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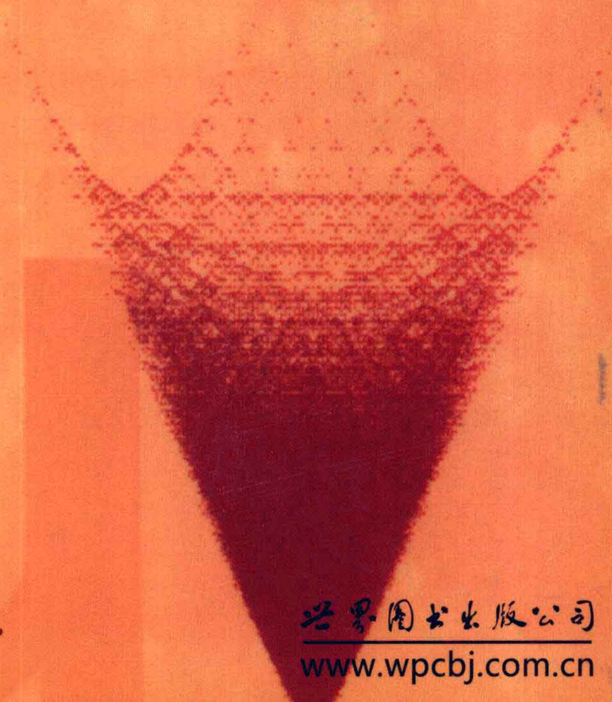
Mark Gross, Daniel Huybrechts & Dominic Joyce

Calabi-Yau Manifolds and Related Geometries

卡拉比-丘流形和相关几何



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Calabi-Yau Manifolds and Related Geometries

Lectures at a Summer School in Nordfjordeid, Norway, June 2001

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Preface

Each summer since 1996, algebraic geometers and algebraists in Norway have organised a summer school in Nordfjordeid, a small place in the western part of Norway. In addition to the beauty of the place, located between the mountains, close to the fjord and not far from the Norway's largest glacier, a reason for going there is that Sophus Lie was born and spent his few first years in Nordfjordeid, so it has a flavour of both the exotic and pilgrimage. It is also convenient: the municipality of Eid has created a conference centre named after Sophus Lie, aimed at attracting activities to fill the summer term of the local boarding school.

The summer schools are a joint effort of the four universities in Norway — the universities of Tromsø, Trondheim, Bergen and Oslo. They are primarily meant as training for Norwegian graduate students, but have over the years attracted increasing numbers of students from other parts of the world, adding the value of being international to the schools.

The themes of the schools have been varied, but build around some central topics in contemporary mathematics. The format of the school has by now become tradition — three international experts giving independent, but certainly connected, series of talks with exercise sessions in the evening, over five or six days.

In 2001 the organising committee consisted of Stein Arild Strømme from the University of Bergen, Loren Olson from the University of Tromsø, and Kristian Ranestad and Geir Ellingsrud from the University of Oslo. We wanted to make a summer school giving the students insight in some of the new interactions between differential and algebraic geometry. The three topics we finally chose, Riemannian holonomy and calibrated geometry, Calabi-Yau manifolds and mirror symmetry, and Compact hyperkähler manifolds, are parts of the fascinating current development of mathematics, and we think they illustrate well the modern interplay between differential and algebraic geometry.

We were fortunate enough to get positive answers when we asked Dominic Joyce, Mark Gross and Daniel Huybrechts to give the courses, and we are thankful for the great job the three lecturers did, both on stage in Nordfjordeid and by writing up the nice notes which have now developed into this book.

Geir Ellingsrud

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Part I

Riemannian Holonomy Groups and Calibrated Geometry

Dominic Joyce

1 Introduction

The *holonomy group* $\mathrm{Hol}(g)$ of a Riemannian n-manifold (M,g) is a global invariant which measures the constant tensors on the manifold. It is a Lie subgroup of $\mathrm{SO}(n)$, and for generic metrics $\mathrm{Hol}(g) = \mathrm{SO}(n)$. If $\mathrm{Hol}(g)$ is a proper subgroup of $\mathrm{SO}(n)$ then we say g has *special holonomy*. Metrics with special holonomy are interesting for a number of different reasons. They include *Kähler metrics* with holonomy $\mathrm{U}(m)$, which are the most natural class of metrics on complex manifolds, *Calabi–Yau manifolds* with holonomy $\mathrm{SU}(m)$, and *hyperkähler manifolds* with holonomy $\mathrm{Sp}(m)$.

