3) = 0$, and we have an indeterminate form. We calculate

$$f'(x) = 3x^2 - 4x - 2, \qquad F'(x) = 2x.$$

By Theorem 3 (l'Hôpital's Rule):

$$\lim_{x\to 3} \frac{f(x)}{F(x)} = \lim_{x\to 3} \frac{f'(x)}{F'(x)} = \frac{3(9) - 4(3) - 2}{2(3)} = \frac{13}{6}.$$ ◀

Remarks. It is essential that $f(x)$ and $F(x)$ *both* tend to zero as x tends to a before applying l'Hôpital's Rule. If either or both functions tend to finite ③
limits $\neq 0$, or if one tends to zero and the other does not, then the limit of the quotient is found by the method of direct substitution as given in Chapter 4.

It may happen that $f'(x)/F'(x)$ is an indeterminate form as $x \to a$. Then l'Hôpital's Rule may be applied again, and the limit $f''(x)/F''(x)$ may exist as x tends to a. In fact, for some problems l'Hôpital's Rule may be required a number of times before the limit is actually determined. Example 3 below exhibits this point.

Example 2. Evaluate

$$\lim_{x\to a} \frac{x^p - a^p}{x^q - a^q}, \qquad a > 0.$$

Solution. We set $f(x) = x^p - a^p$, $F(x) = x^q - a^q$. Then $f(a) = 0$, $F(a) = 0$. We compute $f'(x) = px^{p-1}$, $F'(x) = qx^{q-1}$. Therefore

$$\lim_{x\to a} \frac{f(x)}{F(x)} = \lim_{x\to a} \frac{f'(x)}{F'(x)} = \lim_{x\to a} \frac{px^{p-1}}{qx^{q-1}} = \frac{p}{q} a^{p-q}.$$ ◀

Example 3. Evaluate

$$\lim_{x\to 0} \frac{x - \sin x}{x^3}.$$

Solution. We set $f(x) = x - \sin x$, $F(x) = x^3$. Since $f(0) = 0$, $F(0) = 0$, we apply l'Hôpital's Rule and get

$$\lim_{x \to 0} \frac{f(x)}{F(x)} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2}.$$

But we note that $f'(0) = 0$, $F'(0) = 0$, and so we apply l'Hôpital's Rule again:

$$f''(x) = \sin x, \qquad F''(x) = 6x.$$

Therefore

$$\lim_{x \to 0} \frac{f(x)}{F(x)} = \lim_{x \to 0} \frac{f''(x)}{F''(x)}.$$

Again we have an indeterminate form: $f''(0) = 0$, $F''(0) = 0$. We continue, to obtain $f'''(x) = \cos x$, $F'''(x) = 6$. Now we find that.

$$\lim_{x \to 0} \frac{f(x)}{F(x)} = \lim_{x \to 0} \frac{f'''(x)}{F'''(x)} = \frac{\cos 0}{6} = \frac{1}{6}.$$ ◀

L'Hôpital's Rule can be extended to the case where *both* $f(x) \to \infty$ and
④ $F(x) \to \infty$ as $x \to a$. The proof of the next theorem, which we omit, is analogous to the proof of Theorem 3.

Theorem 4 (l'Hôpital's Rule). *Suppose that*

$$\lim_{x \to a} f(x) = \infty, \qquad \lim_{x \to a} F(x) = \infty, \qquad \text{and} \qquad \lim_{x \to a} \frac{f'(x)}{F'(x)} = L.$$

Then

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)} = L.$$

Remark. Theorems 3 and 4 hold for one-sided limits as well as for ordinary limits. In many problems a one-sided limit is required even though this statement is not made explicitly. The next example illustrates such a situation.

