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潜水井稳定流的解析解

An Analytical Solution of Unconfineb Steady Flow Toward a Well

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提　　要

本文用 Dagan的二阶线性化理论，求得潜水井稳定流的解析解。用杨式德教授的数值解相验证基本符合。为测定潜水含水层的渗透系数提供了一个理论基础。据该式可以找到裘布衣公式在离井不同距离处的适用深度，于是与引用补给半径的概念相结合，便有可能根据众所周知的裘布衣公式，布置抽水试验和计算渗透系数。

An Analytical Solution of Unconfined Steady Flow Toward a Well

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Abstract

This paper presents an analytical solution of unconfined flow toward a well by Dagan's second order linearization theory. It is proved to agree in the main with Professor Yang Shi De's numerical solution, and furnishes a theoretical base for identifying permeability coefficient of phreatic aquifer. According to that solution the applicable depth of Dupuit solution at various distances from the well may be fixed. Therefore, combined with the concept of presumed radius of replenishment it is possible to design pumping test and to calculate permeability coefficient with the well-known Dupuit solution.

目　　录

潜水井稳定流的解析解

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前　　言

潜水井势场的解析解是测定潜水层渗透系数，计算潜水自由面高度和含水层中各点水头的理论基础。

潜水自由面作为渗流场的一个边界，在问题得到解答之前是未知的，这就为解题带来很大困难。自十九世纪中叶开始地下水的渗流研究以来，直至1949年由杨式德教授在哈佛大学的博士论文中用迭弛法(Relaxation method)根据拉普拉斯方程求得了数值解，才第一次从理论上算得潜水井的自由水面及等势线(载于参考文献1第85页)。1964年柯卡姆(D.Kirkham)提出了潜水井的解析解②，但这个解中的一个系数，系随计算网格的密度而变，实际上仍只能算得离散的值，加之算法繁冗，故在实用上未能推广。

自电子计算机应用以来，德赛（Desai 1970)用有限差分法也算得类似杨式德的数值解。

自1954年由W·S·Boulton开始而后由Dagan（1967）、Stretesova（1972）及Neuman(1974，1975)发展的潜水非稳定流的解析解，则在井的边界上用均匀的流速代替了随Z变化的水头，并将降深限于最小等简化条件，因而反映不出潜水流的特点。

本文用G·Dagan的二阶线性化理论，导出了二阶近似的解析解。并用电子计算机作了计算，计算结果与杨式德的数值解作了对比，基本符合。

一、潜水自由面方程

潜水自由面方程最早是由柯静娜给出的，该方程是根据水质点一旦位于自由面即不再离去的原理推导的。

在流体力学中研究液体的变形或位移时，对其某个质点在时间t的位置常用座标（x、y、z）来辨别，同时又假定该质点在另一参考时刻t的位置为（X、Y、Z），因而可按（X、Y、Z）找到该质点在时间t的位置。又考虑（x，y，z）是参考位置（X、Y、Z）及t的函数，因此只要用：

$$x = x(X、Y、Z、t)$$
$$y = y(X、Y、Z、t)$$
$$z = z(X、Y、Z、t) \qquad (1.1)$$

等函数关系，便可全面的描述液体的形变或其质点的位移。（X、Y、Z）又称物质座标。

当一个水质点在空间作曲线运动时，其速度分量便可按其位移分量的时间变率给出为：

$$U_x = DX/Dt$$
$$U_y = DY/Dt$$
$$U_z = DZ/Dt \qquad (1.2)$$

式中D/Dt代表当物质座标（X、Y、Z）保持为常数时，对时间t的偏微商，也称物质微分法(Material defferenfiation)或称追随质点的微分法(Defferentiatwion folloiwng the particle)。

设f是某一种流体流动时的变量，（例如在潜水渗流的具体问题中，可以将f看作是联系于水质点的势 φ）且是x、y、z及时间t的函数，f的物质微商（Material derivative)就是它对时间的偏微商。由于f＝f（x、y、z、t），而x、y、z又是独立变量X、Y、Z及时间t的函数。因此根据复合函数求微商的方法，可写出f的物质微商表达式：

$$\frac{Df(x,x,y,t)}{Dt}=\frac{\partial f(x,y,z,t)}{\partial t}+\frac{\partial f(x,y,z,t)}{\partial x}\cdot\frac{Dx(X,Y,Z,t)}{Dt}$$

$$+\frac{\partial f(x,y,z,t)}{\partial y}\cdot\frac{Dy(X,Y,Z,t)}{Dt}+\frac{\partial f(x,y,z,t)}{\partial z}$$

$$\times\frac{Dz(X,Y,Z,t)}{Dt} \tag{1.3}$$

利用式（1.2）得

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+U_x\frac{\partial f}{\partial x}+U_y\frac{\partial f}{\partial y}+U_z\frac{\partial f}{\partial z} \tag{1.4}$$

式中$\frac{Df}{Dt}$是指当某一质点在空间运动时联系于此质点的标量f对时间的变化率。如这个质点静止不动了，即$U_x=U_y=U_z=0$，或限制在垂直于f的梯度而运动，即$\partial f/\partial x=\partial f/\partial y=\partial f/\partial z=0$，则式（1.4）右侧后三项均为零，对Df/Dt有贡献的仅是此质点所占空间的某一点上f的时间变率$\partial f/\partial t$。另一情况，如空间各点的f值不随时间而变，则质点的运动便与f的变化方向一致，于是式（1.4）中的$\frac{Df}{Dt}=0$。如f既随时间又随空间而变，则式（1.4）中四项都应保留。

在潜水的渗流中由于自由面上的压力为零，即自由面上某点的势等于该点的高度z。故得：

$$\varphi(x,y,z,t)-z(x,y,z,t)=0 \tag{1.5}$$

又因自由面上的水质点始终不离自由面，（除非该质点脱离含水层）故自由面上任意质点的势的时间变率可用其物质微商$\frac{D\varphi}{Dt}$表示。

式（1.5）表明在自由面上势差$\varphi-z=0$故

$$\frac{D(\varphi-z)}{Dt}=0 \tag{1.6}$$

将式（1.4）代入式（1.6）得：

$$\frac{\partial(\varphi-z)}{\partial t}+U_x\frac{\partial(\varphi-z)}{\partial x}+U_y\frac{\partial(\varphi-z)}{\partial y}+U_z\frac{\partial(\varphi-z)}{\partial z}=0 \tag{1.7}$$

