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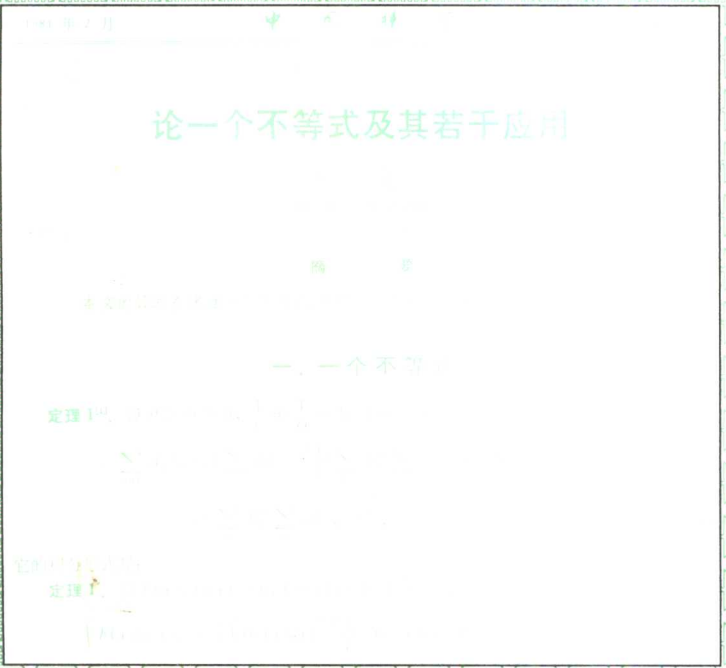
论一个不等式及其若干应用

江西师范大学数学系

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祝贺胡克教授诞生七十周年

胡克教授,男,生于1925年9月,江西奉新人,现任江西省政协常委,中国气功学术研究会名誉理事,江西师范大学数学与计算机学院数学研究所所长,1991年起享受国务院颁发的特殊津贴.

他长期致力于单复函数理论和应用的研究与教学工作,治学严谨,学术造诣很深,是国内外著名的单叶函数专家之一.1979年以来,他在中外杂志上共发表论文70多篇,在单叶函数的偏差定理与系数及其相关问题的研究结果,至今仍居世界领先地位,对单叶函数s(α)族研究取得了公认的"一些极为重要的结果",解决了奇单叶函数相邻数模平方差Duren猜测.在不等式和应用的研究领域中也有卓著的成果,他在"中国科学"上发表的"一个不等式及其若干重要应用"论文,被美国数学评论评价为"是一个卓越的,非凡的不等式".他解决了为时已久的Opial－华罗庚型积分不等式精密性问题等,他在人体功能与数学理论的研究上,提出"人体存在两个进出信息处理系统",得到了有关专家的重视和肯定.中国科学院学部委员胡海昌教授认为是"很有意思"的工作.1985年被德国《数学评论》聘为评论员,1990年载入世界数学家人名录(第九版),1994年载入世界专家人名录(第十一版),多次获江西省科技成果和学校优秀科技成果一、二等奖,1991年获国家科技进步三等奖.

他在教学和育人方面也做出了突出的成绩,培养和造成了一支充满生机和活力,具有较高学术水平的中青年学科梯队,为学科建设做出了重大贡献.他培养的研究生,学术思想活跃,科研和教学能力强,业务水平高,其中大多已赴美攻读博士,由于他教学方法独特,教学和育人成绩卓著,1990年获国家教委教学成果优秀奖,1992年获曾宪梓教育基金会教育三等奖.

为了弘扬胡克教授为科学和教学上取得的卓著成就和对事业的献身精神,值此胡克教授七十寿辰之际,出版他的主要研究成果论文集,以此激励后人学习继承和发扬他为事业献身的精神.

江西师大数学系

目　　录

ON AN INEQUALITY AND ITS APPLICATIONS*

Hu Ke （胡　克）

ABSTRACT

In this paper a fundamental inequality is established and some of its important applications are given.

I. An INEQUALITY

Theorem 1. Let $P \geqslant Q > 0, \frac{1}{P} + \frac{1}{Q} = 1, 1 + e_n - e_m \geqslant 0$ *and* $A_n, B_n \geqslant 0$. xxThen

$$\sum_n A_nB_n \leqslant \Big(\sum_n B_n^Q\Big)^{\frac{1}{Q}-\frac{1}{P}}\Big\{\Big(\sum_n B_n^Q \sum_n A_n^P\Big)^2 - \Big(\sum_n B_n^Q e_n \sum_n A_n^P - \sum_n B_n^Q \sum_n A_n^P e_n\Big)^2\Big\}^{\frac{1}{2P}} \tag{1.1}$$

The integral form is as follows:

Theorem 1. Let $F(x), G(x) \geqslant 0$, *and* $1 + e(x) - e(y) \geqslant 0$. Then

$$\int F(x)G(x)\mathrm{d}x \leqslant \Big(\int G^Q(x)\mathrm{d}x\Big)^{\frac{1}{Q}-\frac{1}{P}}\Big\{\Big(\int G^Q(x)\mathrm{d}x\int F^P(x)\mathrm{d}x\Big)^2 - \Big(\int G^Q(x)e(x)\mathrm{d}x\int F^P(x)\mathrm{d}x - \int G^Q(x)\mathrm{d}x\int F^P(x)e(x)\mathrm{d}x\Big)^2\Big\}^{\frac{1}{2P}} \tag{1.2}$$

Proof. Here we need only to prove that (1.1) is true. Since

$$I^2 = \Big(\sum_n A_nB_n\Big)^2 = \sum_n\sum_m A_nB_nA_mB_m = \sum_n\sum_m A_nB_nA_mB_m(1 - e_n + e_m), \tag{1.3}$$

supposing that $P \neq Q$, by Hölder inequality, we have

$$I^2 \leqslant \sum_n A_nB_n\Big\{\sum_m A_m^P(1 - e_n + e_m)\Big\}^{\frac{1}{P}}\Big\{\sum_m B_m^Q(1 - e_n + e_m)\Big\}^{\frac{1}{Q}} \tag{1.4}$$

$$= \sum_n A_nB_nX_n^{\frac{1}{P}}Y_n^{\frac{1}{Q}},$$

where

$$X_n = \sum_m A_m^P(1 - e_n + e_m),\ Y_n = \sum_m B_m^Q(1 - e_n + e_m).$$

* 原载"SCIENTIA SINICA"第14卷第8期 August 1981.

