高等师范专科学校交流讲义

电动力学简明教程

（理论物理概论之一）

习题解答

《电动力学简明教程》编写组

第一章　电磁现象的普遍规律

1、设u是空间坐标x、y、z的函数，证明：

(1) $\nabla f(u) = \dfrac{df}{du}\nabla u$

(2) $\nabla\cdot\vec{A}(u) = \nabla u\cdot\dfrac{d\vec{A}}{du}$

(3) $\nabla\times\vec{A}(u) = \nabla u\times\dfrac{d\vec{A}}{du}$

证明：

(1) $\nabla f(u) = \vec{i}\dfrac{\partial f(u)}{\partial x} + \vec{j}\dfrac{\partial f(u)}{\partial y} + \vec{k}\dfrac{\partial f(u)}{\partial z}$

$= \vec{i}\dfrac{df}{du}\dfrac{\partial u}{\partial x} + \vec{j}\dfrac{df}{du}\dfrac{\partial u}{\partial y} + \vec{k}\dfrac{df}{du}\dfrac{\partial u}{\partial z}$

$= \dfrac{df}{du}[\vec{i}\dfrac{\partial u}{\partial x} + \vec{j}\dfrac{\partial u}{\partial y} + \vec{k}\dfrac{\partial u}{\partial z}] = \dfrac{df}{du}\nabla u$

(2) $\nabla\cdot\vec{A}(u) = \dfrac{\partial A_x(u)}{\partial x} + \dfrac{\partial A_y(u)}{\partial y} + \dfrac{\partial A_z(u)}{\partial z}$

$= \dfrac{dA_x}{du}\dfrac{\partial u}{\partial x} + \dfrac{dA_y}{du}\dfrac{\partial u}{\partial y} + \dfrac{dA_z}{du}\dfrac{\partial u}{\partial z}$

$= \dfrac{d\vec{A}}{du}\cdot\nabla u = \nabla u\cdot\dfrac{d\vec{A}}{du}$

(3) $\nabla\times\vec{A}(u) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x(u) & A_y(u) & A_z(u) \end{vmatrix}$

$= \vec{i}[\dfrac{\partial A_z(u)}{\partial y} - \dfrac{\partial A_y(u)}{\partial z}]$

$+ \vec{j}[\dfrac{\partial A_x(u)}{\partial z} - \dfrac{\partial A_z(u)}{\partial x}] + \vec{k}[\dfrac{\partial A_y(u)}{\partial x} - \dfrac{\partial A_x(u)}{\partial y}]$

$= \vec{i}[\dfrac{dA_z}{du}\dfrac{\partial u}{\partial y} - \dfrac{dA_y}{\partial u}\dfrac{\partial u}{\partial z}] + \vec{j}[\dfrac{dA_x}{du}\dfrac{\partial u}{\partial z} - \dfrac{dA_z}{du}\dfrac{\partial u}{\partial x}]$

$+ \vec{k}[\dfrac{dA_y}{du}\dfrac{\partial u}{\partial x} - \dfrac{dA_x}{du}\dfrac{\partial u}{\partial y}]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{dA_x}{du} & \frac{dA_y}{du} & \frac{dA_z}{du} \end{vmatrix}$$

$$= \nabla u \times \frac{d\vec{A}}{du}$$

2、设 $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ 为源点 x' 到场点 x 的距离，$\vec{r}$ 的方向规定从源点指向场点。

(1)证明下列结果：

$$\nabla r = -\nabla' r = \frac{\vec{r}}{r}$$

$$\nabla \frac{1}{r} = -\nabla' \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla \times \frac{\vec{r}}{r^3} = 0$$

$$\nabla \cdot \frac{\vec{r}}{r^3} = -\nabla' \cdot \frac{\vec{r}}{r^3} = 0 \qquad (r \neq 0)$$

(2)求 $\nabla\cdot\vec{r}$，$\nabla\times\vec{r}$，$(\vec{a}\cdot\nabla)\vec{r}$，$\nabla(\vec{a}\cdot\vec{r})$，$\nabla\cdot[\vec{E}_0 \sin(\vec{k}\cdot\vec{r})]$ 及 $\nabla\times[\vec{E}_0 \sin(\vec{k}\cdot\vec{r})]$

其中 $\vec{a}$，$\vec{k}$ 及 $\vec{E}_0$ 均为常矢量。

〔证明〕 (1)

(a) $\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{i}\,\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial x}$$

$$+ \vec{j}\,\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial y} + \vec{k}\,\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial z}$$

$$= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\left[\vec{i}(x-x')+\vec{j}(y-y')+\vec{k}(z-z')\right]$$

$$= \frac{\vec{r}}{r}$$

同理 $\nabla' r = \vec{i}\frac{\partial r}{\partial x'} + \vec{j}\frac{\partial r}{\partial y'} + \vec{k}\frac{\partial r}{\partial z'}$

$$= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}[-\vec{i}(x-x')-\vec{j}(y-y')-\vec{k}(z-z')]$$

$$= -\frac{\vec{r}}{r}$$

$$\therefore \nabla r = -\nabla' r = \frac{\vec{r}}{r}$$

(b) $\nabla\frac{1}{r} = -\frac{1}{r^2}\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$

同理 $\nabla'\frac{1}{r} = -\frac{1}{r^2}\nabla' r = -\frac{1}{r^2}(-\frac{\vec{r}}{r}) = \frac{\vec{r}}{r^3}$

(c) $\nabla\times\frac{\vec{r}}{r^3} = \nabla\times(-\nabla\frac{1}{r}) = -\nabla\times(\nabla\frac{1}{r}) = 0$