Calibrated submanifolds are a class of k-dimensional submanifolds N of a Riemannian manifold (M,g) defined using a closed k-form φ on M called a *calibration*. Calibrated submanifolds are automatically *minimal submanifolds*. Manifolds with special holonomy (M,g) generally come equipped with one or more natural calibrations φ, which then define interesting classes of submanifolds in M. One such class is *special Lagrangian submanifolds* (*SL m-folds*) of Calabi–Yau manifolds.

This part of the book is an expanded version of a course of 8 lectures given at Njordfjordeid in June 2001. The first half of the course discussed Riemannian holonomy groups, focussing in particular on Kähler and Calabi–Yau manifolds. The second half discussed calibrated geometry and calibrated submanifolds of manifolds with special holonomy, focussing in particular on SL m-folds of Calabi–Yau m-folds. The final lecture surveyed research on the *SYZ Conjecture*, which explains Mirror Symmetry between Calabi–Yau 3-folds using special Lagrangian fibrations, and so made contact with Mark Gross' lectures.

I have retained this basic format, so that the first half §1–§6 is on Riemannian holonomy, and the second half §7–§12 on calibrated geometry, finishing with the SYZ Conjecture. The principal aim of the first half is to provide a firm grounding in Kähler and Calabi–Yau geometry from the differential geometric point of view, to serve as background for the more advanced, algebro-geometric material discussed in Parts II and III below. Therefore I have treated other subjects such as the exceptional holonomy groups fairly briefly.

In the second half I shall concentrate mainly on SL m-folds in $\mathbb{C}^m$ and Calabi–Yau m-folds. This is partly because of the focus of the book on Calabi–Yau manifolds and the link with Mirror Symmetry, partly because of my own research interests, and partly because more work has been done on special Lagrangian geometry than on other interesting classes of calibrated submanifolds, so there is simply more to say.

This is not intended as an even-handed survey of a field, but is biassed in favour of my own interests, and areas of research I want to promote in future. Therefore my own publications appear more often than they deserve, whilst more significant work is omitted, through my oversight or ignorance. I apologize to other authors who feel left out.

This is particularly true in the second half. Much of sections 9, 11 and 12 is an account of my own research programme into the singularities of SL m-folds. It therefore has a provisional, unfinished quality, with some conjectural material. My excuse for including it in a book is that I believe that a proper understanding of SL m-folds and their singularities will lead to exciting new discoveries — new invariants of Calabi–Yau 3-folds, and new chapters in the Mirror Symmetry story which are obscure at present, but are hinted at in the Kontsevich Mirror Symmetry proposal and the SYZ Conjecture. So I would like to get more people interested in this area.

We begin in §2 with some background from Differential Geometry, and define holonomy groups of connections and of Riemannian metrics. Section 3 explains Berger's classification of holonomy groups of Riemannian manifolds. Section 4 discusses Kähler geometry and Ricci curvature of Kähler manifolds and defines Calabi–Yau manifolds, and §5 sketches the proof of the Calabi Conjecture, and how it is used to construct examples of Calabi–Yau and hyperkähler manifolds via Algebraic Geometry. Section 6 surveys the exceptional holonomy groups G_2 and Spin(7).

The second half begins in §7 with an introduction to calibrated geometry. Section 8 covers general properties of calibrated submanifolds in $\mathbb{R}^n$, and §9 construction of examples of SL m-folds in $\mathbb{C}^m$. Section 10 discusses compact calibrated submanifolds in special holonomy manifolds, and §11 the singularities of SL m-folds. Finally, §12 briefly introduces String Theory and Mirror Symmetry, explains the SYZ Conjecture, and summarizes some research on the singularities of special Lagrangian fibrations.

2 Introduction to Holonomy Groups

We begin by giving some background from differential and Riemannian geometry, principally to establish notation, and move on to discuss connections on vector bundles, parallel transport, and the definition of holonomy groups. Some suitable reading for this section is my book [113, §2–§3].