式中U_x，U_y，U_z都是水质点的实际流速，其与渗速V的关系为：

$$V_x = n\cdot U_x = -k\frac{\partial\varphi}{\partial x}$$

$$V_y = n\cdot U_y = -k\frac{\partial\varphi}{\partial y}$$

$$V_z = n\cdot U_z = -k\frac{\partial\varphi}{\partial z}$$

代入后即得本题所需的自由面方程：

$$n\frac{\partial\varphi}{\partial t} + U_x\frac{\partial\varphi}{\partial x} + U_y\frac{\partial\varphi}{\partial y} + U_z\left(\frac{\partial\varphi}{\partial z} - 1\right) = 0 \qquad (1.8)$$

或

$$\frac{\partial\varphi}{\partial t} = \frac{k}{n}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 - \frac{\partial\varphi}{\partial z}\right\} \qquad (1.9)$$

在轴对称问题中式（1.9）变为：

$$\frac{\partial\varphi}{\partial t} = \frac{k}{n}\left\{\left(\frac{\partial\varphi}{\partial r}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 - \frac{\partial\varphi}{\partial z}\right\} \qquad (1.10)$$

二、Dagan的二阶线性化理论

有了潜水自由面的严格表达式，潜水井的微分方程和定解条件就有可能列出。如柯静娜（1952）对不可压缩的均匀流体通过各向同性、均质，不变形的孔隙介质的饱和渗流，写出了在势场内速势 $\varphi = -k[q/r + z]$ 应满足的方程

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (2.1)$$

在自由面 $z = \eta(x, y, t)$ 上应满足：

$$n\frac{\partial\varphi}{\partial t} + k\frac{\partial\varphi}{\partial z} + (\nabla\varphi)^2 \equiv \eta\frac{\partial\varphi}{\partial t} + k\frac{\partial\varphi}{\partial z} + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = 0 \qquad (2.2)$$

及自由面高度

$$\eta = -\frac{1}{k}\varphi(x, y, z, t) \qquad (2.3)$$

式中k为介质的渗透系数即

$$k = k'\gamma/\mu$$

k′是介质骨架的透水系数，γ为流体的密度，μ为动力粘滞系数，n是有效孔隙系数。

但正如本文一开始提到的，由于方程（2.3）是未知的，再加上方程（2.2）的非线性，使得解潜水井问题变得非常困难。

1964年Dagan引进了二阶线性化理论后，为这一难题的解决提供了可能，Dagan写道[②]“以自由面为边界的理想流体的无旋运动方程和服从达西定律的孔隙介质中具自由面液体的渗流，存在着某种相似性；两者的速势都是调和函数，而在自由面上的运动学（ Kinematic ）条件也是相同的。但动力学（ Dynamics ）条件不同，前者根据伯努力关系产生双曲型方程（守恒运动）而后者由达西定律产生抛物型方程（弥散运动）但两者又都以自由面上势能边界条件的非线性，构成了问题的数学困难。然而在理想流体流动理论中（特别是水的波浪运

动）线性化理论已有相当的进展，一阶，二阶和更高阶的逼近已推导出来。” Dagan 就是将这已推导出的理论，借用到潜水渗流中来。从何解决了上面存在的困难。

这一理论基本假想就是速势 φ和自由面高度 η都可以展开成小参数 ε的幂级数，即：

$$\varphi(x,y,z,t)=\varphi_0(x,y,z,t)+\varepsilon\varphi_1(x,y,z,t)+\varepsilon^2\varphi_2(x,y,z,t)+\cdots\cdots \quad (2.4)$$

$$\eta(x,y,z,t)=\eta_0(x,y)+\varepsilon\eta_1(x,y,t)+\varepsilon^2\eta_2(x,y,t)+\cdots\cdots \quad (2.5)$$

这里ε的意义先暂不予规定，显然它具有摄动的性质。φ_0，η_0 则代表不受扰动的稳定状态。

将式（2.4）代入式（2.1）得：

$$\nabla^2\varphi=\nabla^2\varphi_0+\varepsilon\nabla^2\varphi_1+\varepsilon^2\nabla^2\varphi_2+\cdots\cdots=0 \quad (2.6)$$

由于$\varepsilon\neq0$，故在势场内必得

$$\left.\begin{aligned}&\frac{\partial^2\varphi_0}{\partial x^2}+\frac{\partial^2\varphi_0}{\partial y^2}+\frac{\partial^2\varphi_0}{\partial z^2}=0\\&\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}+\frac{\partial^2\varphi_1}{\partial z^2}=0\\&\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial y^2}+\frac{\partial^2\varphi_2}{\partial z^2}=0\\&\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \quad (2.7)$$

亦即在势场内的零阶，一阶，二阶……的势 φ_0，φ_1，φ_2…… 均满足拉普拉斯方程。

各阶势函数在自由面上应予满足的条件，可从方程（2.2），（2.4）及（2.5）得出。首先在自由面上的φ值及其导数，可用台劳级数在 $z=\eta_0\ (x,y)$ 曲面上展开的方法导出。设z只是在z_0上下一个极小的范围内变化，即 $z-z_0=\delta\eta$，$\delta\eta$是一可正可负的极小值，则

$$\varphi(x,y,z,t)=\varphi(x,y,z_0+\delta\eta,t)=\varphi(x,y,z_0,t)+\left.\frac{\partial\varphi}{\partial z}\right|_{z=z_0}\times\delta\eta+\frac{1}{2!}\left.\frac{\partial^2\varphi}{\partial z^2}\right|_{z=z_0}\times(\delta\eta)^2+\cdots\cdots \quad (2.8)$$