Let $f_n = B_n^{1-\frac{Q}{P}} Y_n^{\frac{1}{Q}-\frac{1}{P}}$, $g_n = A_n Y_n^{\frac{1}{P}}$ and $h_n = B_n^{\frac{Q}{P}} X_n^{\frac{1}{P}}$. (1. 4) may be written in the form

$$I^2 \leqslant \sum_n f_n g_n h_n. \tag{1. 5}$$

We apply Hölder inequality to (1. 5), obtaining

$$I^2 \leqslant (\sum_n f_n^{\alpha})^{\frac{1}{\alpha}} (\sum_n g_n^{\beta})^{\frac{1}{\beta}} (\sum_n h_n^{\gamma})^{\frac{1}{\gamma}}, \tag{1. 6}$$

for $\frac{1}{\alpha} = \frac{1}{Q} - \frac{1}{P}, \frac{1}{\beta} = \frac{1}{P}$ and $\frac{1}{\gamma} = \frac{1}{P}$ with $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{P} + \frac{1}{Q} = 1.$

It is noted that:

$$\sum_n f_n^{\alpha} = \sum_n B_n^Q \sum_m B_m^Q (1 - e_n + e_m) = (\sum_m B_m^Q)^2, \tag{1. 7}$$

$$\sum_n g_n^{\beta} = \sum_n A_n^P \sum_m B_m^Q (1 - e_n + e_m) \tag{1. 8}$$

$$\sum_n h_n^{\gamma} = \sum_n B_n^Q \sum_n A_m^P (1 - e_n + e_m). \tag{1. 9}$$

From (1. 7), (1. 8), (1. 9) and (1. 6), we get inequality (1. 1). The theorem is obvious when P = Q.

I . APPLICATION 1

Theorem 2. If A, B and C are real positive definite matrices of order n, then

$$\frac{1}{|\lambda A + (1-\lambda)B|} \leqslant \frac{1}{|A|^{2\lambda-1}} \left\{ \frac{1}{|A||B|} - \left(\frac{1}{\sqrt{|A+C||B|}} - \frac{1}{\sqrt{|A||B+C|}} \right)^2 \right\}^{1-\lambda} \tag{2. 1}$$

for $\frac{1}{2} \leqslant \lambda \leqslant 1$. The case C = 0 leads to the well known Ky Fan inequality[2].

Proof. If D is a positive definte matrix of order n, then

$$J_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x, Dx)} dx = \frac{\pi^{n/2}}{|D|^{1/2}} \tag{2. 2}$$

where the integration is over the entire n − dimensional space. Thus we have

$$\frac{\pi^{n/2}}{|\lambda A + (1-\lambda)B|^{1/2}} = \int^{\infty} \cdot\cdot \int^{\infty} e^{-\lambda(x, Ax) - (1-\lambda)(x, Bx)} dx. \tag{2. 3}$$

Let us apply the inequality (1. 2) to the integral (2. 3) with $Q = \frac{1}{\lambda}$, $P = \frac{1}{1-\lambda}$, $G(x) = e^{-\lambda(x, Ax)}$, $F(x) = e^{-(1-\lambda)(x, Bx)}$ and $e(x) = {}^{-(x, Cx)}$. By using (2. 2), the inequality follows.

Ⅱ. APPLICATION 2

Assuming the function $f_2(z) = z + \sum_{n=2}^{\infty} b_n z^{2n-1} \in S_2$, the classical result is $|f_2(\rho e^{i\theta})| \leqslant$

$\rho/(1-\rho^2)$. Here we give the following theorem:

Theorem 3. If the function $f_2(z) \in S_2$, then we have

$$|f_2(\rho e^{i\theta})| \leqslant \| f(\rho) \| \leqslant \frac{\rho}{1-\rho^2}\left\{1-(1-\rho^2)^2\rho^{4N-2}\left(N-\sum_{k=1}^{N}|b_k|^2\right)\right)^2\right\}^{\frac{1}{4}}, \quad (3.1)$$

where $$\| f_2(\rho) \| = \rho + \sum_{k=2}^{\infty}|b_k|^2\rho^{2k-1}.$$

Proof. Let $P = Q = 2, A_k = |b_k|\rho^{k-\frac{1}{2}}, B_k = \rho^{k-\frac{1}{2}}, e_k = \rho^{2N-2k+1}$ for $1 \leqslant k \leqslant N$; and $e_k = 0$ for $k > N$ in (1.1). Then we get

$$\| f_2(\rho) \|^4 \leqslant \left(\frac{\rho}{1-\rho^2}\sum_{k=1}^{\infty}|b_k|^2\rho^{2k+1}\right)^2 - \rho^{4N}\left(N\sum_{k=1}^{\infty}|b_k|^2\rho^{2k-1}\right.$$

$$\left.-\frac{\rho}{1-\rho^2}\left|\sum_{k=1}^{N}|b_k|^2\right)^2 . \quad b_1 = 1 \qquad (3.2)$$

Baernstein[3] has proved that

$$\sum_{k=1}^{\infty}|b_k|^2\rho^{2k-1} \leqslant \frac{\rho}{1-\rho^2} \qquad (3.3)$$

Substitute (3.3) into (3.1), then the result follows at once.

IV. APPLICATION 3

Let B be the class of functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ each of which is regular and $f(z_1)f(z_2) \neq 1$ for any two points z_1 and z_2 in the unit circle $|z| < 1$. Jenkins[4] and Shan[5] have proved that: If $f(z) \in B$, then

$$|f(z_0)| \leqslant \frac{|z_0|}{\sqrt{1-|z_0|^2}} \qquad (4.1)$$

Now we extend (4.1) into the following theorem.