2.1 Tensors and Forms

Let M be a smooth n-dimensional manifold, with tangent bundle TM and cotangent bundle T^*M. Then TM and T^*M are *vector bundles* over M. If E is a vector bundle over M, we use the notation $C^\infty(E)$ for the vector space of smooth sections of E. Elements of $C^\infty(TM)$ are called *vector fields*, and elements of $C^\infty(T^*M)$ are called *1-forms*. By taking tensor products of the vector bundles TM and T^*M we obtain the bundles of *tensors* on M. A *tensor* T on M is a smooth section of a bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ for some $k, l \in \mathbb{N}$.

It is convenient to write tensors using the *index notation*. Let U be an open set in M, and $(x^1, \dots, x^n)$ coordinates on U. Then at each point $x \in U$,

$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are a basis for T_xU. Hence, any vector field v on U may be uniquely written $v = \sum_{a=1}^n v^a \frac{\partial}{\partial x^a}$ for some smooth functions $v^1, \ldots, v^n : U \to \mathbb{R}$. We denote v by v^a, which is understood to mean the collection of n functions $v^1, \ldots, v^n$, so that a runs from 1 to n.

Similarly, at each $x \in U$, $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ are a basis for T_x^*U. Hence, any 1-form α on U may be uniquely written $\alpha = \sum_{b=1}^n \alpha_b \mathrm{d}x^b$ for some smooth functions $\alpha_1, \ldots, \alpha_n : U \to \mathbb{R}$. We denote α by α_b, where b runs from 1 to n. In the same way, a general tensor T in $C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$ is written $T^{a_1 \ldots a_k}_{b_1 \ldots b_l}$, where

$$T = \sum_{\substack{1 \leqslant a_i \leqslant n,\ 1 \leqslant i \leqslant k \\ 1 \leqslant b_j \leqslant n,\ 1 \leqslant j \leqslant l}} T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \frac{\partial}{\partial x^{a_k}} \otimes \mathrm{d}x^{b_1} \otimes \cdots \otimes \mathrm{d}x^{b_l}.$$

The k^{th} exterior power of the cotangent bundle T^*M is written $\Lambda^k T^*M$. Smooth sections of $\Lambda^k T^*M$ are called *k-forms*, and the vector space of k-forms is written $C^\infty(\Lambda^k T^*M)$. They are examples of tensors. In the index notation they are written $T_{b_1 \ldots b_k}$, and are antisymmetric in the indices $b_1, \ldots, b_k$. The *exterior product* $\wedge$ and the *exterior derivative* d are important natural operations on forms. If α is a k-form and β an l-form then $\alpha \wedge \beta$ is a $(k+l)$-form and $\mathrm{d}\alpha$ a $(k+1)$-form, which are given in index notation by

$$(\alpha \wedge \beta)_{a_1 \ldots a_{k+l}} = \alpha_{[a_1 \ldots a_k} \beta_{a_{k+1} \ldots a_{k+l}]} \quad \text{and} \quad (\mathrm{d}\alpha)_{a_1 \ldots a_{k+1}} = \frac{\partial}{\partial x^{[a_1}} \alpha_{a_2 \ldots a_{k+1}]},$$

where $[\cdots]$ denotes antisymmetrization over the enclosed group of indices.

Let v, w be vector fields on M. The *Lie bracket* $[v, w]$ of v and w is another vector field on M, given in index notation by

$$[v, w]^a = v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \tag{1}$$

Here we have used the *Einstein summation convention*, that is, the repeated index b on the right hand side is summed from 1 to n. The important thing about this definition is that it is independent of choice of coordinates $(x^1, \ldots, x^n)$.

2.2 Connections on Vector Bundles and Curvature

Let M be a manifold, and $E \to M$ a vector bundle. A *connection* ∇^E on E is a linear map $\nabla^E : C^\infty(E) \to C^\infty(E \otimes T^*M)$ satisfying the condition

$$\nabla^E(\alpha e) = \alpha \nabla^E e + e \otimes \mathrm{d}\alpha,$$

whenever $e \in C^\infty(E)$ is a smooth section of E and α is a smooth function on M.

If ∇^E is such a connection, $e \in C^\infty(E)$, and $v \in C^\infty(TM)$ is a vector field, then we write $\nabla^E_v e = v \cdot \nabla^E e \in C^\infty(E)$, where '$\cdot$' contracts together the

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TM and T^*M factors in v and $\nabla^E e$. Then if $v \in C^\infty(TM)$ and $e \in C^\infty(E)$ and α, β are smooth functions on M, we have

$$\nabla^E_{\alpha v}(\beta e) = \alpha\beta\nabla^E_v e + \alpha(v \cdot \beta)e.$$

Here $v \cdot \beta$ is the Lie derivative of β by v. It is a smooth function on M, and could also be written $v \cdot \mathrm{d}\beta$.