(d) $\nabla\cdot\frac{\vec{r}}{r^3} = \frac{\partial}{\partial x}(\frac{x-x'}{r^3}) + \frac{\partial}{\partial y}(\frac{y-y'}{r^3}) + \frac{\partial}{\partial z}(\frac{z-z'}{r^3})$

$$= (x-x')\frac{\partial}{\partial x}(\frac{1}{r^3}) + \frac{1}{r^3} + (y-y')\frac{\partial}{\partial y}(\frac{1}{r^3}) + \frac{1}{r^3} + (z-z')\frac{\partial}{\partial z}(\frac{1}{r^3}) + \frac{1}{r^3}$$

$$= -\frac{3(x-x')^2}{r^5} - \frac{3(y-y')^2}{r^5} - \frac{3(z-z')^2}{r^5} + \frac{3}{r^3}$$

$$= -\frac{3[(x-x')^2+(y-y')^2+(z-z')^2]}{r^5} + \frac{3}{r^3}$$

$$= -\frac{3}{r^3} + \frac{3}{r^3} = 0 \qquad (r \neq 0)$$

又证： $\nabla\cdot\frac{\vec{r}}{r^3} = (\nabla\frac{1}{r^3})\cdot\vec{r} + \frac{1}{r^3}(\nabla\cdot\vec{r}) = \frac{d}{dr}(\frac{1}{r^3})(\nabla r)\cdot\vec{r} + \frac{3}{r^3}$

$$= -\frac{3}{r^4}(\frac{\vec{r}}{r})\cdot\vec{r} + \frac{3}{r^3} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

同理可证 $\nabla'\cdot\frac{\vec{r}}{r^3} = 0$

(2) (a) $\nabla\cdot\vec{r} = \frac{\partial(x-x')}{\partial x} + \frac{\partial(y-y')}{\partial y} + \frac{\partial(z-z')}{\partial z} = 3$

(b) $\nabla\times\vec{r} = \vec{i}[\frac{\partial(z-z')}{\partial y} - \frac{\partial(y-y')}{\partial z}] + \vec{j}[\frac{\partial(x-x')}{\partial z} - \frac{\partial(z-z')}{\partial x}]$

$$+ \vec{k}[\frac{\partial(y-y')}{\partial x} - \frac{\partial(x-x')}{\partial y}] = 0$$

(c) $(\vec{a}\cdot\nabla)\vec{r}=(a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}+a_z\frac{\partial}{\partial z})[(x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}]$

$$=\vec{i}a_x\frac{\partial(x-x')}{\partial x}+\vec{j}a_y\frac{\partial(y-y')}{\partial y}+\vec{k}a_z\frac{\partial(z-z')}{\partial z}$$

$$=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}=\vec{a}$$

(d) $\nabla(\vec{a}\cdot\vec{r})=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})[a_x(x-x')+a_y(y-y')+a_z(z-z')]$

$$=\vec{i}a_x\frac{\partial(x-x')}{\partial x}+\vec{j}a_y\frac{\partial(y-y')}{\partial y}+\vec{k}a_z\frac{\partial(z-z')}{\partial z}$$

$$=\vec{i}a_x+\vec{j}a_y+\vec{k}a_z=\vec{a}$$

或应用公式计算

$$\nabla(\vec{a}\cdot\vec{r})=\vec{a}\times(\nabla\times\vec{r})+(\vec{a}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{a})+(\vec{r}\cdot\nabla)\vec{a}$$

$$=0+\vec{a}+0+0=\vec{a}$$

(e) $\nabla\cdot[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]$

$$=\vec{E}_0\cdot\nabla\sin(\vec{k}\cdot\vec{r})+(\nabla\cdot\vec{E}_0)\sin(\vec{k}\cdot\vec{r})$$

$$=E_{0x}\frac{\partial}{\partial x}\sin(\vec{k}\cdot\vec{r})+E_{0y}\frac{\partial}{\partial y}\sin(\vec{k}\cdot\vec{r})+E_{0z}\frac{\partial}{\partial z}\sin(\vec{k}\cdot\vec{r})$$

$$=E_{0x}\frac{d[\sin(\vec{k}\cdot\vec{r})]}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial x}+E_{0y}\frac{d[\sin(\vec{k}\cdot\vec{r})]}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial y}$$

$$+E_{0z}\frac{d[\sin(\vec{k}\cdot\vec{r})]}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial z}$$

$$=\cos(\vec{k}\cdot\vec{r})[E_{0x}k_x+E_{0y}k_y+E_{0z}k_z]$$

$$=\vec{k}\cdot\vec{E}_0\cos(\vec{k}\cdot\vec{r})$$

(f) $\nabla\times[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_{0x}\sin(\vec{k}\cdot\vec{r}) & E_{0y}\sin(\vec{k}\cdot\vec{r}) & E_{0z}\sin(\vec{k}\cdot\vec{r})\end{vmatrix}$$

$$=\vec{i}[\frac{\partial E_{0z}\sin(\vec{k}\cdot\vec{r})}{\partial y}-\frac{\partial E_{0y}\sin(\vec{k}\cdot\vec{r})}{\partial z}]+\vec{j}[\frac{\partial E_{0x}\sin(\vec{k}\cdot\vec{r})}{\partial z}-\frac{\partial E_{0z}\sin(\vec{k}\cdot\vec{r})}{\partial x}]$$

$$+\vec{k}[\frac{\partial E_{0y}\sin(\vec{k}\cdot\vec{r})}{\partial x}-\frac{\partial E_{0x}\sin(\vec{k}\cdot\vec{r})}{\partial y}]$$

$$=\vec{i}\left[E_{oz}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\cdot\frac{\partial(\vec{k}\cdot\vec{r})}{\partial y}-E_{oy}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial z}\right]$$

$$+\vec{j}\left[E_{ox}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial z}-E_{oz}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial x}\right]$$

$$+\vec{k}\left[E_{oy}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\cdot\frac{\partial(\vec{k}\cdot\vec{r})}{\partial x}-E_{ox}\frac{d\sin(\vec{k}\cdot\vec{r})}{d(\vec{k}\cdot\vec{r})}\frac{\partial(\vec{k}\cdot\vec{r})}{\partial y}\right]$$

$$=\cos(\vec{k}\cdot\vec{r})\{\vec{i}(k_yE_{oz}-k_zE_{oy}]+\vec{j}(k_zE_{ox}-k_xE_{oz})+\vec{k}(k_xE_{oy}-k_yE_{ox})\}$$