Theorem 4. If $f(z) \in B$, then we have

$$|f(z_0)| \leqslant \frac{|z_0|}{\sqrt{1-|z_0|^2}}\{1-[(1-|z_0|^2)|z_0|^{2n-2}-|a_n|^2]^2\}^{\frac{1}{4}}, n = 1,2,\cdots, \qquad (4.2)$$

for each point $z_0(|z_0| < 1)$. The sign of equality holds if and only if $|a_n| = |z_0|^{n-1}\sqrt{1-|z_0|^2}$, $(n = 1,2,\cdots)$.

Proof. By Lebejef's theorem

$$\sum_{n=1}^{\infty}|a_n|^2 \leqslant 1, \qquad (4.3)$$

Taking $P = Q = 2$, $A_k = a_k, B_k = \rho^{k-1}, e_k = 0$ for $k \neq n$, and $e_n = 1$ in the inequality (1.1), thus we get

$$\Big|\sum_{k=1}^{\infty}|a_k|\rho^{k-1}\Big|^4 \leqslant \Big(\frac{1}{1-\rho^2}\sum_{k=1}^{\infty}|a_k|^2\Big)^2 - \Big(\frac{1}{1-\rho^2}|a_n|^2 - \rho^{2n-2}\sum_{k=1}^{\infty}|a_k|^2\Big)^2$$

$$\leqslant \Big(\frac{1}{1-\rho^2}\Big)^2 - \Big(\frac{1}{1-\rho^2}|a_n|^2 - \rho^{2n-2}\Big)^2. \qquad (4.4)$$

This is the result desirable.

V. APPLICATION 4

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and

$$\log\frac{f(z)-f(\zeta)}{z-\zeta}\,\frac{z\zeta}{f(z)f(\zeta)} = \sum_{m,n=1}^{\infty} d_{m,n} z^m \zeta^n. \qquad (5.1)$$

Then we have the following theorem:

Theorem 5. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, then we have

$$A_l^2 + 2B^2 \leqslant \frac{8^2\rho^2}{(1+\rho)^4}\Big\{\log\frac{\rho}{(1-\rho)^2|f(\rho e^{i\theta})|}\Big\}^2, l = 1,2, \qquad (5.2)$$

where

$$B = \log\Big(\Big|\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\Big|\rho\frac{1-\rho}{1+\rho}\Big[\frac{\rho}{|f(\rho e^{i\theta})|(1-\rho)^2}\Big]^{\left(\frac{1-\rho}{1+\rho}\right)^2}\Big),$$

$$A_1 = \frac{1}{m}\Big|F_m\Big(\frac{1}{f(z)}\Big) - \Big(\frac{1}{z^m} + z^m\Big)\Big|^2, (z = \rho e^{i\theta})$$

$$A_2 = \frac{2}{m[1 + R(md_{m,m})]}\Big|R_e\Big\{F_m\Big(\frac{1}{f(z)}\Big) - \Big(\frac{1}{z^m} + z^m\Big)\Big\}\Big|^2,$$

and $F_m(t)$ is a Faber polynomial of t with degree m.

It should be observed that our inequalities have improved that of [7].

Proof. For the proof, we may confine ourselves to the functions f(z) of Löwner

$$f(z) = \lim_{t\to\infty} e^t f(z,t), \qquad (5.3)$$

where $f(z,t) = e^{-t}\{z + \sum_{n=2}^{\infty} a_n(t) z^n\}$ being the solution of the differential equation

$$\frac{\partial}{\partial t} f(z,t) = -f(z,t)\frac{1 + k(t)f(z,t)}{1 - k(t)f(z,t)}, |k(t)| = 1,$$

with initial condition f(z,0) = z.

Let
$$\frac{k(t)f(z,t)}{1 - k(t)f(z,t)} = \sum_{n=1}^{\infty} b_n(t) z^n;$$

the following relations can be verified:

$$d_{m,n} = -2\int_0^{\infty} b_m(t) b_n(t)\,dt, \qquad (5.4)$$

$$\int_0^\infty b_m(t)\,\overline{b_n(t)}\mathrm{d}t = \begin{cases} 0, & m \neq n, \\ \frac{1}{2n}, & m = n, \end{cases} \tag{5.5}$$

$$\log \frac{\rho}{(1-\rho)^2|f(z)|} = 2\int_0^\infty \frac{|f| + R(kf)}{|1-kf|^2}\mathrm{d}t$$
$$= 2\int_0^\infty \frac{|1-kf|^2}{|f| - Rkf}|I(\frac{kf}{1-kf})|^2\mathrm{d}t \tag{5.6}$$

$$\log|\frac{f'(z)}{f^2(z)}\rho^2|(\frac{1}{1-\rho^2}) = 4\int_0^\infty (I(\frac{kf}{1-kf}))^2\mathrm{d}t. \tag{5.7}$$

and

$$\frac{|f| - R(kf)}{|1-kf|^2} \leqslant \frac{2\rho}{(1+\rho)^2}, \tag{5.8}$$

where $f = f(z,t), |z| = \rho.$

From (5.4), (5.5) and (5.8), we have

$$\frac{1}{m}g_m(z) = \frac{1}{m}\left|R\left[F_m\left(\frac{1}{f(z)}\right) - \frac{1}{z^m} - z^m\right]\right|$$
$$= 2\left|\int_0^\infty (I_1 b_m(t))\left(\sum_{n=1}^\infty b_n(t)z^n - \sum_{n=1}^\infty \overline{b_n(t)z^n}\right)\mathrm{d}t\right.$$
$$= 4\left|\int_0^\infty (Ib_m(t))I\left(\frac{kf}{1-kf}\right)\mathrm{d}t\right|$$
$$\leqslant 4\frac{\sqrt{2\rho}}{1+\rho}\int_0^\infty |Ib_m(t)|\frac{|1-kf|}{\sqrt{|f| - R(kf)}}\left|I\left(\frac{kf}{1-kf}\right)\right|\mathrm{d}t. \tag{5.9}$$