又从式（2.5）得：

$$\delta\eta=\varepsilon\eta_1+\varepsilon^2\eta_1+\cdots\cdots \quad (2.9)$$

以式（2.4）及（2.9）代入（2.8）得：

$$\begin{aligned}\varphi(x,y,\eta_0,t)&=\varphi_0(x,y,\eta_0,t)+\varepsilon\varphi_1(x,y,\eta_0,t)+\varepsilon^2\varphi_2(x,y,\eta_0,t)+\cdots\cdots\\&+\left[\left.\frac{\partial\varphi_0}{\partial z}\right|_{z=\eta_0}+\varepsilon\left.\frac{\partial\varphi_1}{\partial z}\right|_{z=\eta_0}+\varepsilon^2\left.\frac{\partial\varphi_1}{\partial z}\right|_{z=\eta_0}+\cdots\cdots\right]\times(\varepsilon\eta_1+\varepsilon^2\eta_2+\cdots\cdots)\\&+\frac{1}{2}\left[\left.\frac{\partial^2\varphi_0}{\partial z^2}\right|_{z=\eta_0}+\varepsilon\left.\frac{\partial^2\varphi_1}{\partial z^2}\right|_{z=\eta_0}+\varepsilon^2\left.\frac{\partial^2\varphi_2}{\partial z^2}\right|_{z=\eta_0}+\cdots\cdots\right]\\&\times(\varepsilon^2\eta_1^2+2\varepsilon^2\eta_1\eta_2+\varepsilon^4\eta_0^2+\cdots\cdots)\end{aligned}$$

展开上式，合并同类项得ε的二次以下幂项的系数，见下表。

ε 的零次幂项	ε 的一次幂项	ε 的二次幂项
$\varphi_0(x,y,\eta_0,t)$	$\varphi_1(x,y,\eta_0,t)$ $+\left.\frac{\partial\varphi_0}{\partial z}\right\|_{z=\eta_0}\times\eta_1$	$\varphi_2(x,y,\eta_0,t)$ $+\left.\frac{\partial\varphi_1}{\partial z}\right\|_{z=\eta_0}\times\eta_1$ $+\left.\frac{\partial\varphi_0}{\partial z}\right\|_{z=\eta_0}\times\eta_2$ $+\frac{1}{2}\left.\frac{\partial^2\varphi_0}{\partial z^2}\right\|_{z=\eta_0}\times\eta_1^2$

因此得

$$\varphi(x,y,z,t)\Big|_{Z=\eta}=\varphi_0\Big|_{Z=\eta_0}+\varepsilon\left(\eta_1\frac{\partial\varphi_0}{\partial z}\Big|_{Z=\eta_0}+\varphi_1\Big|_{Z=\eta_0}\right)+\varepsilon^2\left(\eta_2\frac{\partial\varphi_0}{\partial z}\Big|_{Z=\eta_0}+\frac{\eta_1^2}{2}\frac{\partial^2\varphi_0}{\partial z^2}\Big|_{Z=\eta_0}+\eta_1\frac{\partial\varphi_1}{\partial z}\Big|_{Z=\eta_0}+\varphi_2\Big|_{Z=\eta_0}\right)+\cdots\cdots \quad (2.10)$$

由式（2.3）有

$$-k(\eta_0+\varepsilon\eta_1+\varepsilon^2\eta_2+\cdots\cdots)=\varphi(x,y,z,t)$$

将式（2.10）代入上式得

$$-k(\eta_0+\varepsilon\eta_1+\varepsilon^2\eta_2+\cdots\cdots)\equiv\varphi_0\Big|_{Z=\eta_0}+\varepsilon\left(\eta_1\frac{\partial\varphi_0}{\partial z}\Big|_{Z=\eta_0}+\varphi_1\Big|_{Z=\eta_0}\right)+\varepsilon^2\left(\eta_2\frac{\partial\varphi_0}{\partial z}\Big|_{Z=\eta_0}+\frac{\eta_1^2}{2}\frac{\partial^2\varphi_2}{\partial z^2}\Big|_{Z=\eta_0}+\eta_1\frac{\partial\varphi_1}{\partial z}\Big|_{Z=\eta_0}+\varphi_2\Big|_{Z=\eta_0}\right)+\cdots\cdots$$

上列恒等式的相应各项应相等，故得零阶，一阶及二阶的自由面高度：

$$\left.\begin{aligned}
\eta_0&=-\frac{1}{k}\varphi_0\Big|_{Z=\eta_0}\\
\eta_1&=-\frac{\varphi_1}{k+\partial\varphi_0/\partial z}\Big|_{Z=\eta_0}\\
\eta_2&=-\frac{\eta_1\left(\frac{\eta_1}{2}\frac{\partial\varphi_0}{\partial z}+\frac{\partial\varphi_1}{\partial z}\right)+\varphi_2}{k+(\partial\varphi/\partial z)}\Bigg|_{Z=\eta_0}
\end{aligned}\right\} \quad (2.11)$$

为求零阶，一阶及二阶的自由面方程，用类似的方法展开 φ的偏导数：

$$\frac{\partial\varphi}{\partial t}=\frac{\partial\varphi_0}{\partial t}+\varepsilon\left(\eta_1\frac{\partial^2\varphi_0}{\partial z\partial t}+\frac{\partial\varphi_1}{\partial t}\right)+\varepsilon^2\left(\eta_2\frac{\partial^2\varphi_0}{\partial z\partial t}+\frac{\eta_1^2}{2}\frac{\partial^2\varphi_0}{\partial z\partial t}+\eta_1\frac{\partial^2\varphi_1}{\partial z\partial t}+\frac{\partial\varphi_2}{\partial t}\right)+\cdots\cdots \quad (2.12)$$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi_0}{\partial x}+\varepsilon\left(\eta_1\frac{\partial^2\varphi_0}{\partial z\partial x}+\frac{\partial\varphi_1}{\partial x}\right)+\varepsilon^2\left(\eta_2\frac{\partial^2\varphi_0}{\partial z\partial x}+\frac{\eta_1^2}{2}\frac{\partial^3\varphi_0}{\partial z^2\partial x}+\eta_1\frac{\partial^2\varphi_1}{\partial z\partial x}+\frac{\partial\varphi_2}{\partial z}\right)+\cdots\cdots \quad (2.13)$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi_0}{\partial y}+\varepsilon\left(\eta_1\frac{\partial^2\varphi_0}{\partial z\partial y}+\frac{\partial\varphi_1}{\partial y}\right)+\varepsilon^2\left(\eta_2\frac{\partial^2\varphi_0}{\partial z\partial y}+\frac{\eta_1^2}{2}\frac{\partial^3\varphi_0}{\partial z^2\partial y}+\eta_1\frac{\partial^2\varphi_1}{\partial z\partial y}+\frac{\partial\varphi_2}{\partial y}\right)+\cdots\cdots \quad (2.14)$$