$$=\vec{k}\times\vec{E}_o\cos(\vec{k}\cdot\vec{r})$$

3、应用高斯定理证明

$$\iiint_V dV\nabla\times\vec{f}=\oint_S d\vec{S}\times\vec{f}$$

应用斯托克斯定理证明

$$\iint_S d\vec{S}\times\nabla\varphi=\oint_L d\vec{l}\varphi$$

〔证明〕(1) 设$\vec{c}$为任一常矢量

$$\vec{c}\cdot\oint_S d\vec{S}\times\vec{f}=\oint_S\vec{c}\cdot(d\vec{S}\times\vec{f})=\oint_S d\vec{S}\cdot(\vec{f}\times\vec{c})$$

$$=\iiint_V dV\nabla\cdot(\vec{f}\times\vec{c})=\iiint_V dV[(\nabla\times\vec{f})\cdot\vec{c}-\vec{f}\cdot(\nabla\times\vec{c})]$$

$$=\iiint_V dV[(\nabla\times\vec{f})\cdot\vec{c}]=\vec{c}\cdot\iiint dV(\nabla\times\vec{f})$$

因为$\vec{c}$为任意常矢量

∴得证：$\oint_S d\vec{S}\times\vec{f}=\iiint_V dV\nabla\times\vec{f}$

(2) 同样设$\vec{c}$为任一常矢量

$$\vec{c}\cdot\oint_L d\vec{l}\varphi=\oint_L\varphi\vec{c}\cdot d\vec{l}=\iint_S[\nabla\times(\varphi\vec{c})]\cdot d\vec{S}$$

$$=\iint_S[\nabla\varphi\times\vec{c}+\varphi(\nabla\times\vec{c})]\cdot d\vec{S}$$

$$=\iint_S(\nabla\varphi\times\vec{c})\cdot d\vec{S}$$

$$=\iint_S d\vec{S}\cdot(\nabla\varphi\times\vec{c})$$

$$=\iint_S \vec{c}\cdot(d\vec{s}\times\nabla\varphi)=\vec{c}\cdot\iint_S d\vec{s}\times\nabla\varphi$$

因为$\vec{c}$为一常矢易

$\therefore$ 得证 $\oint_L d\vec{e}\varphi=\iint_S d\vec{s}\times\nabla\varphi$

4、有一内外半径分别为r_1和r_2的空心介质球，介质的介电常数为ε，使介质内均匀带静止自由电荷ρ_f，求

(1) 空间各点的电场；

(2) 极化体电荷和极化面电荷分布

〔解〕(1) 根据高斯定理

$$\oint_S \varepsilon\vec{E}\cdot d\vec{s}=q$$

1) $r<r_1$ $q=0$，$\vec{E}=0$

2) $r_1<r<r_2$

$$4\pi r^2\varepsilon E=\frac{4}{3}\pi(r^3-r_1^3)\rho_f$$

$$E=\frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^2}\qquad \vec{E}=\frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^3}\vec{r}$$

3) $r>r_2$

$$4\pi r^2\varepsilon_0 E=\frac{4}{3}\pi(r_2^3-r_1^3)$$

$$E=\frac{(r_2^3-r_1^3)\rho_f}{3\varepsilon_0 r^2}\qquad \vec{E}=\frac{(r_2^3-r_1^3)\rho_f}{3\varepsilon_0 r^3}\vec{r}$$

(2) 根据公式 $\vec{P}=\chi_e\varepsilon_0\vec{E}$ 和 $\rho'=-\nabla\cdot\vec{P}$

当 $r_1<r<r_2$

$$\rho'=-\nabla\vec{P}=-\nabla\cdot\varepsilon_0\chi_e\vec{E}$$

$$=-\nabla\cdot(\varepsilon-\varepsilon_0)\vec{E}$$

$$=-\nabla\cdot(\varepsilon-\varepsilon_0)\frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^3}\vec{r}$$

$$=-\frac{(\varepsilon-\varepsilon_0)\rho_f}{3\varepsilon}\nabla\cdot\frac{(r^3-r_1^3)}{r^3}\vec{r}$$

$$= -\frac{(\varepsilon-\varepsilon_0)\rho_f}{3\varepsilon}\nabla\cdot(1-\frac{r_1^3}{r^3})\vec{r}$$

$$= -\frac{(\varepsilon-\varepsilon_0)}{3\varepsilon}\rho_f[\nabla\cdot\vec{r}-r_1^3\nabla\cdot\frac{\vec{r}}{r_3}]$$

$$= -\frac{(\varepsilon-\varepsilon_0)\rho_f}{3\varepsilon}[3-0] = -(1-\frac{\varepsilon_0}{\varepsilon})\rho_f$$

在 $r<r_1$ 和 $r>r_2$ 区间 $\vec{P}=0$

在 $r=r_1$ 的面上

$$\sigma' = -\vec{n}\cdot(\vec{P_2}-\vec{P_1})$$

$$= -\vec{n}\cdot[0-(\varepsilon-\varepsilon_0)\frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^3}\vec{r}]$$

$$= -\vec{n}\cdot[0-\frac{(\varepsilon-\varepsilon_0)(r_1^3-r_1^3)\rho_f}{3\varepsilon r_1^3}\vec{r_1}]$$

$$= 0$$

在 $r=r_2$ 的面上

$$\sigma' = -\vec{n}\cdot(\vec{P_2}-\vec{P_1})$$

$$= -\vec{n}\cdot[0-\frac{(\varepsilon-\varepsilon_0)(r_2^3-r_1^3)\rho_f}{3\varepsilon r_2^3}\vec{r_2}]$$

$$= \frac{(\varepsilon-\varepsilon_0)(r_2^3-r_1^3)}{3\varepsilon r_2^3}(\vec{n}\cdot\vec{r_2})\rho_f$$