Example 4. Evaluate

$$\lim_{x \to 0} \frac{\ln x}{\ln (2e^x - 2)}.$$

Solution. First note that x must tend to zero through positive values since
⑤ otherwise the logarithm function is not defined. We set

$$f(x) = \ln x, \qquad F(x) = \ln (2e^x - 2).$$

Then $f(x) \to -\infty$ and $F(x) \to -\infty$ as $x \to 0^+$. Therefore

$$\lim_{x \to 0^+} \frac{f(x)}{F(x)} = \lim_{x \to 0^+} \frac{1/x}{e^x/(e^x - 1)} = \lim_{x \to 0^+} \frac{e^x - 1}{xe^x}$$

$$= \lim_{x \to 0^+} \frac{1 - e^{-x}}{x}.$$

We still have an indeterminate form, and we take derivatives again. We obtain

$$\lim_{x\to 0^+} \frac{\ln x}{\ln(2e^x - 2)} = \lim_{x\to 0^+} \frac{e^{-x}}{1} = 1. \quad ◀$$

Remark. Theorems 3 and 4 are valid when $a = +\infty$ or $-\infty$. That is, if we have an indeterminate expression for $f(\infty)/F(\infty)$, then

$$\lim_{x\to+\infty} \frac{f(x)}{F(x)} = \lim_{x\to+\infty} \frac{f'(x)}{F'(x)},$$

and a similar statement holds when $x \to -\infty$. The next example exhibits this type of indeterminate form.

Example 5. Evaluate

$$\lim_{x\to+\infty} \frac{8x}{e^x}.$$

Solution

$$\lim_{x\to+\infty} \frac{8x}{e^x} = \lim_{x\to+\infty} \frac{8}{e^x} = 0. \quad ◀$$

Remarks. Indeterminate forms of the type $0 \cdot \infty$ or $\infty - \infty$ can often be evaluated by transforming the expression into a quotient of the form $0/0$ or ∞/∞. Limits involving exponential expressions may often be evaluated by taking logarithms. Of course, algebraic or trigonometric reductions may be made at any step. The next examples illustrate the procedure.

Example 6. Evaluate

$$\lim_{x\to\pi/2} (\sec x - \tan x).$$

Solution. We employ trigonometric reduction to change $\infty - \infty$ into a standard form. We have

$$\lim_{x\to\pi/2} (\sec x - \tan x) = \lim_{x\to\pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x\to\pi/2} \frac{-\cos x}{-\sin x} = 0. \quad ◀$$

Example 7. Evaluate

$$\lim_{x\to 0} (1 + x)^{1/x}.$$

Solution. We have 1^∞, which is indeterminate. Set $y = (1 + x)^{1/x}$ and take logarithms. Then

$$\ln y = \ln(1 + x)^{1/x} = \frac{\ln(1 + x)}{x}.$$

By l'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = \lim_{x\to 0} \frac{1}{1 + x} = 1.$$