Let $F(t) = |Ib_m(t)|, G(t) = \dfrac{|1-kf|}{\sqrt{|f| - R_e|kf|}}|I(\dfrac{kf}{1-kf})|$ and $e(t) = \dfrac{|f| - R_e(kf)}{|1-kf|^2}$ $\dfrac{(1+\rho)^2}{2\rho} < 1$. Applying the inequality (1.2) with $P = Q = 2$ to (5.9), and by Cauchy inequality, we have

$$\frac{1}{4^5 m^4}\frac{(1+\rho)^4}{\rho^2}g_m^4(z) \leqslant \left\{\int_0^\infty (Ib_m(t))^2\mathrm{d}t\int_0^\infty \frac{|1-kf|^2}{|f| - R_e(kf)|}\left(I\left(\frac{kf}{1-kf}\right)\right)^2\mathrm{d}t\right\}^2$$
$$-\left\{\int_0^\infty (Ib_m(t))^2 e(t)\mathrm{d}t\int_0^\infty \frac{|1-kf|^2}{|f| - R_e(kf)}(I(\frac{kf}{1-kf}))^2\mathrm{d}t\right\},$$
$$-\frac{(1+\rho)^2}{2\rho}\int_0^\infty (Ib_m(t))^2\mathrm{d}t\int_0^\infty\left(I\left(\frac{kf}{1-kf}\right)\right)^2\mathrm{d}t\right\}^2, \tag{5.10}$$

and

$$\frac{1}{4^2m^2}g_m^2(z) \leqslant \left\{\int_0^\infty Ib_m(t)\left|\frac{\sqrt{|f| - R_e(kf)}}{|1-kf|}\right|I\left(\frac{kf}{1-kf}\right)\left|\frac{|1-kf|}{\sqrt{|f| - R_e fk}}\mathrm{d}t\right\}^2.$$
$$\leqslant \int_0^\infty (Ib_m(t))^2\frac{|f| - Re(kf)}{|1-kf|^2}\mathrm{d}t\int_0^\infty \frac{|1-kf|^2}{|f|^2 - Re(kf)}\left(I\left(\frac{kf}{1-kf}\right)\right)^2\mathrm{d}t. \tag{5.11}$$

From (5.10) and (5.11), it follows that

$$\frac{1}{4^4 m^4} g_m^4(z) \leqslant \frac{4\rho^2}{(1+\rho)^4}\left\{\int_0^\infty (Ib_m(t))^2 dt \int_0^\infty \frac{|1-kf|^2}{|f|-R_e(kf)}\left[I\left(\frac{kf}{1-kf}\right)\right]^2 dt\right\}^2$$

$$+ x^2 - \left\{x - \int_0^\infty (Ib_m(t))^2 dt \int_0^\infty \left[I\left(\frac{kf}{1-kf}\right)\right]^2 dt\right\}^2, \qquad (5,12)$$

where

$$x = \int_0^\infty (Ib_m(t))^2 \frac{|f| - R_e(kf)}{|1-kf|^2} dt \int_0^\infty \frac{|1-kf|^2}{|f| - Re(kf)}\left\{I\left(\frac{kf}{1-kf}\right)\right\}^2 dt$$

$$\leqslant \frac{2\rho}{(1+\rho)^2}\int_0^\infty (Ib_m(t))^2 dt \int_0^\infty \frac{|1-kf|^2}{|f| - R_e(kf)}\left\{I\left(\frac{kf}{1-kf}\right)\right\}^2 dt$$

$$= \frac{\rho}{2(1+\rho)^2}\left\{\int_0^\infty [Ib_m(t)]^2 dt\right\} \log \frac{\rho}{|f(z)|(1-\rho)^2}$$

$$= \frac{\rho}{2(1+\rho)^2}\left\{\int_0^\infty (|b_m(t)|^2 - [Rb_m(t)])^2 dt\right\} \log \frac{\rho}{|f(z)|(1-\rho)^2}$$

$$= \frac{\rho}{4(1+\rho)^2}\left[\frac{1}{m} + Rd_{m,m}\right] \log \frac{\rho}{|f(z)|(1-\rho)^2} \qquad (5.13)$$

Since $\varphi(x) = x_2 - (x - B)^2 (B > 0)$ is an increasing function of x, we may substitute (5.13) into (5.12). The case $l = 2$ of the theorem follows at once. The proof Theorem 5 for the case $l = 1$ is similar to that of $l = 2$.

Ⅵ. APPLICATION 5

Let $H_\delta(\delta > 0)$ be the class of functions $f(z) = \sum_{n=0}^\infty a_n z^n$, each of which is regular in the unit circle $|z| < 1$ and satisfies

$$\overline{\lim_{\rho\to 1}} \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^\delta d\theta \leqslant H_\delta(f) < \infty.$$

Fradman[8] has proved that

$$|a_n| \leqslant (n+1)^{\frac{1}{\delta}-1} H^{\frac{1}{\delta}}_\delta(f), n = 0,1,2,\cdots, \qquad (6.1)$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})| d\theta \leqslant \left(\frac{1}{1-\rho}\right)^{\frac{1}{\delta}-1} H^{\frac{1}{\delta}}_\delta(f), \qquad (6.2)$$

for $0 < \delta < 1$.

Here we improve (6.1) and (6.2) to the following theorem.