There exists a unique, smooth section $R(\nabla^E) \in C^\infty\big(\mathrm{End}(E) \otimes \Lambda^2 T^*M\big)$ called the *curvature* of ∇^E, that satisfies the equation

$$R(\nabla^E) \cdot (e \otimes v \wedge w) = \nabla^E_v \nabla^E_w e - \nabla^E_w \nabla^E_v e - \nabla^E_{[v,w]} e \qquad (2)$$

for all $v, w \in C^\infty(TM)$ and $e \in C^\infty(E)$, where $[v, w]$ is the Lie bracket of v, w.

Here is one way to understand the curvature of ∇^E. Define $v_i = \partial/\partial x^i$ for $i = 1, \ldots, n$. Then v_i is a vector field on U, and $[v_i, v_j] = 0$. Let e be a smooth section of E. Then we may interpret $\nabla^E_{v_i} e$ as a kind of *partial derivative* $\partial e/\partial x^i$ of e. Equation (2) then implies that

$$R(\nabla^E) \cdot (e \otimes v_i \wedge v_j) = \frac{\partial^2 e}{\partial x^i \partial x^j} - \frac{\partial^2 e}{\partial x^j \partial x^i}. \qquad (3)$$

Thus, *the curvature $R(\nabla^E)$ measures how much partial derivatives in E fail to commute.*

Now let ∇ be a connection on the tangent bundle TM of M, rather than a general vector bundle E. Then there is a unique tensor $T = T^a_{bc}$ in $C^\infty\big(TM \otimes \Lambda^2 T^*M\big)$ called the *torsion* of ∇, satisfying

$$T \cdot (v \wedge w) = \nabla_v w - \nabla_w v - [v, w] \quad \text{for all } v, w \in C^\infty(TM).$$

A connection ∇ with zero torsion is called *torsion-free*. Torsion-free connections have various useful properties, so we usually restrict attention to torsion-free connections on TM.

A connection ∇ on TM extends naturally to connections on all the bundles of tensors $\bigotimes^k TM \otimes \bigotimes^l T^*M$ for $k, l \in \mathbb{N}$, which we will also write ∇. That is, we can use ∇ to differentiate not just vector fields, but any tensor on M.

2.3 Parallel Transport and Holonomy Groups

Let M be a manifold, $E \to M$ a vector bundle over M, and ∇^E a connection on E. Let $\gamma : [0,1] \to M$ be a smooth curve in M. Then the pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t \in [0,1]$, where E_x is the fibre of E over $x \in M$. The connection ∇^E pulls back under γ to give a connection on $\gamma^*(E)$ over $[0,1]$.

Definition 2.1 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma : [0,1] \to M$ is (piecewise) smooth, with $\gamma(0) = x$

and $\gamma(1) = y$, where $x, y \in M$. Then for each $e \in E_x$, there exists a unique smooth section s of $\gamma^*(E)$ satisfying $\nabla^E_{\dot\gamma(t)} s(t) = 0$ for $t \in [0,1]$, with $s(0) = e$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is a well-defined linear map, called the *parallel transport map*.

We use parallel transport to define the *holonomy group* of ∇^E.

Definition 2.2 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Fix a point $x \in M$. We say that γ is a *loop based at x* if $\gamma : [0,1] \to M$ is a piecewise-smooth path with $\gamma(0) = \gamma(1) = x$. The parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map, so that P_γ lies in $\mathrm{GL}(E_x)$, the group of invertible linear transformations of E_x. Define the *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E based at x to be

$$\mathrm{Hol}_x(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset \mathrm{GL}(E_x). \tag{4}$$

The holonomy group has the following important properties.