Therefore, $\lim_{x\to 0} \ln y = 1$, and we conclude that

$$\lim_{x\to 0} y = \lim_{x\to 0} (1 + x)^{1/x} = e. \quad ◀$$

PROBLEMS

⑥ **In each of Problems 1 through 42, find the limit.**

1. $\lim_{x\to -2} \frac{2x^2+5x+2}{x^2-4}$

2. $\lim_{x\to 2} \frac{x^3-x^2-x-2}{x^3-8}$

3. $\lim_{x\to 1} \frac{x^3-3x+2}{x^3-x^2-x+1}$

4. $\lim_{x\to 2} \frac{x^4-3x^2-4}{x^3+2x^2-4x-8}$

5. $\lim_{x\to +\infty} \frac{2x^3-x^2+3x+1}{3x^3+2x^2-x-1}$

6. $\lim_{x\to 4} \frac{x^3-8x^2+2x+1}{x^4-x^2+2x-3}$

7. $\lim_{x\to +\infty} \frac{x^3-3x+1}{2x^4-x^2+2}$

8. $\lim_{x\to +\infty} \frac{x^4-2x^2-1}{2x^3-3x^2+3}$

9. $\lim_{x\to 0} \frac{\tan 3x}{\sin x}$

10. $\lim_{x\to 0} \frac{\sin 7x}{x}$

11. $\lim_{x\to 0} \frac{e^{2x}-2x-1}{1-\cos x}$

12. $\lim_{x\to 0} \frac{e^{3x}-1}{1-\cos x}$

13. $\lim_{x\to 0} \frac{\ln x}{e^x}$

14. $\lim_{x\to +\infty} \frac{\ln x}{x^h}, \quad h>0$

15. $\lim_{x\to 0} \frac{\ln(1+2x)}{3x}$

16. $\lim_{x\to 0} \frac{3^x-2^x}{x}$

17. $\lim_{x\to 0} \frac{3^x-2^x}{x^2}$

18. $\lim_{x\to 0} \frac{3^x-2^x}{\sqrt{x}}$

19. $\lim_{x\to \pi/2} \frac{1-\sin x}{\cos x}$

20. $\lim_{x\to 2} \frac{\sqrt{2x}-2}{\ln(x-1)}$

21. $\lim_{x\to \pi/2} \frac{\ln \sin x}{1-\sin x}$

22. $\lim_{x\to \pi/2} \frac{\cos x}{\sin^2 x}$

23. $\lim_{x\to +\infty} \frac{x^3}{e^x}$

24. $\lim_{x\to \pi/2} \frac{\tan x}{\ln \cos x}$

25. $\lim_{x\to +\infty} \frac{\sin x}{x}$

26. $\lim_{x\to \pi/2} \frac{\sin x}{x}$

27. $\lim_{x\to 0} \frac{x-\arctan x}{x-\sin x}$

28. $\lim_{x\to 0} \sqrt{x}\ln x$

29. $\lim_{x\to 0} x\cot x$

30. $\lim_{x\to \pi/2} (x-\pi/2)\sec x$

31. $\lim_{x\to +\infty} \frac{\arctan x}{x}$

32. $\lim_{\theta\to 0} \left(\csc\theta - \frac{1}{\theta}\right)$

33. $\lim_{x\to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$

34. $\lim_{x\to 0} x^x$

35. $\lim_{x\to 0} x^{4x}$

36. $\lim_{x\to +\infty} \left(1+\frac{k}{x}\right)^x$

37. $\lim_{x\to 0} x^{(x^2)}$

38. $\lim_{x\to 0} (\cot x)^x$

39. $\lim_{x\to 0} x^{(1/\ln x)}$

40. $\lim_{x\to +\infty} \frac{x^p}{e^x}, \quad p>0$

41. $\lim_{x\to 0^+} (\sin x)^{\tan x}$

42. $\lim_{x\to 0} \frac{\ln x}{x^h}$, h is a real number

43. a) Prove that

$$\lim_{x\to 0^+} x^3 e^{1/x} = +\infty.$$

b) Prove that for every positive integer n,

$$\lim_{x\to 0^+} x^n e^{1/x} = +\infty.$$

c) Find the result if in part (b) we have $x \to 0^-$ instead of $x \to 0^+$.

44. By direct methods, find the value of $\lim_{x\to+\infty}(x \sin x)/(x^2 + 1)$. What happens if l'Hôpital's Rule is used? Explain.

45. Prove the following form of l'Hôpital's Rule:
If

$$\lim_{x\to+\infty} f(x) = 0, \quad \lim_{x\to+\infty} g(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L.$$

[*Hint:* Consider $\lim_{x\to 0^+} f(1/x)/g(1/x)$.]

46. Suppose that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = \lim_{x\to 0} f''(x) = \lim_{x\to 0} f'''(x) = 0,$$

and that

$$\lim_{x\to 0} \frac{x^2 f'''(x)}{f''(x)} = 2.$$

Find $\lim_{x\to 0} x^2 f'(x)/f(x)$.

47. If the second derivative f'' of a function f exists at a value x_0, show that

$$\lim_{h\to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).$$

48. Let $P(x)$ and $Q(x)$ be polynomials of degree m and n, respectively. Analyze

$$\lim_{x\to+\infty} \frac{P(x)}{Q(x)}$$

according as $m > n$ or $m = n$ or $m < n$.