Theorem 6. If $f(z) \in H_\delta$, then we have

$$|a_n| \leqslant (n+1)^{\frac{1}{\delta}-1} H^{\frac{1}{\delta}}_\delta(f)\left\{1 - \left(|a_0|^{\frac{\delta}{2\delta-1}} H^{\frac{-\delta}{\frac{\delta}{\delta}-1}}(f) - \frac{1}{n+1}\right)^2\right\}^{\frac{1}{4}K(\delta)}, \qquad (6.3)$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})| d\theta \leqslant \left(\frac{1}{1-\rho^{\frac{2\delta}{1-\delta}}}\right)^{\frac{1}{\delta}-1} H^{\frac{1}{\delta}}_\delta(f)\{1 - [|a_0|^{\frac{\delta}{2\delta-1}} H^{\frac{-\delta}{\frac{\delta}{\delta}-1}}(f)$$

$$- (1 - \rho^{\frac{2\delta}{1-\delta}})]^2\}^{\frac{1}{4}K(\delta)}. \qquad (6.4)$$

where

$$K(\delta) = 2 - \frac{1}{\delta}, \frac{1}{2} < \delta \leqslant \frac{2}{3}; \frac{1}{\delta} - 1 \text{ for } \frac{1}{3} < \delta < 1. \tag{6.5}$$

For the proof of Theorem 6, we require the following theorems.

Hausdorff — Young Theorem. Let $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$ be regular in $|z| < 1$ and $\varphi^P(z) \in L(0, 2\pi), 1 < P \leqslant 2$, Then

$$\sum_{n=0}^{\infty} |b_n|^{\frac{P}{P-1}} \leqslant \{\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{i\theta})|^P d\theta\}^{\frac{1}{P-1}}. \tag{6.6}$$

Lemma. *If* $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n \in H_{2\delta}, \frac{1}{2} < \delta \leqslant 1$, *then*

$$I_n(\varphi) = \sum_{k=0}^{n} |b_k|^2 \rho^{2k} \leqslant \left\{\frac{1 - \rho^{C_\delta(n+1)}}{1 - \rho^{C_\delta}}\right\}^{\frac{1}{\delta}-1} H_{2\delta}^{\frac{1}{\delta}}(\varphi)\{1 - \left[|b_0|^{\frac{2\delta}{2\delta-1}} H_{2\delta}^{\frac{-1}{2\delta-1}}(\varphi) - \frac{1 - \rho^{C_\delta}}{1 - \rho^{C_\delta(n+1)}}\right]^2\}^{\frac{1}{2}K(\delta)}, \tag{6.7}$$

where $C_\delta = 2\delta/1 - \delta$, and $K(\delta)$ is defined in (6.5).

Proof. If $\frac{1}{2} < \delta \leqslant \frac{2}{3}$, we take $\frac{1}{Q} = \frac{1}{\delta} - 1, \frac{1}{P} = 2 - \frac{1}{\delta}$ $A_n = |b_n|^2, B_n = \rho^{2n}, e_0 = 1$, and $e_n = 1 (n \geqslant 1)$ in (1.1) This gives

$$I_n(\varphi) \leqslant \left(\frac{1 - \rho^{C_\delta(n+1)}}{1 - \rho^{C_\delta}}\right)^{\frac{2}{\delta}-3}\left\{\left(\frac{1 - \rho^{C_\delta(n+1)}}{1 - \rho^{C_\delta}}\right) \cdot \sum_{k=0}^{n} |b_k|^{\frac{2\delta}{2\delta-1}}\right)^2 - \left(|b_0|^{\frac{2\delta}{2\delta-1}} \frac{1 - \rho^{C_\delta(n+1)}}{1 - \rho^{C_\delta}} - \sum_{k=0}^{n} |b_k|^{\frac{2\delta}{2\delta-1}}\right)^2\}^{\frac{1}{2}K(\delta)} \tag{6.8}$$

By substituting (6.6) into (6.8), the lemma for $\frac{1}{2} < \delta \leqslant \frac{2}{3}$ follows immediately.

If $\frac{2}{3} < \delta < 1$, we put $\frac{1}{Q} = 2 - \frac{1}{\delta}, \frac{1}{P} = \frac{1}{\delta} - 1, B_k = |b_k|^2, Ak = \rho^{2k}, e_0 = 1$ and $e_k = 0 (k \geqslant 1)$. The proof is similar to that of $\frac{1}{2} < \delta < \frac{2}{3}$.

Proof of Theorem 6. It is well known that $f(z) = B(z)F(z)$, where $B(z)$ is the Blacki function and $F(z) \neq 0$ in the unit circle $|z| < 1$. Then $\varphi_1(z) = B(z)F^{\frac{1}{2}}(z)$ *and* $\varphi_2(z) = F^{\frac{1}{2}}(z)$ are regular in $|z| < 1$ and $f(z) = \varphi_1(z)\varphi_2(z)$. We suppose $\varphi(z) = \sum_{k=0}^{\infty} b_k^{(l)} z^k (l = 1, 2)$.

It is trivial that $\varphi_1 \in H_{2\delta}$. And we have

$$H_{2\delta}(\varphi) = H_\delta(f), \tag{6.9}$$

$$a_n = \sum_{k=0}^{n} b_k^{(1)} b_{n-k}^{(2)}. \tag{6.10}$$

$$\left.\begin{aligned} a_0 &= b_0^{(1)} b_0^{(2)}, \\ b_0^{(1)} &= B(0) b_0^{(2)}. \end{aligned}\right\} \tag{6.11}$$

Thus, we obtain

$$|a_0| = |b_0^{(2)}|^2|B(0)| \leqslant |b_0^{(2)}|^2, \tag{6.12}$$

$$|b_0^{(1)}|^2 = |b_0^{(1)}||B(0)||b_0^{(2)}| \leqslant |b_0^{(1)}b_0^{(2)}| = |a_0|. \tag{6.13}$$

Hence we get that: (i) if $|a|^{\frac{\delta}{2\delta-1}} > A$, then

$$|b_0^{(2)}|^{\frac{2\delta}{2\delta-1}} - A \geqslant |a_0|^{\frac{\delta}{2\delta-1}} - A \geqslant 0$$

(*ii*) if $|a_0^{\frac{\delta}{2\delta-1}}| < A$, then

$$A - |b_0^{(1)}|^{\frac{2\delta}{2\delta-1}} \geqslant A - |a_0|^{\frac{\delta}{2\delta-1}} \geqslant 0;$$

which gives

$$\{1 - (|b_0^{(1)}|^{\frac{2\delta}{2\delta-1}} - A)^2\}\{1 - (|b_0^{(2)}|^{\frac{2\delta}{2\delta-1}} - A)^2\} \leqslant \{1 - (|a_0|^{\frac{2\delta}{2\delta-1}} - A)^2\}, \tag{6.14}$$

when

$(|b_0^{(l)}|^{\frac{2\delta}{2\delta-1}} - A)^2 \leqslant 1, (l = 1,2).$

By (6.10), we obtain

$$|a_n\rho^n|^2 = |\sum_{k=0}^{n} b_k^{(1)}\rho^k b_{n-k}^{(2)}\rho^{n-k}|^2 \leqslant \sum_{k=0}^{n} |b_k^{(1)}|^2\rho^{2k} \sum_{k=0}^{n} |b_k^{(2)}|^2\rho^{2k}$$

$$= I_n(\varphi_1)I_n(\varphi_2) \tag{6.15}$$

Combining (6.7), (6.14) (6.15), and putting $\rho \to 1$, we have (6.13).