$$\frac{\partial\varphi}{\partial z}=\frac{\partial\varphi_0}{\partial z}+\varepsilon\left(\eta_1\frac{\partial^2\varphi_0}{\partial z^2}+\frac{\partial\varphi_1}{\partial z}\right)+\varepsilon^2\left(\eta_2\frac{\partial^2\varphi_0}{\partial z^2}+\frac{\eta_1^2}{2}\frac{\partial^3\varphi_0}{\partial z^3}+\eta_1\frac{\partial^2\varphi_1}{\partial z^2}+\frac{\partial\varphi_0}{\partial z}\right)+\cdots\cdots \quad (2.15)$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^2=\left(\frac{\partial\varphi_0}{\partial x}\right)^2+2\varepsilon\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x}+2\varepsilon\eta_1\frac{\partial\varphi_0}{\partial x}\frac{\partial}{\partial z}\frac{\partial\varphi_0}{\partial x}+2\varepsilon^2\eta_1\frac{\partial^2\varphi_0}{\partial z\partial x}\frac{\partial\varphi_1}{\partial x}+\varepsilon^2\left(\frac{\partial\varphi_1}{\partial x}\right)^2$$

$$+2\varepsilon^2\eta_2\frac{\partial\varphi_0}{\partial x}\frac{\partial^2\varphi_2}{\partial z\partial x}$$

$$+\varepsilon^2\eta_1^2\frac{\partial\varphi_0}{\partial x}\frac{\partial^3\varphi_0}{\partial z^2\partial x}+\varepsilon^2\eta_1^2\left(\frac{\partial^2\varphi_0}{\partial z\partial x}\right)^2+2\varepsilon^2\eta_1\frac{\partial\varphi_0}{\partial x}\frac{\partial^2\varphi_1}{\partial z\partial x}+2\varepsilon^2\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_2}{\partial x}$$

$$+O(\varepsilon^3)+O(\varepsilon^4)+\cdots\cdots \quad (2.16)$$

$\left(\frac{\partial\varphi}{\partial y}\right)^2$ 与 $\left(\frac{\partial\varphi}{\partial z}\right)^2$ 的展开式与式（2.16）相似就不再重复。

由于

$$\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\frac{\partial^2\varphi_0}{\partial z\partial x}\right)=\frac{\partial\varphi_0}{\partial x}\frac{\partial^3\varphi_0}{\partial z^2\partial x}+\left(\frac{\partial^2\varphi_0}{\partial z\partial x}\right)^2$$

及

$$\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x}\right)=\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_0}{\partial z\partial x}-\frac{\partial\varphi_0}{\partial x}\frac{\partial^2\varphi_1}{\partial z\partial x}$$

式（2.16）又可改写为：

$$\left(\frac{\partial\varphi}{\partial x}\right)^2=\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\varepsilon\left[2\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x}+\eta_1\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\right)^2\right]+\varepsilon^2\left[\left(\frac{\partial\varphi_1}{\partial x}\right)^2\right.$$

$$+2\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_2}{\partial x}+\eta_2\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\eta_1^2\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\frac{\partial}{\partial z}\frac{\partial\varphi_0}{\partial x}\right)$$

$$\left.+2\eta_1\frac{\partial}{\partial z}\left(\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x}\right)\right]+\cdots\cdots \tag{2.17}$$

将式（2.12），（2.13），（2.14），（2.15）及（2.17）等代入式（2.2）合并同类项，并考虑φ_0是稳定的，即$\frac{\partial\varphi_0}{\partial t}=0$故得自由面的展开式

$$(\nabla\varphi_0)^2+k\frac{\partial\varphi_0}{\partial z}+\varepsilon\left[\eta_1\frac{\partial\varphi_1}{\partial t}+2\nabla\varphi_0\cdot\nabla_1\varphi+\eta_1\frac{\partial}{\partial z}(\nabla\varphi_0)^2\right.$$

$$\left.+k\left(\eta_1\frac{\partial^2\varphi_0}{\partial z^2}+\frac{\partial\varphi_1}{\partial z}\right)\right]+\varepsilon^2\left[\eta\left(\eta_1\frac{\partial^2\varphi_1}{\partial z\partial t}+\frac{\partial\varphi_2}{\partial t}\right)+2\nabla\varphi_0\cdot\nabla\varphi_2\right.$$

$$+(\nabla\varphi_1)^2+\eta_2\frac{\partial}{\partial z}(\nabla\varphi_0)^2+\left[\eta_1^2\frac{\partial}{\partial z}\left(\nabla\varphi_0\cdot\frac{\partial}{\partial z}\nabla\varphi_1\right)+2\eta_1\frac{\partial}{\partial z}(\nabla\varphi_0\cdot\nabla\varphi_1)\right.$$

$$\left.+k\left(\eta_2\frac{\partial^2\varphi_0}{\partial z^2}+\frac{\eta_1^2}{2}\frac{\partial^3\varphi_0}{\partial z^3}+\eta_1\frac{\partial^2\varphi_1}{\partial z^2}+\frac{\partial\varphi_2}{\partial z}\right)\right]+\cdots\cdots=0 \tag{2.18}$$

