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微分方程題解

編者 龔晨

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AN ELEMENTARY TREATISE

ON

DIFFERENTIAL EQUATIONS

BY

ABRAHAM COHEN

東華書社印行

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KEY TO COHEN'S DIFFERENTIAL EQUATIONS.

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CHAPTER I

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

§1—4 P. 9.

Ex. 3. $y = cx + \sqrt{1-c^2}$ (1)

$\frac{d}{dx}(1)$, $y' = c$ (2) $\therefore y = xy' + \sqrt{1-y'^2}$ (3)

Ex. 4. $(x-a)^2 + (y-b)^2 = r^2$ (1)

$\frac{d}{dx}(1)$, $(x-a) + (y-b)y' = 0$ (2)

$\frac{d}{dx}(2)$ $1 + y'^2 + (y-b)y'' = 0$ (3)

$\therefore y - b = \frac{-(1+y'^2)}{y''}$ (4) $x - a = \frac{y'(1+y'^2)}{y''}$ (5)

(4) and (5) subs into (1)

$$\frac{y'^2(1+y'^2)^2}{y''^2} + \frac{(1+y'^2)^2}{y''^2} = r^2 \quad (6)$$

Hence $(1+y'^2)^3 = r^2 y''^2$ (7)

Ex. 5. $y = c_1 x^2 + c_2$ (1)

$\frac{d}{dx}(1)$, $y' = 2c_1 x$ (2)

$\frac{d}{dx}(2)$, $y'' = 2c_1$ (3)

$(2)/(3)$ $\frac{y'}{y''} = x$ (4)

Hence $xy'' - y' = 0$ (5)

② Ex.6 $x^2+y^2+cx=0$ (1) $\frac{d}{dx}(1)$, $2x+2yy'+c=0$ (2)

$\therefore c=-2x-2yy'$ (3) (3) Subs. into (1)

We have $y^2-x^2-2xyy'=0$ (4)

Ex.7. $c_1x^2+c_2y^2=1$ (1)

$\frac{d}{dx}(1)$, $2c_1x+2c_2yy'=0$ (2)

$\frac{d}{dx}(2)$, $2c_1+2c_2yy''+2c_2y'^2=0$ (3).

From (2) $\frac{c_1}{c_2}=-\frac{yy'}{x}$ (4)

From (3) $\frac{c_1}{c_2}=-(y'^2+yy'')$ (5)

$\therefore yy'=x(y'^2+yy'')$ (6)

Hence $xyy''+xy'^2-yy'=0$ (7)

Ex.8. $y=ae^{kx}+be^{lx}$ (1)

$\frac{d}{dx}(1)$, $y'=kae^{kx}+lbe^{lx}$ (2)

$\frac{d}{dx}(2)$, $y''=k^2ae^{kx}+l^2be^{lx}$ (3)

eliminant a and b between equations (1) (2) and (3)

We have

$$\begin{vmatrix} e^{kx} & e^{lx} & y \\ ke^{kx} & le^{lx} & y' \\ k^2e^{kx} & l^2e^{lx} & y'' \end{vmatrix}=0 \quad (4)$$

i.e.

$$\begin{vmatrix} 1 & 1 & y \\ k & l & y' \\ k^2 & l^2 & y'' \end{vmatrix}=0 \quad (5)$$

Hence

$$y''-(k+l)y'+kly=0 \quad (6)$$

Ex 9.
$$x^2 = 2cy + c^2 \quad (1)$$

$\frac{d}{dx}(1)$, $\quad x = cy' \quad (2) \quad \therefore c = xx' \quad (3)$

(3) Subs into (1) $\quad x^2 = 2xyx' + x^2x'^2 \quad (4)$

Hence $\quad xy'^2 - 2yy' - x = 0 \quad (5)$

Ex 10. $\quad y^2 = 4c(x+c) \quad (1)$

$\frac{d}{dx}(1)$, $\quad 2yy' = 4c \quad (2) \quad c = \frac{yy'}{2} \quad (3)$

$y^2 = 2yy'(x + \frac{yy'}{2}) \quad (4)$ Hence $yy'^2 - 2xyy' - y = 0 \quad (5)$

Ex 11. $\quad y = \sin(x+c) \quad (1) \quad y' = \cos(x+c) \quad (2)$

$(1)^2 + (2)^2$, $\quad \therefore y^2 + y'^2 = 1 \quad (3)$

Ex 12. $\quad y = \sin cx \quad (1)$

$\sin^{-1} y = cx \quad (2)$, $\quad \frac{d}{dx}(2)$,

$$\frac{y'}{\sqrt{1-y^2}} = c \quad (3)$$

(3) Subs. into (2),

$$\sin^{-1} y = \frac{xy'}{\sqrt{1-y^2}} \quad (4)$$

Hence $\quad y = \sin \frac{xy'}{\sqrt{1-y^2}} \quad (5)$

④

CHAPTER II.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

§II—1. p. 15.