- It is a *Lie subgroup* of $\mathrm{GL}(E_x)$. To show that $\mathrm{Hol}_x(\nabla^E)$ is a subgroup of $\mathrm{GL}(E_x)$, let γ, δ be loops based at x, and define loops $\gamma\delta$ and γ^{-1} by

$$\gamma\delta(t) = \begin{cases} \delta(2t) & t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma^{-1}(t) = \gamma(1-t) \quad \text{for } t \in [0,1].$$

  Then $P_{\gamma\delta} = P_\gamma \circ P_\delta$ and $P_{\gamma^{-1}} = P_\gamma^{-1}$, so $\mathrm{Hol}_x(\nabla^E)$ is closed under products and inverses.
- It is *independent of basepoint* $x \in M$, in the following sense. Let $x, y \in M$, and let $\gamma : [0,1] \to M$ be a smooth path from x to y. Then $P_\gamma : E_x \to E_y$, and $\mathrm{Hol}_x(\nabla^E)$ and $\mathrm{Hol}_y(\nabla^E)$ satisfy $\mathrm{Hol}_y(\nabla^E) = P_\gamma \mathrm{Hol}_x(\nabla^E) P_\gamma^{-1}$.
  Suppose E has fibre $\mathbb{R}^k$, so that $\mathrm{GL}(E_x) \cong \mathrm{GL}(k, \mathbb{R})$. Then we may regard $\mathrm{Hol}_x(\nabla^E)$ as a subgroup of $\mathrm{GL}(k, \mathbb{R})$ defined up to conjugation, and it is then independent of basepoint x.
- If M is simply-connected, then $\mathrm{Hol}_x(\nabla^E)$ is connected. To see this, note that any loop γ based at x can be continuously shrunk to the constant loop at x. The corresponding family of parallel transports is a continuous path in $\mathrm{Hol}_x(\nabla^E)$ joining P_γ to the identity.

The holonomy group of a connection is closely related to its curvature. Here is one such relationship. As $\mathrm{Hol}_x(\nabla^E)$ is a Lie subgroup of $\mathrm{GL}(E_x)$, it has a *Lie algebra* $\mathfrak{hol}_x(\nabla^E)$, which is a Lie subalgebra of $\mathrm{End}(E_x)$. It can be shown that the curvature $R(\nabla^E)_x$ at x lies in the linear subspace $\mathfrak{hol}_x(\nabla^E) \otimes \Lambda^2 T_x^* M$ of $\mathrm{End}(E_x) \otimes \Lambda^2 T_x^* M$. Thus, *the holonomy group of a connection places a linear restriction upon its curvature.*

Now let ∇ be a connection on TM. Then from §2.2, ∇ extends to connections on all the tensor bundles $\bigotimes^k TM \otimes \bigotimes^l T^*M$. We call a tensor S on M *constant* if $\nabla S = 0$. The constant tensors on M are determined by the holonomy group $\mathrm{Hol}(\nabla)$.

Theorem 2.3 *Let M be a manifold, and ∇ a connection on TM. Fix $x \in M$, and let $H = \mathrm{Hol}_x(\nabla)$. Then H acts naturally on the tensor powers $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Suppose $S \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$ is a constant tensor. Then $S|_x$ is fixed by the action of H on $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Conversely, if $S|_x \in \bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$ is fixed by H, it extends to a unique constant tensor $S \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$.*

The main idea in the proof is that if S is a constant tensor and $\gamma : [0,1] \to M$ is a path from x to y, then $P_\gamma(S|_x) = S|_y$. Thus, constant tensors are invariant under parallel transport.

2.4 Riemannian Metrics and the Levi-Civita Connection

Let g be a *Riemannian metric* on M. We refer to the pair (M, g) as a *Riemannian manifold.* Here g is a tensor in $C^\infty(S^2T^*M)$, so that $g = g_{ab}$ in index notation with $g_{ab} = g_{ba}$. There exists a unique, torsion-free connection ∇ on TM with $\nabla g = 0$, called the *Levi-Civita connection*, which satisfies

$$\begin{aligned} 2g(\nabla_u v, w) &= u \cdot g(v,w) + v \cdot g(u,w) - w \cdot g(u,v) \\ &\quad + g([u,v],w) - g([v,w],u) - g([u,w],v) \end{aligned}$$

for all $u, v, w \in C^\infty(TM)$. This result is known as the *fundamental theorem of Riemannian geometry.*

The curvature $R(\nabla)$ of the Levi-Civita connection is a tensor $R^a{}_{bcd}$ on M. Define $R_{abcd} = g_{ae}R^e{}_{bcd}$. We shall refer to both $R^a{}_{bcd}$ and R_{abcd} as the *Riemann curvature* of g. The following theorem gives a number of symmetries of R_{abcd}. Equations (6) and (7) are known as the *first* and *second Bianchi identities*, respectively.