式中

$$(\nabla\varphi_0)^2=\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\left(\frac{\partial\varphi_0}{\partial y}\right)^2+\left(\frac{\partial\varphi_0}{\partial z}\right)^2$$

$$\nabla\varphi_0\cdot\nabla\varphi_1=\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_0}{\partial y}\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_0}{\partial z}\frac{\partial\varphi_1}{\partial z}$$

最终对不同幂的ε可以从式（2.7），（2.18）及（2.11）得出零阶、一阶、二阶的微分方程、自由面边界条件及自由面高度方程：

$\nabla^2\varphi_0=0$； $z<\eta_0$

$$(\nabla\varphi_0)^2 + k\frac{\partial\varphi_0}{\partial z} = 0 \qquad z=\eta_0 \tag{2.19}$$

$$\eta_0 = -\frac{\varphi_0}{k}$$

$$\nabla^2\varphi_1 = 0 \tag{2.20}$$

$$\eta\frac{\partial\varphi_1}{\partial t} + 2\nabla\varphi_0\cdot\nabla\varphi_1 + \eta_1\frac{\partial}{\partial z}(\nabla\varphi_0)^2$$

$$+ k\left(\eta_1\frac{\partial^2\varphi_0}{\partial z^2} + \frac{\partial\varphi_1}{\partial z}\right) = 0 \tag{2.21}$$

$$z = \eta_0$$

$$\eta_1 = -\frac{\varphi_1}{k+\partial\varphi_0/\partial z} \tag{2.22}$$

$$\nabla^2\varphi_2 = 0 \tag{2.23}$$

$$\eta\left(\eta_1\frac{\partial^2\varphi_1}{\partial z\partial t} + \frac{\partial\varphi_2}{\partial t}\right) + 2\nabla\varphi_0\cdot\nabla\varphi_2 + (\nabla\varphi_1)^2 + \eta_2\frac{\partial}{\partial z}(\nabla\varphi_0)^2$$

$$+\eta_1^2\frac{\partial}{\partial z}\left(\nabla\varphi_0\cdot\frac{\partial}{\partial z}\nabla\varphi_0\right) + 2\eta_1\frac{\partial}{\partial z}(\nabla\varphi_0\cdot\nabla\varphi_1) + k\left(\eta_2\frac{\partial^2\varphi_0}{\partial z^2}\right.$$

$$\left.+\frac{\eta_1^2}{2}\frac{\partial^3\varphi_0}{\partial z^3} + \eta_1\frac{\partial^2\varphi_1}{\partial z^2} + \frac{\partial\varphi_2}{\partial z}\right) = 0 \tag{2.24}$$

$$z = \eta_0$$

$$\eta_2 = \frac{\eta_1\left(\frac{\eta_1}{2}\frac{\partial^2\varphi_0}{\partial z^2} + \frac{\partial\varphi_1}{\partial z}\right) + \varphi_2}{k + (\partial\varphi_0/\partial z)} \tag{2.25}$$

显然这样的方程可以推导到更高阶项。

方程（2.19）对于φ_0是稳定的确定方程。方程（2.20）～（2.22）及（2.23）～(2.25)是基于稳定的自由面$z=\eta_0$的一阶、二阶项的线性化方程，虽然在式（2.21）中出现二次项$(\nabla\varphi_0)^2$及在（2.24）中也出现了二次方项$(\nabla\varphi_0)^2$及$(\nabla\varphi_1)^2$，但由于在解φ_1时φ_0是已经解出的已知函数，在解φ_2时φ_1、φ_0也是已知函数，故已保证方程是线性的。如上推导，将φ展开为小参数ε的幂级数，使自由面方程线性化，并被转移到$z=0$的已知平面上，这样便解决了解潜水井的巨大困难。

三、稳定的潜水井

在图（1）所示的模式中（注意为推导方便图中将座标原点定在潜水自由面的最大降落点，即井边渗漏面的上端）令$\varphi=p/r+z$，则φ的微分方程和定解条件可列出如下：

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3.1}$$

$r=R$； $\qquad\qquad\qquad\qquad\qquad\qquad \varphi=+h_*$ （3.2）

$r=r_w$， $\quad -(h_0-h_*)\leqslant z\leqslant -h_s$； $\quad \varphi=-h_s$ （3.3）

$r=r_w$， $\quad -h_s\leqslant z\leqslant 0$； $\quad \varphi=z$ （3.4）

$$r_w < r < R, \qquad z = -(h_0 - h_s); \qquad \frac{\partial\varphi}{\partial z} = 0 \tag{3.5}$$

$$\text{自由面,} \quad -\frac{\partial\varphi}{\partial z} + \left(\frac{\partial\varphi}{\partial r}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = 0 \tag{3.6}$$

式（3.6）是将$\partial\varphi/\partial t = 0$代入式（1.10）得出的。令

$$\varepsilon = h_*/R$$

即令小参数ε为自由面的最大降落h_*与含水层的半径R之比，因ε的数量级一般小于10^{-2}。故在实际工作中，用二阶线性化理论已可保证使获得的解有足够的精度，并对于非扰动状态，则因$h_* = 0$，而使渗流处于静止状态故得：

$$\varphi_0 = 0$$

$$\eta_0 = 0$$

因此图（1）中的

$$\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 \tag{3.7}$$

及自由面高度

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 \tag{3.8}$$

于是可根据二阶线性化理论将图（1）的势场分解成图（2）、图（3）两个势场。第一个势场的微分方程和边界条件为：

$$\frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_1}{\partial r} + \frac{\partial^2\varphi_1}{\partial z^2} = 0 \tag{3.9}$$

$$r = R; \qquad \varphi_1 = +\frac{h_*}{\varepsilon} = +R \tag{3.10}$$

$$r = r_w, \qquad -(h_0 - h_*) \leqslant z \leqslant -h_s; \qquad \varphi_1 = \frac{-h_s}{\varepsilon} = -\frac{h_s}{h_*}R \tag{3.11}$$

$$r = r_w, \qquad -h_s < z \leqslant 0; \qquad \varphi_1 = \frac{+z}{\varepsilon} = +\frac{R}{h_*}z \tag{3.12}$$

$$r_w \leqslant r \leqslant R, \qquad z = -(h_0 - h_*); \qquad \frac{\partial\varphi_1}{\partial z} = 0 \tag{3.13}$$