We see that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(\rho e^{i\theta})|d\theta)\right\}^2 = \left\{\frac{1}{2\pi}\int_0^{2\pi}|\varphi_1(\rho e^{i\theta})\varphi_2(\rho e^{i\theta})|d\theta\right\}^2$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}|\varphi_1(\rho e^{i\theta})|^2 d\theta \frac{1}{2\pi}\int_0^{2\pi}|\varphi_2(\rho e^{i\theta})|^2 d\theta)$$

$$= \lim_{n\to\infty} I_n(\varphi_1)I_n(\varphi_2) \tag{6.16}$$

Then (6.4) is deduced at once from (6.7), (6.14) and (6.16). Thus the proof of Theorem 6 is completed.

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ON HILBERT'S INEQUALITY*

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Abstract

This paper gives some improvements of Hilbert's inequality. The main results are:

(i) $$\left|\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y\right|^4 \leqslant \pi^4\left\{\left(\int_0^{\infty}f^2(x)\mathrm{d}x\right)^2\right.$$

$$\left.\left(\int_0^{\infty}f^2(x)k(x)\mathrm{d}x\right)^2\right\}\left\{\left(\int_0^{\infty}g^2(x)\mathrm{d}x\right)^2-\left(\int_0^{\infty}g^2(x)k(x)\mathrm{d}x\right)^2\right\}$$

where $k(x)=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+t^2}c(t^2x)\mathrm{d}t-c(x), 1-c(x)+c(y)\geqslant 0$. *and* $f, g \geqslant 0$, The case $k(x)=0$ is Hibert's integral form.

(ii) $$\left|\sum_{r,s=1}^{n}\frac{a_rb_r}{r+s}\right|^2+\left|\sum_{\substack{r,s=1\\ r\neq s}}^{n}\frac{a_rb_s}{r-s}\right|^2 \leqslant \pi^2\left(\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2\right).$$

Take zero in place of the second term of the left part, then it reduces to Hilbert's discrete form. And

(iii) An improvment of Hardy — Hilbert Inequality.

Hilber's inequality is an important theorem for analytic function which may be put into two forms; integral and discrete

(A) $$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y \leqslant \pi\sqrt{\int_0^{\infty}f^2(x)\mathrm{d}x\int_0^{\infty}g^2(x)\mathrm{d}x}, f, g \geqslant 0$$

(B) $$\left|\sum_{r,s=1}^{n}\frac{a_rb_s}{r+s}\right| \leqslant \pi\sqrt{\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2}.$$

The following inequality is in the name of Hilbert:

(C) $$\left|\sum_{\substack{r,s=1\\ r\neq s}}^{n}\frac{a_rb_s}{r-s}\right|^2 \leqslant A\sqrt{\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2},$$

where $A=3\pi$, If a's and b's are real numbers, then $A=2\pi$; if $b=a$, then $A=\pi$(see [1,3]).

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Theorem 1. Let positiev functions $f(x)$, $g(x) \in L^2(0,\infty)$. Then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\right)^2 \leqslant \pi^4\left\{\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty (f^2(x)k(x)\mathrm{d}x)^2\right\}\left\{\left(\int_0^\infty g^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty g^2(x)k(x)\mathrm{d}x\right)^2\right\},$$

where $k(x) = \frac{2}{\pi}\int_0^\infty \frac{c(xt^2)}{1+t^2}\mathrm{d}t - c(x)$ *and* $1 - c(x) + c(y) \geqslant 0.$

Proof First, suppose that g = f and let c(x) be a real function such that $1 - c(x) + c(y) \geqslant 0$. By Cauchy — Schwarz inequality, we have

$$J = \int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}\mathrm{d}x\mathrm{d}y = \int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}[1 - c(x) + c(y)]\mathrm{d}x\mathrm{d}y \leqslant \sqrt{J_1J_2} \tag{2.2}$$

where

$$J_1 = \int_0^\infty\int_0^\infty \frac{f(y)}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}}(1 - c(x) + c(y))\mathrm{d}x\mathrm{d}y,$$

$$J_2 = \int_0^\infty\int_0^\infty \frac{f^2(x)}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}}(1 - c(x) + c(y))\mathrm{d}x\mathrm{d}y.$$

Since

$$J_1 = \pi\int_0^\infty f^2(y)\mathrm{d}y - \pi\int_0^\infty k(y)f^2(y)\mathrm{d}y, J_2 = \pi\int_0^\infty f^2(x)\mathrm{d}x + \int_0^\infty k(x)f^2(x)\mathrm{d}x, \tag{2.3}$$

the theorem follows from (2. 3) and (2. 2)

If $g \neq f$, note that

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y = \int_0^\infty\left(\int_0^\infty e^{-xw}f(x)\mathrm{d}x\int_0^\infty e^{-yw}g(y)\mathrm{d}y\right)\mathrm{d}w$$

$$\leq \sqrt{\int_0^\infty\left(\int_0^\infty e^{-xw}f(x)\mathrm{d}x\right)^2\mathrm{d}w\int_0^\infty\left(\int_0^\infty e^{-yw}g(y)\mathrm{d}y\right)^2\mathrm{d}w}$$

$$= \sqrt{\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}\mathrm{d}x\mathrm{d}y\int_0^\infty\int_0^\infty \frac{g(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y}.$$

Using Theorem 1 for g = f, the theorem follows at once.