Theorem 2.4 *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection of g, and R_{abcd} the Riemann curvature of g. Then*

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, \tag{5}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{6}$$

and

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{7}$$

Let (M, g) be a Riemannian manifold, with Riemann curvature $R^a{}_{bcd}$. The *Ricci curvature* of g is $R_{ab} = R^c{}_{acb}$. It is a component of the full Riemann curvature, and satisfies $R_{ab} = R_{ba}$. We say that g is *Einstein* if $R_{ab} = \lambda g_{ab}$ for some constant $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} = 0$. Einstein and Ricci-flat metrics are of great importance in mathematics and physics.

2.5 Riemannian Holonomy Groups

Let (M, g) be a Riemannian manifold. We define the ***holonomy group*** $\mathrm{Hol}_x(g)$ of g to be the holonomy group $\mathrm{Hol}_x(\nabla)$ of the Levi-Civita connection ∇ of g, as in §2.3. Holonomy groups of Riemannian metrics, or ***Riemannian holonomy groups***, have stronger properties than holonomy groups of connections on arbitrary vector bundles. We shall explore some of these.

Firstly, note that g is a *constant tensor* as $\nabla g = 0$, so g is invariant under $\mathrm{Hol}(g)$ by Theorem 2.3. That is, $\mathrm{Hol}_x(g)$ lies in the subgroup of $\mathrm{GL}(T_xM)$ which preserves $g|_x$. This subgroup is isomorphic to $\mathrm{O}(n)$. Thus, $\mathrm{Hol}_x(g)$ may be regarded as a *subgroup of* $\mathrm{O}(n)$ *defined up to conjugation*, and it is then independent of $x \in M$, so we will often write it as $\mathrm{Hol}(g)$, dropping the basepoint x.

Secondly, the holonomy group $\mathrm{Hol}(g)$ constrains the Riemann curvature of g, in the following way. The Lie algebra $\mathfrak{hol}_x(\nabla)$ of $\mathrm{Hol}_x(\nabla)$ is a vector subspace of $T_xM \otimes T_x^*M$. From §2.3, we have $R^a{}_{bcd}|_x \in \mathfrak{hol}_x(\nabla) \otimes \Lambda^2 T_x^*M$.

Use the metric g to identify $T_xM \otimes T_x^*M$ and $\otimes^2 T_x^*M$, by equating $T^a{}_b$ with $T_{ab} = g_{ac}T^c{}_b$. This identifies $\mathfrak{hol}_x(\nabla)$ with a vector subspace of $\otimes^2 T_x^*M$ that we will write as $\mathfrak{hol}_x(g)$. Then $\mathfrak{hol}_x(g)$ lies in $\Lambda^2 T_x^*M$, and $R_{abcd}|_x \in \mathfrak{hol}_x(g) \otimes \Lambda^2 T_x^*M$. Applying the symmetries (5) of R_{abcd}, we have:

Theorem 2.5 *Let (M, g) be a Riemannian manifold with Riemann curvature R_{abcd}. Then R_{abcd} lies in the vector subspace $S^2\mathfrak{hol}_x(g)$ in $\Lambda^2 T_x^*M \otimes \Lambda^2 T_x^*M$ at each $x \in M$.*

Combining this theorem with the Bianchi identities, (6) and (7), gives strong restrictions on the curvature tensor R_{abcd} of a Riemannian metric g with a prescribed holonomy group $\mathrm{Hol}(g)$. These restrictions are the basis of the classification of Riemannian holonomy groups, which will be explained in §3.

2.6 Exercises

2.1 Let M be a manifold and u, v, w be vector fields on M. The *Jacobi identity* for the Lie bracket of vector fields is

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$

Prove the Jacobi identity in coordinates $(x^1, \ldots, x^n)$ on a coordinate patch U. Use the coordinate expression (1) for the Lie bracket of vector fields.

2.2 In §2.3 we explained that if M is a manifold, $E \to M$ a vector bundle and ∇^E a connection, then $\mathrm{Hol}(\nabla^E)$ is connected when M is simply-connected. If M is not simply-connected, what is the relationship between the fundamental group $\pi_1(M)$ and $\mathrm{Hol}(\nabla^E)$?

2.3 Work out your own proof of Theorem 2.3.

3 Berger's Classification of Holonomy Groups

Next we describe Berger's classification of Riemannian holonomy groups, and briefly discuss the possibilities in the classification. A reference for the material of this section is [113, §3]. Berger's original paper is [18], but owing to language and notation most will now find it difficult to read.