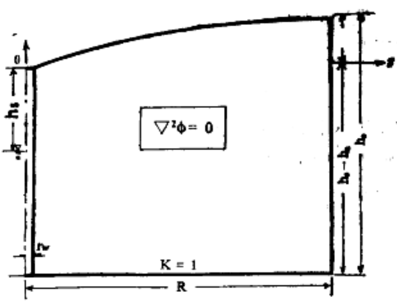
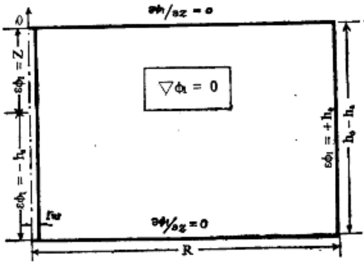


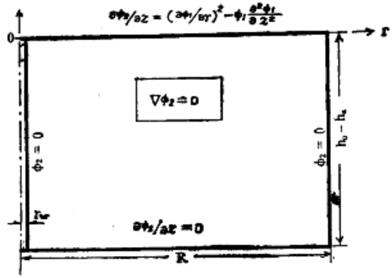
图 (1)

$r_w \leqslant r \leqslant R, \quad z = 0;$ $$\frac{\partial \varphi_1}{\partial z} = 0 \qquad (3.14)$$

式（3.14）是用$\frac{\partial \varphi_1}{\partial t} = 0$，$\varphi_0 = 0$，$\frac{\partial \varphi_0}{\partial z} = 0$，$k = -1$代入式（2.21）而得。这里应予指出：由于Dagan的线性理论是从“速势”$\varphi = -k\left(\frac{p}{r} + z\right)$出发的，而本文是从势$\varphi = \frac{p}{r} + z$出发的，两种φ相差$-k$倍，故在应用式（2.21）和（2.24）时应令其中的$k = -1$。



图（2）



图（3）

第二个势场的微分方程和边界条件为：

$$\frac{\partial^2 \varphi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \varphi_2}{\partial z^2} = 0 \qquad (3.15)$$

$r = R;$ $\qquad \varphi_2 = 0 \qquad (3.16)$

$r = r_w;$ $\qquad \varphi_2 = 0$

$z = -(h_0 - h_w);$ $\qquad \frac{\partial \varphi_2}{\partial z} = 0 \qquad (3.18)$

$z = 0;$ $\qquad \frac{\partial \varphi_2}{\partial z} = \left(\frac{\partial \varphi_1}{\partial r}\right) - \varphi_1 \frac{\partial^2 \varphi_1}{\partial z^2} \qquad (3.19)$

式(3.19)是用$\partial\varphi_2/\partial t = = 0$，$\partial\varphi_1/\partial t = 0$，$\varphi_0 = 0$，

$\partial\varphi_1/\partial z = 0 \Big|_{z=0}$，$k = -1$代入式（2.24）而得。

先解第一个势场，由（3.9），（3.13），（3.14）得：

$$\varphi_1 = a_0 + b_0 \ln r + \sum_{p=1}^{\infty} b_p U_p(r) \cos \frac{p\pi z}{h_0 - h_w} \qquad (3.20)$$

式中

$$U_p(r) = K_0\left(\frac{p\pi r}{h_0 - h_w}\right) - \frac{K_0\left(\frac{p\pi R}{h_0 - h_w}\right)}{I_0\left(\frac{p\pi R}{h_0 - h_w}\right)} I_0\left(\frac{p\pi r}{h_0 - h_w}\right) \qquad (3.21)$$

由式（3.10）得

$$\varphi_1 = a_0 + b_0 \ln r = R$$

$$a_0 = R - b_0 \ln R \tag{3.22}$$

代入式（3.20）得

$$\varphi_1 = R - b_0 \ln\frac{R}{r} + \sum_{p=1}^{\infty} b_p U_p(r) \cos\frac{p\pi z}{h_0 - h_a} \tag{3.23}$$

为确定b_p，将式（3.11），（3.12）代入式（3.23）得

$$-b_0 \ln\frac{R}{r_w} + \sum_{p=1}^{\infty} b_p U_p(r_w) \cos\frac{p\pi z}{h_0 - h_a}$$

$$= \begin{cases} -\left(1+\frac{h_a}{h_s}\right)R; \quad -(h_0 - h_a) \leqslant z \leqslant -h_s & (3.24) \\ -\left(1-\frac{z}{h_s}\right)R; \quad -h_s \leqslant z \leqslant 0 & (3.25) \end{cases}$$

式中（3.24）及（3.25）是一付里叶余弦级数，其系数为

$$2b_0 \ln\frac{R}{r_w} = \frac{2}{h_0 - h_a}\int_{-h_s}^{0}\left(1-\frac{z}{h_s}\right)R\,dz + \frac{2}{h_0 - h_a}\int_{-(h_0-h_a)}^{-h_s}\left(1+\frac{h_a}{h_s}\right)R\,dz$$

$$= \frac{R}{h_s(h_0 - h_a)}\left[2(h_0 - h_a)(h_s + h_a) - h_s^2\right]$$

$$\therefore \quad b_0 = \frac{R}{2h_s(h_0 - h_a)\ln\frac{R}{r_w}}\left[2(h_0 - h_a)(h_s + h_a) - h_s^2\right] \tag{3.26}$$

$$U_p(r_w) b_p = \frac{2R}{h_0 - h_a}\left\{\int_{-h_s}^{0}\left(1-\frac{z}{h_s}\right)\cos\frac{p\pi z}{h_0 - h_a}dz + \int_{-(h_0-h_a)}^{-h_s}\left(1+\frac{z}{h_s}\right)\cos\frac{p\pi z}{h_0 - h_a}dz\right\}$$