$$= \frac{(\varepsilon-\varepsilon_0)(r_2^3-r_1^3)}{3\varepsilon r_2^2}\rho_f$$

5、内外半径分别为 r_1 和 r_2 的无穷长中空导体园柱，沿轴向流有稳恒均匀自由电流 J_f，导体的磁导率为 μ，求磁感应强度和磁化电流。

〔解〕　根据沿轴向流有稳恒电流的无穷长中空园柱导体的磁场的轴对称性质，由安培环路定律

$$\oint_L \vec{H}\cdot d\vec{l} = I \quad 得：$$

i) $r<r_1$，$I=0$　　$\vec{H}=\vec{B}=0$

ii) $r_1<r<r_2$　　$H\cdot 2\pi r = \pi(r^2-r_1^2)J_f$

$$H_\varphi = \frac{(r^2 - r_1^2)}{2r} J_f \qquad \vec{H} = \frac{(r^2 - r_1^2) J_f}{2r} \vec{e}_\varphi$$

$$\vec{B} = \mu \vec{H} = \frac{\mu (r^2 - r_1^2) J_f}{2r} \vec{e}_\varphi$$

$$H_r = H_z = 0 \qquad B_r = B_z = 0$$

iii) $r > r_2$ $\quad H_\varphi \cdot 2\pi r = \pi (r_2^2 - r_1^2) j_f$

$$H_\varphi = \frac{(r_2^2 - r_1^2)}{2r} j_f \qquad \vec{H} = \frac{(r_2^2 - r_1^2)}{2r} j_f \vec{e}_\varphi$$

$$H_r = H_z = 0$$

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 (r_2^2 - r_1^2)}{2r} j_f \vec{e}_\varphi$$

在 $r_1 < r < r_2$ 处：

$$\vec{M} = \chi_M \vec{H} = (\mu_r - 1) \vec{H} = (\frac{\mu}{\mu_0} - 1) \vec{H}$$

应用柱坐标中旋度计算公式：

磁化体电流密度

$$\vec{J}' = \nabla \times \vec{M}$$

$$= (\frac{1}{r} \frac{\partial M_z}{\partial \varphi} - \frac{\partial M_\varphi}{\partial z}) \vec{e}_r + (\frac{\partial M_r}{\partial z} - \frac{\partial M_z}{\partial r}) \vec{e}_\varphi + [\frac{1}{r} \frac{\partial (r M_\varphi)}{\partial r} - \frac{1}{r} \frac{\partial M_r}{\partial \varphi}] \vec{e}_z$$

$$M_\varphi = (\frac{\mu}{\mu_0} - 1) H_\varphi = (\frac{\mu}{\mu_0} - 1) \frac{(r^2 - r_1^2) j_f}{2r}$$

$$M_r = M_z = 0$$

$$\therefore \vec{J}' = \frac{1}{r} \frac{\partial}{\partial r} (r M_\varphi) \vec{e}_z$$

$$= \frac{1}{r} \frac{\partial}{\partial r} [r (\frac{\mu}{\mu_0} - 1) \frac{(r^2 - r_1^2)}{2r} j_f] \vec{e}_z$$

$$= (\frac{\mu}{\mu_0} - 1) j_f \vec{e}_z = (\frac{\mu}{\mu_0} - 1) \vec{j}_f$$

根据边界条件，磁化面电流密度

$$\vec{i}' = \vec{n} \times (\vec{M}_2 - \vec{M}_1)$$

在 $r = r_1$ 表面：$\vec{M}_1 = \vec{M}_2 = 0 \quad \vec{i}' = 0$

在 $r = r_2$ 表面：$\vec{M}_2 = 0 \qquad \vec{r} = \vec{e}_r$

$$\vec{i}' = -\vec{n}\times\vec{M}_1 = -\left(\frac{\mu}{\mu_0}-1\right)\frac{(r_2^2-r_1^2)}{2r_2} J_f\,\vec{e}_r\times\vec{e}_\varphi$$

$$= -\left(\frac{\mu}{\mu_0}-1\right)\frac{(r_2^2-r_1^2)}{2r_2} J_f\,\vec{e}_z$$

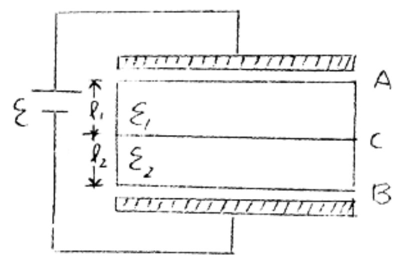
$$= -\left(\frac{\mu}{\mu_0}-1\right)\frac{(r_2^2-r_1^2)}{2r_2}\vec{J}_f$$

6、平行板电容中内有两层介质，它的厚度分别为 l_1 和 l_2，介电常数为 ε_1 和 ε_2，今在两板接上电动势为 $\mathcal{E}$ 的电池。

求 (1) 电容两板上的自由电荷面密度

(2) 介质分界面上的自由电荷面密度

若介质是漏电的，电导率分别为 σ_1 和 σ_2，当电流达到稳恒时，上述两问题的结果如何？



〔解〕 (1) 当介质为绝缘介质，且充电过程结束时，电容两板间电压等于电池电动势 $U_{AB}=\mathcal{E}$，两层介质中的场强分别为 $\vec{E}_1$、$\vec{E}_2$

则 $\mathcal{E} = l_1E_1 + l_2E_2$

A 极板 $\sigma_{fA} = D_1 = \varepsilon_1 E_1 \quad E_1 = \dfrac{\sigma_{fA}}{\varepsilon_1}$

B 极板 $\sigma_{fB} = -D_2 = -\varepsilon_2 E_2\,,\quad E_2 = -\dfrac{\sigma_{fB}}{\varepsilon_2}$

而 $\sigma_{fA} = -\sigma_{fB}$

$$\therefore \mathcal{E} = l_1\frac{\sigma_{fA}}{\varepsilon_1} + l_2\left(-\frac{\sigma_{fB}}{\varepsilon_2}\right) = l_1\frac{\sigma_{fA}}{\varepsilon_1} + l_2\frac{\sigma_{fA}}{\varepsilon_2}$$

$$= \sigma_{fA}\left(\frac{l_1}{\varepsilon_1}+\frac{l_2}{\varepsilon_2}\right)$$

$$\therefore \sigma_{fA} = \frac{\mathcal{E}}{l_1/\varepsilon_1 + l_2/\varepsilon_2}$$

$$\sigma_{fB} = -\sigma_{fA} = -\frac{\mathcal{E}}{l_1/\varepsilon_1 + l_2/\varepsilon_2}$$