Example 1. If $c(x) = \cos\sqrt{x}$, then $k(x) = \frac{1}{2}(e^{-\sqrt{x}} - \cos\sqrt{x})$.

2. If $c(x) = 1$, then $k(x) = 0$.

Theorem 2. Let $a_n \geqslant 0$ and

$$A(x) = \sum_{n=0}^\infty a_nx^n,\ A^*(x) = \sum_{n=0}^\infty \frac{a_nx^n}{n!}.$$

Then

$$\left\{\int_0^1 A^2(x)\mathrm{d}x\right\}^2 \leqslant \pi^2\left\{\left[\int_0^\infty (e^{-x}A^*(x))^2\mathrm{d}x\right]^2 - \left[\int_0^\infty (e^{-x}A^*(x))^2k(x)\mathrm{d}x\right]^2\right\},$$

where $k(x)$ is defined by (2. 1)

The case $k(x) = 0$ is Widder's Theorem. [2]

Proof Since

$$A(x) = \int_0^\infty e^{-t} A^*(xt)\mathrm{d}t = \frac{1}{x}\int_0^\infty e^{-\frac{u}{x}} A^*(u))\mathrm{d}u$$

and

$$\int_0^1 A^2(x)\mathrm{d}x = \int_0^1 \frac{1}{x^2}\mathrm{d}x\left\{\int_0^\infty e^{-\frac{u}{x}} A^*(u))\mathrm{d}u\right\}^2$$

$$= \int_1^\infty \mathrm{d}y \left\{\int_0^\infty e^{-uy} A^*(u))\mathrm{d}u\right\}^2 = \int_0^\infty \mathrm{d}w\left\{\int_0^\infty e^{-uw} a(u)\mathrm{d}u\right\}^2$$

where $a(u) = e^{-u} A^*(u)$, by Theorem 1 we have

$$\int_0^\infty A^2(x)\mathrm{d}x = \int_0^\infty\int_0^\infty \frac{a(u)a(v)}{u+v}\mathrm{d}u\mathrm{d}v \leqslant \pi \sqrt{\left(\int_0^\infty a^2(u)\mathrm{d}u\right)^2 - \left(\int_0^\infty a^2(u)k(u)\mathrm{d}u\right)^2}.$$

Thus the theorem is proved.

Theorem 3. Let a's and b's be arbitrary complex numbers. Then

$$\left|\sum_{r,s=1}^{n} \frac{a_r b_s}{r+s}\right|^2 + \left|\sum_{\substack{r,s=1\\ r\neq s}}^{n} \frac{a_r b_s}{r-s}\right|^2 \leqslant \pi^2 \sum_{r=1}^{n} |a_r|^2 \sum_{s=1}^{n} |b_s|^2. \quad (3.1)$$

Proof It is easy to deduce tnat

$$\int_{-\pi}^{\pi} t\left[\sum_{r=1}^{n} (-1)^r (a_r \cos(r^t) - b_r \sin(rt))\right]^2 \mathrm{d}t = 2\pi(S - T), \quad (3.2)$$

where $$S = \sum_{r,s=1}^{n} \frac{a_r b_s}{r+s}, T = \sum_{\substack{r,s=1\\ r\neq s}}^{n} \frac{a_r b_s}{r-s}.$$

Therefore

$$2\pi|S - T| \leq \pi\int_{-\pi}^{\pi}\left|\sum_{r=1}^{n} (-1)^r (a_r \cos(rt) - b_r \sin(rt))\right|^2 \mathrm{d}t$$

$$= \pi^2 \sum_{r=1}^{n} (|a_r|^2 + |b_r|^2). \quad (3.3)$$

It is import ant to notice that: $(i) \sum_{r,s=1}^{n} \frac{a_r b_s}{r+s} = \sum_{r,s=1}^{n} \frac{b_r a_s}{r+s}, \sum_{\substack{r,s=1\\ r\neq s}}^{n} \frac{a_r b_s}{r-s} = -\sum_{r\neq s}^{n} \frac{a_s b_r}{r-s}$,

(ii) Interchange a's and b's in (3.3), *we obtain*

$$2\pi|S + T| \leqslant \pi^2\left(\sum_{r=1}^{n} |a_r|^2 + \sum_{r=1}^{n} |b_r|^2\right). \quad (3.4)$$

Squaring (3.3) and (3.4) and then adding them, we have

$$|S|^2 + |T|^2 \leqslant \frac{\pi^2}{4}\left\{\sum_{r=1}^{n} (|a_r|^2 + |b_r|^2\right\}^2. \quad (3.5)$$

Take $a_r \Big/ \sqrt{\sum_{k=1}^{n} |a_k|^2}, b_r \Big/ \sqrt{\sum_{k=1}^{n} |b_k|^2}$ in place of a_r and b_r respectively in (3.5), then the result follows.

Corollary. If a's and b's are real numbers, then

$$|S| + |T| \leqslant \pi \sqrt{\sum_{r=1}^{n} a_r^2 \sum_{k=1}^{n} b_k^2}.$$

(3. 3) and (3. 4) in combination yield the assertionof (3. 6)

Theorem 4. Let $k(x,y)$ be a homogenous form of degree -1 and$k(x,y) \geqslant 0$.