由于

$$\int_{-h_s}^{0}\frac{z}{h_s}\cos\frac{p\pi z}{h_0 - h_a}dz = \left(\frac{h_0 - h_a}{\pi}\right)^2\int_{-h_s\pi/(h_0-h_a)}^{0}\frac{\pi z}{h_s(h_0 - h_a)}\cos\frac{p\pi z}{h_0 - h_a}d\left(\frac{\pi z}{h_0 - h_a}\right)$$

$$= \left(\frac{h_0 - h_a}{\pi}\right)^2\frac{1}{h_s}\int_{-h_s\pi/(h_0-h_a)}^{0} x\cos px\,dx$$

$$= \frac{1}{h_s}\left(\frac{h_0 - h_a}{\pi}\right)^2\left[\frac{1}{p}x\sin px + \frac{1}{p^2}\cos px\right]_{-h_s\pi/(h_0-h_a)}^{0}$$

$$= \frac{1}{h_s}\left(\frac{h_0 - h_a}{\pi}\right)^2\left(-\frac{1}{p}\frac{h_s\pi}{h_0 - h_a}\sin\frac{p\pi h_s}{h_0 - h_a} + \frac{1}{p^2} - \frac{1}{p^2}\cos\frac{p\pi h_s}{h_0 - h_a}\right)$$

故

$$U_p(r_w)b_p = \frac{2R}{h_\cdot(h_\cdot - h_\cdot)}\left\{ + \frac{h_\cdot - h_\cdot}{p\pi}\sin\frac{p\pi h_\cdot}{h_\cdot - h_\cdot} + \left(\frac{h_\cdot - h_\cdot}{\pi}\right)^2\right.$$

$$\times \frac{1}{h_\cdot}\left[\frac{1}{p}\frac{h_\cdot \pi}{h_\cdot - h_\cdot}\text{Sin}\frac{p\pi h_\cdot}{h_\cdot - h_\cdot} - \frac{1}{p^2}\cos\frac{p\pi h_\cdot}{h_\cdot - h_\cdot} + \frac{1}{p^2}\right]$$

$$\left. - \left(1 + \frac{h_\cdot}{h_\cdot}\right) \times \frac{h_\cdot - h_\cdot}{p\pi}\sin\frac{p\pi h_\cdot}{h_\cdot - h_\cdot}\right\}$$

$$= - \frac{2R}{p^2\pi^2 h_\cdot}\left(1 - \cos\frac{p\pi h_\cdot}{h_\cdot - h_\cdot}\right)$$

得

$$b_p = \frac{2R}{p^2\pi^2 h_\cdot U_p(r_w)}\left(1 - \cos\frac{p\pi h_\cdot}{h_\cdot - h_\cdot}\right) \qquad (3.27)$$

将式（3.26），（3.27）代入式（3.23）即得第一个势场

$$\varphi_1 = R - \frac{R\ln\frac{R}{r}}{2h_\cdot(h_\cdot - h_\cdot)\ln\frac{R}{r_w}}[2(h_\cdot - h_\cdot)(h_\cdot + h_\cdot) - h_\cdot^2]$$

$$+ \frac{2R(h_\cdot - h_\cdot)}{\pi^2 h_\cdot}\sum_{p=1}^{\infty}\frac{1 - \cos\frac{p\pi h_\cdot}{h_\cdot - h_\cdot}}{p^2}\frac{U_p(r)}{U_p(r_w)} \times \cos\frac{p\pi z}{h_\cdot - h_\cdot} \qquad (3.28)$$

再求第二个势场的φ_2，据式（3.15），（3.16），（3.17），（3.18）得解：

$$\varphi_2 = \sum_{m,n}^{\infty} a_\cdot [J_\cdot(u_\cdot\frac{r}{r_w})Y_\cdot(u_\cdot\frac{R}{r_w}) - J_\cdot(u_\cdot\frac{R}{r_w})Y_\cdot(u_\cdot\frac{r}{r_w})]$$

$$\times \frac{\text{Ch}\frac{u_\cdot}{r_w}(h_\cdot - h_\cdot + z)}{\text{Sh}\frac{u_\cdot}{r_w}(h_\cdot - h_\cdot)} \qquad (3.29)$$

式中$u_\cdot$是方程

$$J_\cdot(u_\cdot)Y_\cdot(u_\cdot\frac{R}{r_w}) - J_\cdot(u_\cdot\frac{R}{r_w})Y_\cdot(u_\cdot) = 0 \qquad (3.30)$$

的根。

由式（3.29）得

$$\frac{\partial\varphi_2}{\partial z}\bigg|_{z=0} = \sum_{m}^{\infty} a_\cdot [J_\cdot(u_\cdot\frac{r}{r_w})Y_\cdot(u_\cdot\frac{R}{r_w}) - J_\cdot(u_\cdot\frac{R}{r_w})Y_\cdot(u_\cdot\frac{r}{r_w})]$$

$$\times \frac{u_\cdot}{r_w} \qquad (3.31)$$

由式（3.28）得

$$\frac{\partial\varphi_1}{\partial r}\bigg|_{z=0} = \frac{R}{2rh_\cdot(h_\cdot - h_\cdot)\ln\frac{R}{r_w}}[2(h_\cdot - h_\cdot)(h_\cdot + h_\cdot) - h_\cdot^2]$$

$$+ \frac{2R(h_\cdot - h_\cdot)}{\pi^2 h_\cdot}\sum_{p=1}^{\infty}\frac{1 - \text{Cos}\frac{p\pi h_\cdot}{h_\cdot - h_\cdot}}{p^2}\frac{U'_p(r)}{U_p(r_w)} \qquad (3.32)$$