中间介质分界面（两边电位移连续 $\vec{D}_2 = \vec{D}_1$

介质分界面自由面电荷密度 $\sigma_{fc} = \vec{n}\cdot(\vec{D}_2 - \vec{D}_1) = 0$

束缚面电荷密度

$$\sigma_{pc} = \vec{n}\cdot(\vec{P}_1 - \vec{P}_2)$$

$$= \vec{n} \cdot (\chi_{e1}\varepsilon_0 \vec{E}_1 - \chi_{e2}\varepsilon_0 \vec{E}_2)$$

$$= \varepsilon_0 \sigma_{fA} \left(\frac{\chi_{e1}}{\varepsilon_1} - \frac{\chi_{e2}}{\varepsilon_2} \right)$$

$$= \frac{\varepsilon_0 \mathcal{E}}{l_1/\varepsilon_1 + l_2/\varepsilon_2} \left[\frac{(1-\varepsilon_{r1})}{\varepsilon_1} - \frac{(1-\varepsilon_{r2})}{\varepsilon_2} \right]$$

$$= \frac{\varepsilon_0 \mathcal{E}}{l_1/\varepsilon_1 + l_2/\varepsilon_2} \left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2} \right)$$

当介质漏电时，两介质中电流连续

$$\vec{j} = \sigma_1 \vec{E}_1 = \sigma_2 \vec{E}_2 \quad E_1 = J/\sigma_1 \quad E_2 = J/\sigma_2$$

$$\mathcal{E} = E_1 l_1 + E_2 l_2 = \frac{J}{\sigma_1} l_1 + \frac{J}{\sigma_2} l_2 = J\left(\frac{l_1}{\sigma_1} + \frac{l_2}{\sigma_2}\right)$$

$$J = \frac{\mathcal{E}}{\frac{l_1}{\sigma_1} + \frac{l_2}{\sigma_2}}$$

$$\sigma_{fA} = \vec{n} \cdot \vec{D}_1 = D_1 = \varepsilon_1 E_1 = \frac{\varepsilon_1 J}{\sigma_1} = \frac{\varepsilon_1 \mathcal{E}}{\sigma_1 (l_1/\sigma_1 + l_2/\sigma_2)}$$

$$\sigma_{fB} = \vec{n} \cdot (-\vec{D}_2) = -D_2 = -\varepsilon_2 E_2 = -\varepsilon_2 \frac{J}{\sigma_2} - \frac{\varepsilon_2 \mathcal{E}}{\sigma_2 (l_1/\sigma_1 + l_2/\sigma_2)}$$

$$\sigma_{fC} = \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = D_2 - D_1 = -\sigma_{fB} - \sigma_{fA}$$

$$= \frac{\varepsilon_2 \mathcal{E}}{\sigma_2 (l_1/\sigma_1 + l_2/\sigma_2)} - \frac{\varepsilon_1 \mathcal{E}}{\sigma_1 (l_1/\sigma_1 + l_2/\sigma_2)}$$

$$= \frac{\mathcal{E}}{(l_1/\sigma_1 + l_2/\sigma_2)} (\varepsilon_2/\sigma_2 - \varepsilon_1/\sigma_1)$$

$$= \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)}{(l_1\sigma_2 + l_2\sigma_1)} \mathcal{E}$$

7、证明

(1) 当两种绝缘介质的分界面上不带面自由电荷时，电力线的曲折满足 $tg\theta_2 / tg\theta_1 = \varepsilon_2 / \varepsilon_1$

其中 ε_1、ε_2 分别为两种介质的介电常数，θ_1 和 θ_2 分别为界面两侧电力线与法线的夹角。

(2) 当两种导电介质内流有稳恒电流时，分界面上电力线曲折满足 $tg\theta_2 / tg\theta_1 = \sigma_2 / \sigma_1$

其中 σ_1 和 σ_2 分别为两种介质的电导率。

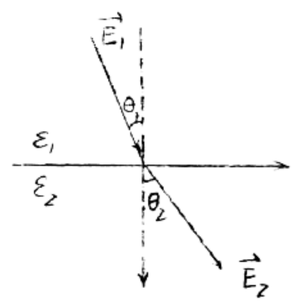
〔证明〕(1) 根据边值关系

$\vec{n}\cdot(\vec{D}_2-\vec{D}_1)=\sigma_f=0$　　$\vec{n}\times(\vec{E}_2-\vec{E}_1)=0$

得 $D_{2n}=D_{1n}$　$\varepsilon_1E_{1n}=\varepsilon_2E_{2n}$　$\varepsilon_1E_1\cos\theta_1=\varepsilon_2E_2\cos\theta_2$　(1)

及 $E_1\sin\theta_1=E_2\sin\theta_2$　(2)

由 (2) ÷ (1) 得证 $tg\theta_2/tg\theta_1=\varepsilon_2/\varepsilon_1$



(2) 根据稳恒电流条件

$\nabla\cdot\vec{J}=0$

$\iiint\limits_V(\nabla\cdot\vec{J})dV=\oint\limits_S\vec{J}\cdot d\vec{S}=0$

作包含界面在内的封闭曲面，使界面两边曲面无限靠拢界面，则

$\vec{J}_1\cdot\Delta\vec{S}_1+\vec{J}_2\cdot\Delta\vec{S}_2=0$

∴有 $J_{1n}=J_{2n}$　$\sigma_1E_{1n}=\sigma_2E_{2}$

$\sigma_1E_1\cos\theta_1=\sigma_2E_2\cos\theta_2$　(3)

且由　$\vec{n}\times(\vec{E}_2-\vec{E}_1)=0$

有　$E_2\sin\theta_2=E_1\sin\theta_1$　(4)