Let $f(x)$, $g(x) \geqslant 0, p \geqslant q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $1 - b(x)c(y) + c(x)b(y) \geqslant 0$ and

$$\int_0^\infty k(1,y)y^{-\frac{1}{q}}\mathrm{d}y = \int_k^\infty (x,1)x^{-\frac{1}{p}}\mathrm{d}x = k,$$

then

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$\leqslant k^{\frac{1}{q}}\Big\{\int_0^\infty g^q(x)\mathrm{d}x\Big\}^{\frac{1}{q}-\frac{1}{p}}\Big\{\Big(k\int_0^\infty f^p(x)\mathrm{d}x\int_0^\infty g^q(x)\mathrm{d}x\Big)^2$$
$$-\Big(\int_0^\infty f^p(x)C(x)\mathrm{d}x\int_0^\infty g^q(x)b(x)\mathrm{d}x$$
$$-\int_0^\infty f^p(x)B(x)\mathrm{d}x\int_0^\infty g^q(x)c(x)\mathrm{d}x\Big)^2\Big\}^{\frac{1}{2p}} \tag{4. 1}$$

where

$$B(x) = \int_0^\infty k(1,w)w^{-\frac{1}{q}}b(xw)\mathrm{d}w,\ C(x) = \int_0^\infty k(1,w)w^{-\frac{1}{q}}c(xw)\mathrm{d}w,$$

$b(x) = c(x)$ is Hardy $-$ Hilbert Theorem.

The proof of this theorem is based on the following inequality [4]

$$\int_0^\infty F(x)G(x)\mathrm{d}x \leqslant \Big(\int_0^\infty G^q(x)\mathrm{d}x\Big)^{\frac{1}{q}-\frac{1}{p}}\Big\{\Big(\int_0^\infty F^p(x)\mathrm{d}x\int_0^\infty G^q(x)\mathrm{d}x\Big)^2$$
$$-\Big(\int_0^\infty F^p(x)c(wx)\mathrm{d}x\int_0^\infty G^q(x)b(wx)\mathrm{d}x - \int_0^\infty F^p(x)b(wx)\mathrm{d}x\int_0^\infty G^q(x)c(wx)\mathrm{d}x\Big)^2\Big\}^{\frac{1}{2p}} \tag{4. 2}$$

where $F,G \geqslant 0$, $1 - b(x)c(y) = b(y)c(x) \geqslant 0$

Now we come to prove the theorem. By (4. 2) we have

$$I = \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y = \int_0^\infty k(1,w)\mathrm{d}w\int_0^\infty f(x)g(wx)\mathrm{d}x$$
$$\leqslant \int_0^\infty k(1,w)\mathrm{d}w\Big(\int_0^\infty g^q(wx)\mathrm{d}x\Big)^{\frac{1}{q}-\frac{1}{p}}\Big\{\Big(\int_0^\infty f^p(x)\mathrm{d}x\int_0^\infty g^q(wx)\mathrm{d}x\Big)^2$$
$$-\Big(\int_0^\infty f^p(x)c(wx)\mathrm{d}x\int_0^\infty g^q(wx)b(wx)\mathrm{d}x - \int_0^\infty f^p(x)b(wx)\mathrm{d}x\int_0^\infty g^q(wx)c(wx)\mathrm{d}x\Big)^2\Big\}^{\frac{1}{2p}}$$

$$= \int_0^\infty k(1,w) w^{-\frac{1}{q}} \mathrm{d}w \left(\int_0^\infty g^q(x) \mathrm{d}x \right)^{\frac{1}{q}-\frac{1}{p}} \left\{ \left(\int_0^\infty f^p(x) \mathrm{d}x \int_0^\infty g^q(x) \mathrm{d}x \right)^2 \right.$$

$$\left. - \left(\int_0^\infty f^p(x) c(wx) \mathrm{d}x \int_0^\infty g^q(x) b(x) \mathrm{d}x - \int_0^\infty f^p(x) b(wx) \mathrm{d}x \int_0^\infty g^q(x) c(x) \mathrm{d}x \right)^2 \right\}^{\frac{1}{2p}}$$

$$= \left(\int_0^\infty g^q(x) \mathrm{d}x \right)^{\frac{1}{q}-\frac{1}{p}} \int_0^\infty K(1,w) w^{-\frac{1}{q}} [J_1(w) J_2(w)]^{\frac{1}{2p}} \mathrm{d}w. \tag{4.3}$$

where

$$J_1(w) = \int_0^\infty f^p(x) \mathrm{d}x \int_0^\infty g^q(x) \mathrm{d}x - \int_0^\infty f^p(x) c(wx) \mathrm{d}x \int_0^\infty g^q(x) b(x) \mathrm{d}x + \int_0^\infty f^p(x) b(wx) \mathrm{d}x \int_0^\infty g^q(x) c(x) \mathrm{d}x,$$

$$J_2(w) = \int_0^\infty f^p(x) \mathrm{d}x \int_0^\infty g^q(x) \mathrm{d}x + \int_0^\infty f^p(x) c(wx) \mathrm{d}x \int_0^\infty g^q(x) b(x) \mathrm{d}x - \int_0^\infty f^p(x) b(wx) \mathrm{d}x \int_0^\infty g^q(x) c(x) \mathrm{d}x.$$

Applying Holder inequality to (4.3), we have

$$\int_0^\infty k(1,w) w^{-\frac{1}{q}} J_1^{\frac{1}{2p}}(w) J_2^{\frac{1}{2p}}(w) \mathrm{d}w$$

$$\leqslant \left(\int_0^\infty k(1,w) w^{-\frac{1}{q}} \mathrm{d}w \right)^{\frac{1}{q}} \left\{ \int_0^\infty k(1,w) w^{-\frac{1}{q}} J_1(w) \mathrm{d}w \int_0^\infty k(1,w) w^{-\frac{1}{q}} J_2(w) \mathrm{d}w \right\}^{\frac{1}{2p}}$$

$$= k^{\frac{1}{q}} \left\{ \left(k \int_0^\infty f^p(x) \mathrm{d}x \int_0^\infty g^q(x) \mathrm{d}x \right)^2 - \left(\int_0^\infty f^q(x) C(x) \mathrm{d}x \int_0^\infty g^q(x) b(x) \mathrm{d}x \right. \right.$$

$$\left. \left. - \int_0^\infty f^p(x) B(x) \mathrm{d}x \int_0^\infty g^q(x) c(x) \mathrm{d}x \right)^2 \right\}^{\frac{1}{2p}}. \tag{4.4}$$

In virture of (4.3) and (4.4), the proof is complete.

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