式中

$$U'_p = -\frac{p\pi}{h_0 - h_s}\left[K_1\left(\frac{p\pi r}{h_0 - h_s}\right) + \frac{K_0\left(\frac{p\pi R}{h_0 - h_s}\right)}{I_0\left(\frac{p\pi R}{h_0 - h_s}\right)} \times I_1\left(\frac{p\pi r}{h_0 - h_s}\right)\right]$$

$$\frac{\partial^2 \varphi_1}{\partial z^2}\bigg|_{z=0} = -\frac{2R}{h_s(h_0 - h_s)}\sum_{p=1}^{\infty}\left(1-\cos\frac{p\pi h_s}{h_0 - h_s}\right)\frac{U_p(r)}{U_p(r_w)} \tag{3.33}$$

$$\varphi_1\bigg|_{z=0} = R - \frac{R\ln\frac{R}{r}}{2h_s(h_0-h_s)\ln\frac{R}{r_w}}\left[2(h_0-h_s)(h_0+h_s) - h_s^2\right]$$

$$+\frac{2R(h_0-h_s)}{\pi^2 h_s}\sum_{p=1}^{\infty}\frac{1}{p^2}\left(1-\cos\frac{p\pi h_s}{h_0-h_s}\right)\frac{U_p(r)}{U_p(r_w)} \tag{3.34}$$

将式（3.31），（3.32），（3.33），（3.34）代入边界条件（3.19）得

$$\sum_{m=1}^{\infty} a_m \frac{u_m}{r_w}\left[J_1\left(u_m\frac{r}{r_w}\right)Y_0\left(u_m\frac{R}{r_w}\right) - J_0\left(u_m\frac{R}{r_w}\right)Y_1\left(u_m\frac{r}{r_w}\right)\right] = f(r) \tag{3.36}$$

式中

$$f(r) = \left\{\frac{R}{2rh_s(h_0-h_s)\ln\frac{R}{r_w}}\left[2(h_0-h_s)(h_0+h_s) - h_s^2\right]\right.$$

$$\left.+\frac{2R(h_0-h_s)}{\pi^2 h_s}\sum_{p=1}^{\infty}\frac{1-\cos\frac{p\pi h_s}{h_0-h_s}}{p^2}\frac{u_p'(r)}{u_p(r_w)}\right\}^2$$

$$+\left\{R - \frac{R\ln\frac{R}{r}}{2h_s(h_0-h_s)\ln\frac{R}{r_w}}\left[2(h_0-h_s)(h_0+h_s) - h_s^2\right]\right.$$

$$\left.+\frac{2R(h_0-h_s)}{\pi^2 h_s}\sum_{p=1}^{\infty}\frac{1}{p^2}\left(1-\cos\frac{p\pi h_s}{h_0-h_s}\right)\frac{u_p(r)}{u_p(r_w)}\right\}$$

$$\times\left\{\frac{2R}{h_s(h_0-h_s)}\sum_{p=1}^{\infty}\left(1-\cos\frac{p\pi h_s}{h_0-h_s}\right)\frac{u_p(r)}{u_p(r_w)}\right\} \tag{3.37}$$

可以证明式（3.36）左端级数的方括号中的函数按权r组成正交系；将该函数简记为

$$\varphi(u_m, r) = J_1\left(u_m\frac{r}{r_w}\right)Y_0\left(u_m\frac{R}{r_w}\right) - J_0\left(u_m\frac{R}{r_w}\right)Y_1\left(u_m\frac{r}{r_w}\right) \tag{3.38}$$

及

$$\varphi(u_n, r) = J_0\left(u_n\frac{r}{r_w}\right)Y_0\left(u_n\frac{R}{r_w}\right) - J_0\left(u_n\frac{R}{r_w}\right)Y_0\left(u_n\frac{r}{r_w}\right)$$

它们都是贝塞尔方程

$$ry'' + y' = -\left(\frac{u}{r_w}\right)^2 ry$$

的解，故

$$r\Psi'' + \Psi' = -\left(\frac{u_m}{r_w}\right)^2 r\Psi \qquad (3.39)$$

$$r\phi'' + \phi' = -\left(\frac{u_n}{r_w}\right)^2 r\phi \qquad (3.40)$$

将式（3.39）乘以 φ及式（3.40）乘以 Ψ后相减得

$$r(\Psi\phi'' - \phi\Psi'') + (\phi\Psi' - \Psi\phi') = -\frac{u_m{}^2 - u_n{}^2}{r_w{}^2} r\phi\Psi$$

或

$$r(\Psi\phi' - \phi\Psi')' + (\phi\Psi' - \Psi\phi') = -\frac{u_m{}^2 - u_n{}^2}{r_w{}^2} r\phi\Psi$$

或

$$[r(\Psi\phi' - \phi\Psi')] = -\frac{u_m{}^2 - u_n{}^2}{r_w{}^2} r\phi\Psi$$

两边积分得

$$[r(\Psi\phi' - \phi\Psi')]_{r=r_w}^{r=R} = -\frac{u_m{}^2 - u_n{}^2}{r_w{}^2}\int_{r_w}^{R} r\phi\Psi dr$$

由于式(3.30)知 $\Psi'\Big|_{r=r_w} = \phi'\Big|_{r=r_w} = 0$

又因 $\Psi\Big|_{r=R} = \phi\Big|_{r=R} = 0$

故知当 $m \neq n$ 时必得

$$\int_{r_w}^{R} r\phi\Psi dr = 0$$

因此函数f(r)可以用式(3.38)展成付里叶级数，其系数为

$$a_m = \frac{\int_{r_w}^{R} rf(r)\phi(u_m, r)dr}{\frac{u_m}{r_w}\int_{r_w}^{R} r\phi^2(u_m, r)dr} \qquad (3.41)$$

式中的

$$\int_{r_w}^{R} r\phi^2(u_m, r)dr = \frac{1}{2}\left\{Y_0{}^2\left(u_m\frac{R}{r_w}\right)\right\}R^2\left[J_1{}^2\left(u_m\frac{R}{r_w}\right) + J_0{}^2\left(u_m\frac{R}{r_w}\right)\right]$$

$$-r_w{}^2\left[J_1{}^2(u_m) + J_0{}^2(u_m)\right]\Big\} - 2J_0\left(u_m\frac{R}{r_w}\right)$$