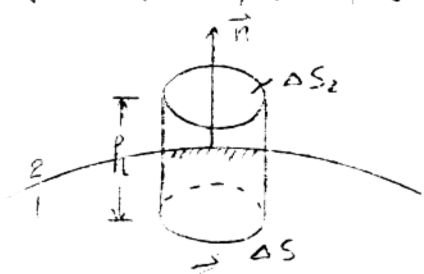
(4) ÷ (3) 得证　$tg\theta_2/tg\theta_1=\sigma_2/\sigma_1$

8、证明 (1) $\vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0$

(2) $\vec{n}\times(\vec{E}_2-\vec{E}_1)=0$

〔证明〕(1) 根据磁学高斯定理 $\oint\limits_S\vec{B}\cdot d\vec{S}=0$

在界面上取任一小面元ΔS，过ΔS周界作一扁平柱体垂直于界面，柱体两底面ΔS_1，ΔS_2分别在介质1、2中，且平行于界面，取这个扁平柱体的表面为高斯面，于是有：



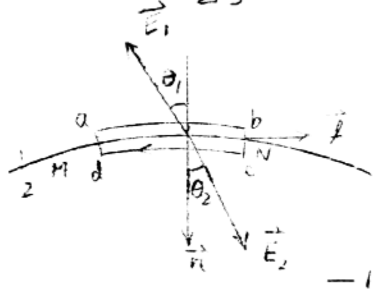
$\vec{B}_2\cdot\Delta\vec{S}_2+\vec{B}_1\cdot\Delta\vec{S}_1+\vec{B}_{侧}\cdot\Delta\vec{S}_{侧}=0$

当扁平圆柱体高$h\to0$时，上下底面ΔS_1、ΔS_2都无限趋近于界面，侧面面积$\Delta S_{侧}\to0$

$\therefore\ \vec{B}_2\cdot\Delta S\vec{n}-\vec{B}_1\cdot\Delta S\vec{n}=0$

得证：$\vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0$

(2) 根据麦氏方程



$$\oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

取一狭长封闭回路a、b、c、d使它包围一段界面截线MN。Δl_{ab}、Δl_{cd}都很短，且与截线MN平行，可以认为ab与cd边上各点的场强都相同，于是：

$$\vec{E}_1\cdot\Delta\vec{l}_{ab}+\vec{E}_2\cdot\Delta\vec{l}_{cd}+\overline{E}\cdot\Delta\vec{l}_{bc}+\overline{E}\cdot\Delta\vec{l}_{da}=-\left(\frac{d}{dt}\overline{B}\right)\cdot\Delta\vec{S}$$

当ab与cd无限趋近于MN时，$\Delta l_{bc}\to 0$，$\Delta l_{ba}\to 0$，封闭回路所包围面积$\Delta\vec{S}\to 0$，且$\frac{d}{dt}\vec{B}$有限值

$$\therefore\ \vec{E}_1\cdot\Delta\vec{l}_{ab}+\vec{E}_2\cdot\Delta\vec{l}_{cd}=0$$

设沿界面线MN方向即沿界面线的切线方向单位矢量为$\vec{t}$，则$\Delta\vec{l}_{ab}$的方向与$\vec{t}$相同，$\Delta\vec{l}_{cd}$的方向与$\vec{t}$相反，它们长均为Δl，于是 $\vec{E}_1\cdot\Delta l\vec{t}-\vec{E}_2\cdot\Delta l\vec{t}=0$

$$\vec{t}\cdot(\vec{E}_2-\vec{E}_1)\Delta l=0 \qquad \vec{t}\cdot(\vec{E}_2-\vec{E}_1)=0$$

若设该点界面法线$\vec{n}$与对应点在两介质中的场强$\vec{E}_1$、$\vec{E}_2$的夹角分别为θ_1和θ_2，则上式可写为

$$E_2\cos\left(\frac{\pi}{2}-\theta_2\right)-E_1\cos\left(\frac{\pi}{2}-\theta_1\right)=0$$

或 $E_2\sin\theta_2-E_1\sin\theta_1=0$

于是得证 $\vec{n}\times\vec{E}_2-\vec{n}\times\vec{E}_1=0$ 即 $\vec{n}\times(\vec{E}_2-\vec{E}_1)=0$

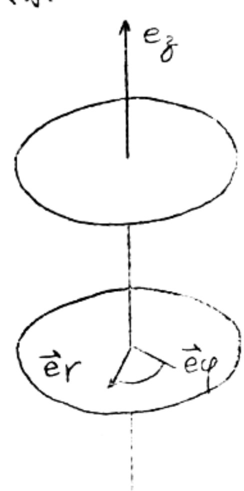
9、在一平行板电容器的两板上加以$u=u_0\cos\omega t$的电压若平板为圆形，半径为a，板间距离为d，试求

(1) 两板间的位移电流

(2) 电容器内离轴为r处的磁场强度

(3) 电容器内的能流密度

(4) 能流密度的平均值



〔解〕 取柱坐标

$$(1)\ I_d=\iint_S\vec{J}_d\cdot d\vec{S}$$
$$=\iint_S\varepsilon_0\frac{\partial\vec{E}}{\partial t}\cdot d\vec{S}$$
$$=\iint_S\varepsilon_0\frac{\partial}{\partial t}\left(\frac{u}{d}\vec{e}_z\right)\cdot d\vec{S}$$
$$=\varepsilon_0\frac{\partial}{\partial t}\left(\frac{u_0}{d}\cos\omega t\right)\pi a^2$$

$$= -\frac{\varepsilon_0 \omega u_0}{d}\pi a^2 \sin\omega t$$

(2) $\oint \vec{H}\cdot d\vec{l} = I_d = \iint \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S}$

$$2\pi r\cdot H = -\frac{\varepsilon_0 \omega u_0}{d}\pi r^2 \sin\omega t$$

$$H = -\frac{\varepsilon_0 \omega u_0 r}{2d}\sin\omega t$$

$$\vec{H} = \frac{\varepsilon_0 \omega u_0 r}{2d}\sin\omega t\,\vec{e}_\varphi \qquad \vec{e}_\varphi = \vec{e}_z \times \vec{e}_r$$