2. CONVERGENT AND DIVERGENT SERIES

In Chapter 4, Section 10, the idea of a sequence of numbers was introduced. We begin by repeating some of the material presented there. The numbers

$$b_1, b_2, b_3, \ldots, b_{12}, b_{13}, b_{14}$$

form a sequence of fourteen numbers. Since this set contains both a first and last element, the sequence is termed **finite.** In all other circumstances it is called **infinite.** The subscripts not only identify the location of each element but also serve to associate a positive integer with each member of the sequence. In other (7) words, *a sequence is a function* with *domain* a portion (or all) of the positive integers and with *range* in the collection of real numbers. If we use J to denote the collection of positive integers and R the set of real numbers, then a sequence is a function $f: J \to R$.

If the domain is an infinite collection of positive integers, e.g., all positive integers, we write

$$a_1, a_2, \ldots, a_n, \ldots,$$

the final dots indicating the never-ending character of the sequence. Simple examples of infinite sequences are

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \tag{1}$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \tag{2}$$

$$2, 4, 6, \ldots, 2n, \ldots. \tag{3}$$

Definition. *Given the infinite sequence*

$$a_1, a_2, \ldots, a_n, \ldots,$$

we say that **this sequence has the limit** c *if, for each* $\epsilon > 0$, *there is a positive integer* N (*the size of* N *depending on* ϵ) *such that*

⑧ $$|a_n - c| < \epsilon \qquad \text{for all } n > N.$$

We also write $a_n \to c$ as $n \to \infty$ and, equivalently,

$$\lim_{n\to\infty} a_n = c.$$

In the sequence (1) above, we have

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad \ldots, \quad a_n = \frac{1}{n}, \ldots$$

and $\lim_{n\to\infty} a_n = 0$. The sequence (2) has the form

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{2}{3}, \quad \ldots, \quad a_n = \frac{n}{n+1}, \quad \ldots$$

and $\lim_{n\to\infty} a_n = 1$. The sequence (3) does not tend to a limit.

An expression such as

$$u_1 + u_2 + u_3 + \cdots + u_{24}$$

is called a *finite series*. The **sum** of such a series is obtained by adding the 24 terms. We now extend the notion of a finite series by considering an expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

which is nonterminating and which we call an **infinite series.*** Our first task is to give a meaning, if possible, to such an infinite succession of additions. ⑨

Definition. *Given the infinite series* $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$, *the quantity* $s_k = u_1 + u_2 + \cdots + u_k$ *is called the* **kth partial sum** *of the series. That is,*

$$s_1 = u_1, \qquad s_2 = u_1 + u_2, \qquad s_3 = u_1 + u_2 + u_3,$$

etc. Each partial sum is obtained simply by a **finite** *number of additions.*

Definition. *Given the series*

$$u_1 + u_2 + u_3 + \cdots + u_n \cdots \tag{4}$$

with the sequence of partial sums

$$s_1, s_2, s_3, \ldots, s_n, \ldots,$$

we define the **sum of the series** (4) *to be*

$$\lim_{n\to\infty} s_n \tag{5}$$

whenever the limit exists.

Using the $\sum$ notation for sum, we can also write

$$\sum_{n=1}^{\infty} u_n = \lim_{n\to\infty} s_n.$$

If the limit (5) does not exist, then the sum (4) *is not defined.*

Definitions. *If the limit* (5) *exists, the series* $\sum_{n=1}^{\infty} u_n$ *is said to* **converge** *to that limit; otherwise the series is said to* **diverge.**

Remark. The expression $\sum_{n=1}^{\infty} u_n$ is a shorthand notation for the formal series expression (4). However, the symbol $\sum_{n=1}^{\infty} u_n$ is also used as a synonym for the numerical value of the series when it converges. There will be no difficulty in recognizing which meaning we are employing in any particular case. We could obtain more precision by using the (more cumbersome) notation described in the footnote below.