3.1 Reducible Riemannian Manifolds

Let (P, g) and (Q, h) be Riemannian manifolds with positive dimension, and $P \times Q$ the product manifold. Then at each (p, q) in $P \times Q$ we have $T_{(p,q)}(P \times Q) \cong T_pP \oplus T_qQ$. Define the *product metric* $g \times h$ on $P \times Q$ by $g \times h|_{(p,q)} = g|_p + h|_q$ for all $p \in P$ and $q \in Q$. We call $(P \times Q, g \times h)$ a *Riemannian product.*

A Riemannian manifold (M, g') is said to be *(locally) reducible* if every point has an open neighbourhood isometric to a Riemannian product $(P \times Q, g \times h)$, and *irreducible* if it is not locally reducible. It is easy to show that the holonomy of a product metric $g \times h$ is the product of the holonomies of g and h.

Proposition 3.1 *If* $(P \times Q, g \times h)$ *is the product of Riemannian manifolds* (P, g), (Q, h), *then* $\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h)$.

Here is a kind of converse to this.

Theorem 3.2 *Let* M *be an* n*-manifold, and* g *an irreducible Riemannian metric on* M. *Then the representation of* $\mathrm{Hol}(g)$ *on* $\mathbb{R}^n$ *is irreducible.*

To prove the theorem, suppose $\mathrm{Hol}(g)$ acts reducibly on $\mathbb{R}^n$, so that $\mathbb{R}^n$ is the direct sum of representations $\mathbb{R}^k$, $\mathbb{R}^l$ of $\mathrm{Hol}(g)$ with $k, l > 0$. Using parallel transport, one can define a splitting $TM = E \oplus F$, where E, F are vector subbundles with fibres $\mathbb{R}^k, \mathbb{R}^l$. These vector subbundles are *integrable*, so locally $M \cong P \times Q$ with $E = TP$ and $F = TQ$. One can then show that the metric on M is the product of metrics on P and Q, so that g is locally reducible.

3.2 Symmetric Spaces

Next we discuss Riemannian symmetric spaces.

Definition 3.3 A Riemannian manifold (M, g) is said to be a *symmetric space* if for every point $p \in M$ there exists an isometry $s_p : M \to M$ that is an involution (that is, s_p^2 is the identity), such that p is an isolated fixed point of s_p.

Examples include $\mathbb{R}^n$, spheres $\mathcal{S}^n$, projective spaces $\mathbb{CP}^m$ with the Fubini–Study metric, and so on. Symmetric spaces have a transitive group of isometries.

Proposition 3.4 *Let (M, g) be a connected, simply-connected symmetric space. Then g is complete. Let G be the group of isometries of (M, g) generated by elements of the form $s_q \circ s_r$ for $q, r \in M$. Then G is a connected Lie group acting transitively on M. Choose $p \in M$, and let H be the subgroup of G fixing p. Then H is a closed, connected Lie subgroup of G, and M is the homogeneous space G/H.*

Because of this, symmetric spaces can be classified completely using the theory of Lie groups. This was done in 1925 by Élie Cartan. From Cartan's classification one can quickly deduce the list of holonomy groups of symmetric spaces.

A Riemannian manifold (M, g) is called *locally symmetric* if every point has an open neighbourhood isometric to an open set in a symmetric space, and *nonsymmetric* if it is not locally symmetric. It is a surprising fact that Riemannian manifolds are locally symmetric if and only if they have *constant curvature.*

Theorem 3.5 *Let (M, g) be a Riemannian manifold, with Levi-Civita connection ∇ and Riemann curvature R. Then (M, g) is locally symmetric if and only if $\nabla R = 0$.*

3.3 Berger's Classification

In 1955, Berger proved the following result.

Theorem 3.6 (Berger) *Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,
(ii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,
(iii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,
(iv) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,
(v) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,
(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*
(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Notice the three simplifying assumptions on M and g: that M is simply-connected, and g is irreducible and nonsymmetric. Each condition has consequences for the holonomy group $\mathrm{Hol}(g)$.

- As M is simply-connected, $\mathrm{Hol}(g)$ is connected, from §2.3.
- As g is irreducible, $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$ by Theorem 3.2.
- As g is nonsymmetric, $\nabla R \not\equiv 0$ by Theorem 3.5.

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