(3) $\vec{S} = \vec{E}\times\vec{H}$

$$= \frac{u_0}{d}\cos\omega t\,\vec{e}_z \times \frac{\varepsilon_0 \omega u_0 r}{2d}\sin\omega t\,\vec{e}_\varphi$$

$$= \frac{\varepsilon_0 \omega u_0^2}{2d^2}\sin\omega t\cos\omega t\,\vec{e}_z\times\vec{e}_\varphi$$

$$= -\frac{\varepsilon_0 \omega u_0^2 r}{2d^2}\sin\omega t\cos\omega t\,\vec{e}_r$$

(4) $\bar{S} = \frac{1}{T}\int_0^T S\,dt$

$$= \frac{\varepsilon_0 \omega u_0^2 r}{2dT}\int_0^T \sin\omega t\cos\omega t\,dt$$

$$= \frac{\varepsilon_0 \omega u_0^2 r}{2d^2 T}\frac{1}{2}\sin^2\omega t\Big|_0^T = 0$$

10、同轴传输线内导线半径为 a，外导线半径为 b，两导线间为均匀绝缘介质，导线载有电流 I，两导线间的电压为 V。

(1)忽略导线的电阻，计标介质中的能流 $\vec{S}$ 和传输功率；

(2)计及内导线的有限电阻率，计标通过内导线表面进入导线内的能流，证明它等于导线的损耗功率。

解：(1) 以距对称轴为 r 的半径作一圆周（$a<r<b$），应用安培环路定理，由对称性得 $2\pi r H_\theta = I$，因而

$$H_\theta = \frac{I}{2\pi r}$$

导线表面上一般带有电荷，设内导线单位长度的电荷（电荷线密度）为 λ，应用高斯定理，由对称性可得 $2\pi r E_r = \frac{1}{\varepsilon}\lambda$，因而

$$E_r = \frac{\lambda}{2\pi\varepsilon r}$$

能流密度为

$$\vec{S}=\vec{E}\times\vec{H}=E_rH_\theta\vec{e}_z=\frac{I\lambda}{4\pi^2\varepsilon r^2}\vec{e}_z$$

式中$\vec{e}_z$为沿导线轴向单位矢量。

两导线间的电压为

$$V=\int_a^b E_r dr=\frac{\lambda}{2\pi\varepsilon}\ln\frac{b}{a}$$

因而 $$\vec{S}=\frac{UI}{2\pi\ln\frac{b}{a}}\frac{1}{r^2}\vec{e}_z$$

把$\vec{S}$对两导线间圆环状截面积分得传输功率

$$P=\int_a^b S\cdot 2\pi r dr=\int_a^b\frac{UI}{\ln\frac{b}{a}}\frac{1}{r}dr=UI$$

UI 即为通常在电路问题中的传输功率表示式，这功率是在场中传输的。

(2)设导线内的电导率为σ，由欧姆定律，在导线内部有

$$\vec{E}=\frac{\vec{J}}{\sigma}=\frac{I}{\pi a^2\sigma}\vec{e}_z$$

由于电场是连续的，因此在紧贴内导线表面的介质内，电场除有径向分量E_r外，还有切向分量E_z

$$E_z\Big|_{r=a}=\frac{I}{\pi a^2\sigma}$$

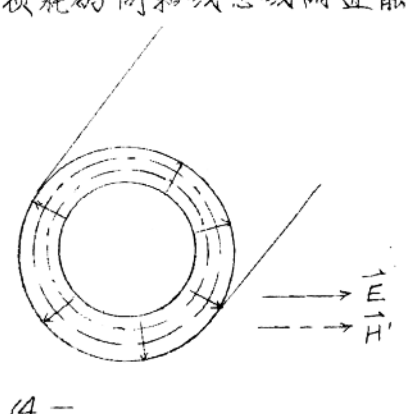
因此能流除有沿z轴传输的分量S_z外，还有沿径向进入导线内的分量$-S_r$

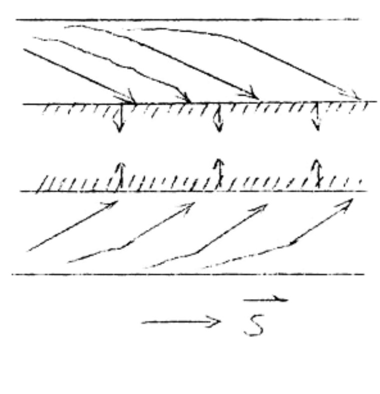
$$-S_r=E_zH_\theta\Big|_{r=a}=\frac{I^2}{2\pi^2a^3\sigma}$$

进入长度为Δl的导线内部的功率为

$$S_r\cdot 2\pi a\Delta l=I^2\frac{\Delta l}{\pi a^2\sigma}=I^2R$$

式中R为该段导线的电阻，I^2R正是该段导线内的损耗功率在有损耗的同轴线芯线附近能流如图所示。





第二章 静电场和稳恒磁场

1、设处于基态的氢原子中的电子电荷以体密度 $\rho(r)=\frac{-e}{\pi a^3}e^{-\frac{2r}{a}}$ 分布，a 是玻尔原子半径，e 是电子电荷。求电子电荷场的电势 φ_e 及场强 E_{er}，若将质子电荷看作集中于原点，求原子中的总电势 φ 及总场强 $\vec{E}$。

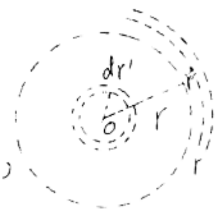
〔解〕由题意知，由于电荷依球对称分布，将电荷分成厚度为 dr 的球层，球层中所含电荷为 dq'，设场点为 P，$op=r$

P 以内（$r'<r$）的球层元在 P 点产生的电势为 $\frac{dq'}{4\pi\varepsilon_0 r}$

P 以外（$r'>r$）的球层元在 P 点产生的电势为 $\frac{dq}{4\pi\varepsilon_0 r'}$

$$\therefore \varphi_e(r)=\frac{1}{4\pi\varepsilon_0}\left(\int_0^r\frac{dq'}{r}+\int_r^\infty\frac{dq'}{r'}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\int_0^r\frac{\rho(r')4\pi r'^2dr'}{r}+\int_r^\infty\frac{\rho(r')4\pi r'^2dr'}{r'}\right)$$



$$=\frac{1}{\varepsilon_0 r}\int_0^r\rho(r')r'^2dr'+\frac{1}{\varepsilon_0}\int_r^\infty\rho(r')r'dr' \quad (1)$$