The sequence of terms

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, ar^n, \ldots$$

forms a *geometric progression.* Each term (except the first) is obtained by multiplication of the preceding term by r, the *common ratio.* The partial sums of the ⑩
geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

---

* The definition given here is informal. A more formal definition is as follows: An **infinite series** is an ordered pair $(\{u_n\}, \{s_n\})$ of infinite sequences in which $s_k = u_1 + \cdots + u_k$ for each k. The infinite series $(\{u_n\}, \{s_n\})$ is denoted by $u_1 + u_2 + \cdots + u_n + \cdots$ or $\sum_{n=1}^{\infty} u_n$. When no confusion can arise we also denote by $\sum_{n=1}^{\infty} u_n$ the limit of the sequence $\{s_n\}$ when it exists.

$$s_1 = a,$$
$$s_2 = a + ar,$$
$$s_3 = a + ar + ar^2,$$
$$s_4 = a + ar + ar^2 + ar^3,$$

and, in general,

$$s_n = a(1 + r + r^2 + \cdots + r^{n-1}).$$

For example, with $a = 2$ and $r = \frac{1}{2}$,

$$s_n = 2\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}\right).$$

The identity

$$(1 + r + r^2 + \cdots + r^{n-1})(1 - r) = 1 - r^n,$$

which may be verified by straightforward multiplication, leads to the formula

$$s_n = a\frac{1 - r^n}{1 - r}$$

for the nth partial sum. The example $a = 2$, $r = \frac{1}{2}$ gives

$$s_n = 2\frac{1 - 2^{-n}}{\frac{1}{2}} = 4 - \frac{1}{2^{n-2}}.$$

In general, we may write

$$s_n = a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r}r^n, \qquad r \neq 1. \tag{6}$$

The next theorem is a direct consequence of formula (6).

Theorem 5. *A geometric series*

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

converges if $-1 < r < 1$ *and diverges if* $|r| \geq 1$. *In the convergent case we have*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}. \tag{7}$$

Proof. From (6) we see that $r^n \to 0$ if $|r| < 1$, yielding (7); also, $r^n \to \infty$ if $|r| > 1$. For $r = 1$, the partial sum s_n is na, and s_n does not tend to a limit as $n \to \infty$. If $r = -1$, the partial sum s_n is a if n is odd and 0 if n is even. ◀

The next theorem is useful in that it exhibits a limitation on the behavior of
11 the terms of a convergent series.

Theorem 6. *If the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{8}$$

converges, then

$$\lim_{n\to\infty} u_n = 0.$$

Proof Writing

$$s_n = u_1 + u_2 + \cdots + u_n,$$
$$s_{n-1} = u_1 + u_2 + \cdots + u_{n-1},$$

we have, by subtraction, $u_n = s_n - s_{n-1}$. Letting c denote the sum of the series, we see that $s_n \to c$ as $n \to \infty$; also, $s_{n-1} \to c$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = c - c = 0. \quad ◀$$

Remark. The converse of Theorem 6 is not necessarily true. Later we shall show (by example) that it is possible both for u_n to tend to 0 and for the series to diverge.

The following corollary, a restatement of Theorem 6, is useful in establishing the divergence of infinite series.

Corollary. *If u_n does not tend to zero as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ is divergent.*

Convergent series may be added, subtracted, and multiplied by constants, as the next theorem shows. 12

Theorem 7. *If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ both converge and c is any number, then the series*

$$\sum_{n=1}^{\infty} (cu_n), \qquad \sum_{n=1}^{\infty} (u_n + v_n), \qquad \sum_{n=1}^{\infty} (u_n - v_n)$$

all converge, and

$$\sum_{n=1}^{\infty} (cu_n) = c\sum_{n=1}^{\infty} u_n,$$
$$\sum_{n=1}^{\infty} (u_n \pm v_n) = \sum_{n=1}^{\infty} u_n \pm \sum_{n=1}^{\infty} v_n.$$

Proof For each n, we have the following equalities for the partial sums:

$$\sum_{j=1}^{n} (cu_j) = c\sum_{j=1}^{n} u_j;$$
$$\sum_{j=1}^{n} (u_j \pm v_j) = \sum_{j=1}^{n} u_j \pm \sum_{j=1}^{n} v_j.$$