以 $\rho(r')=-\frac{e}{\pi a^3}e^{-\frac{2r'}{a}}$ 代入(1)式得

$$\varphi_e(r)=-\frac{e}{\pi\varepsilon_0 a^3}\left[\frac{1}{r}\int_0^r e^{-\frac{2r'}{a}}r'^2dr'+\int_r^\infty e^{-\frac{2r'}{a}}r'dr'\right] \quad (2)$$

通过分部积分，可求得：

$$\int_0^r e^{-\frac{2r'}{a}}r'^2dr'=r'^2\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}}\Big|_0^r-\int_0^r 2r'\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}}dr'$$

$$=-\frac{ar^2}{2}e^{-\frac{2r}{a}}+a\int_0^r r'e^{-\frac{2r'}{a}}dr'$$

$$=-\frac{ar^2}{2}e^{-\frac{2r}{a}}+a\left[r'\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}}\Big|_0^r-\int_0^r\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}}dr'\right]$$

$$=-\frac{ar^2}{2}e^{-\frac{2r}{a}}-\frac{a^2r}{2}e^{-\frac{2r}{a}}+\frac{a^2}{2}\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}}\Big|_0^r$$

$$=-\frac{ar^2}{2}e^{-\frac{2r}{a}}-\frac{a^2r}{2}e^{-\frac{2r}{a}}-\frac{a^3}{4}e^{-\frac{2r}{a}}+\frac{a^3}{4}$$

$$=-\frac{a}{2}\left[r^2e^{-\frac{2r}{a}}+are^{-\frac{2r}{a}}+\frac{a^2}{2}e^{-\frac{2r}{a}}-\frac{a^2}{2}\right]$$

$$=-\frac{a}{2}\left[r(r+a)e^{-\frac{2r}{a}}+\frac{a^2}{2}\left(e^{-\frac{2r}{a}}-1\right)\right]$$

同理可得

$$\int_r^\infty e^{-\frac{2r'}{a}}r'dr'=\frac{a^2}{4}\left(1+\frac{2r}{a}\right)e^{-\frac{2r}{a}}$$

$$\therefore \frac{1}{r}\int_0^r e^{-\frac{2r'}{a}} r'^2 dr' + \int_r^\infty e^{-\frac{2r'}{a}} r' dr'$$

$$= -\frac{a}{2}\frac{r^2}{r}e^{-\frac{2r}{a}} - \frac{a^2}{2}\frac{r}{r}e^{-\frac{2r}{a}} - \frac{a^2}{4}\frac{a}{r}e^{-\frac{2r}{a}} + \frac{a^2}{4}\frac{a}{r} + \frac{a^2}{4}e^{-\frac{2r}{a}} + \frac{a}{4}2re^{-\frac{2r}{a}}$$

$$= \frac{a^2}{4}\left[\frac{a}{r}\left(1-e^{-\frac{2r}{a}}\right) - e^{-\frac{2r}{a}}\right] \qquad (3)$$

以(3)代入(2)式，得

$$\varphi_e(r) = -\frac{1}{4\pi\varepsilon_0}\left[\frac{e}{r}\left(1-e^{-\frac{2r}{a}}\right) - \frac{e}{a}e^{-\frac{2r}{a}}\right]$$

$$E_{er}(r) = -\frac{d\varphi_e}{dr} = \frac{1}{4\pi\varepsilon_0}\left[-\frac{e}{r^2}\left(1-e^{-\frac{2r}{a}}\right) + \frac{2e}{ar}e^{-\frac{2r}{a}} + \frac{2e}{a^2}e^{-\frac{2r}{a}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{-\frac{e}{r^2}\left[1-\left(1+\frac{2r}{a}\right)e^{-\frac{2r}{a}}\right] + \frac{2e}{a^2}e^{-\frac{2r}{a}}\right\}$$

原子中的总电势和场强为位于原点的质子电荷产生的电势、场强与电子电荷产生的电势、场强的迭加

即 $\varphi(r) = \varphi_e(r) + \frac{1}{4\pi\varepsilon_0}\frac{e}{r}$

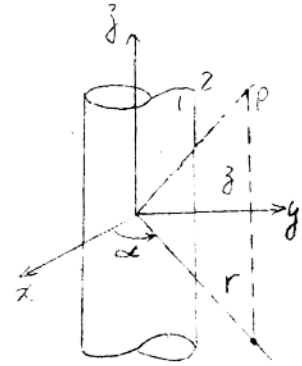
$E(r) = E_{er}(r) + \frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}$

$\vec{E}(r) = \vec{E}_e(r) + \frac{1}{4\pi\varepsilon_0}\frac{e}{r^3}\vec{r}$

2、一无限长圆柱体，半径为a，单位长荷电λ，在下列两种电荷分布下，应用电势的微分方程求柱内外的电势和电场：1)电荷均匀地分布于柱体；2)电荷均匀地分布于柱面。

〔解〕根据问题的轴对称性质，选用柱坐标

泊松方程 $\nabla\varphi = -\frac{\rho}{\varepsilon}$ 的柱坐标形式为



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{\partial^2\varphi}{\partial z^2} = -\frac{\rho}{\varepsilon}$$

其中 $\rho = \begin{cases} \frac{\lambda}{\pi a^2} & (r<a) \\ 0 & (r>a) \end{cases}$

由于场具有轴对称 $\varphi = \varphi(r)$

$\therefore \frac{\partial^2\varphi}{\partial\alpha^2} = 0,\ \frac{\partial^2\varphi}{\partial z^2} = 0$

现分两个区域分别求解

1) 体分布情形

柱内 $r<a$, $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -\frac{\lambda}{\pi\varepsilon_0 a^2}$ (1)