EX.4. $xy\,dx + \sqrt{1+x^2}\,dy = 0$ (1)

Separating of variables, we have

$$\frac{x\,dx}{\sqrt{1+x^2}} + \frac{dy}{y} = 0 \qquad (2)$$

integrating, $\int \frac{x\,dx}{\sqrt{1+x^2}} + \int \frac{dy}{y} = c$ (3)

we get $\sqrt{1+x^2} + \log y = c$ (4).

EX.5. $(x - xy^2)\,dx + (y + x^2y)\,dy = 0$ (1)

$x(1-y^2)dx + y(1+x^2)dy = 0$ (2)

Separating of variables, we have

$$\frac{x\,dx}{1+x^2} + \frac{y\,dy}{1-y^2} = 0 \qquad (3)$$

integrating, $\int \frac{2x\,dx}{1+x^2} + \int \frac{2y\,dy}{1-y^2} = \log c$ (4).

$\log(1+x^2) - \log(1-y^2) = \log c$ (5)

$\frac{1+x^2}{1-y^2} = c$ (6) i.e.

$x^2 + cy^2 = c - 1$ (7)

Ex.6. $(y+3)\,dx + \cot x\,dy = 0$ (1)

Separating of variables,

$$\frac{dx}{\cot x} + \frac{dy}{y+3} = 0 \quad (2)$$

integrating, $\log \sec x + \log(y+3) = \log c$ (3)

$\sec x(y+3) = c$ (4) $\quad \therefore y+3 = c\cos x$ (5)

Hence $\quad y = c\cos x - 3$ (6)

Ex.7. $\sin x \cos y\,dx + \cos x \sin y\,dy = 0$ (1)

Separating of variables.

$$\frac{\sin x}{\cos x}dx + \frac{\sin y}{\cos y}dy = 0 \quad (2)$$

$\tan x\,dx + \tan y\,dy = 0$. (3)

integrating, $\log \sec x + \log \sec y = \log \frac{1}{c}$ (4)

$\therefore \sec x \sec y = \frac{1}{c}$ (5) i.e. $\cos x \cos y = c$ (6)

Ex.8. $\sin x \cos^2 y\,dx + \cos^2 x\,dy = 0$ (1)

Separating of variables,

$$\frac{\sin x}{\cos^2 x}dx + \frac{dy}{\cos^2 y} = 0 \quad (2)$$

$\sec x \tan x\,dx + \sec^2 y\,dy = 0$ (3)

(6) integrating, $\sec x + \tan y = c$ (4)

§ II–2 p16–17.

Ex. 3. $(y^2 - xy)dx + x^2 dy = 0$ (1)

Put $y = vx$ then $dy = v\,dx + x\,dv$

(1) reduces to

$x^2(v^2 - v)dx + x^2(v\,dx + x\,dv) = 0$ (2)

$v^2 dx + x\,dv = 0$ (3)

$\frac{dx}{x} + \frac{dv}{v^2} = 0$ (4) integrating,

$\int \frac{dx}{x} + \int \frac{dv}{v^2} = c$ (5)

$\log x - \frac{1}{v} = c$ (6) i.e. $\log x - \frac{x}{y} = c$ (7)

Ex4. $x\frac{dy}{dx} - y + \sqrt{y^2 - x^2} = 0$ (1)

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(1) reduces to

$x^2 \frac{dv}{dx} + x\sqrt{v^2 - 1} = 0$ (2)

$\frac{dv}{\sqrt{v^2-1}} + \frac{dx}{x} = 0$ (3)

integrating, $\log(v + \sqrt{v^2-1}) + \log x = \log c$ (4)

$vx + \sqrt{v^2x^2 - x^2} = c$ (5)

$y + \sqrt{y^2 - x^2} = c$ (6)

$$\sqrt{y^2-x^2} = c-y \quad (7)$$ each side square,

$$y^2-x^2 = c^2-2cy+y^2 \quad (8)$$ hence

$$x^2-2cy+c^2=0 \quad (9).$$

Ex5. $(2x^2y+3y^3)dx-(x^3+2xy^2)dy=0 \quad (1)$

put $y=vx$ then $dy = x\,dv + v\,dx$

(1) reduces to

$$x^3(2v+3v^3)\,dx - x^3(1+2v^2)(v\,dx+x\,dv) = 0 \quad (2)$$

$$(v+v^3)\,dx - x(1+2v^2)\,dv = 0 \quad (3)$$

$$\frac{dx}{x} - \frac{(1+2v^2)}{v(1+v^2)}\,dv = 0 \quad (4)$$

$$\frac{dx}{x} - \frac{dv}{v} - \frac{v\,dv}{1+v^2} = 0 \quad (5)$$

integrating, $\log x - \log v - \frac{1}{2}\log(1+v^2) = -\log\frac{1}{\sqrt{c}}$ (6)

$$v\sqrt{1+v^2} = \sqrt{c}\,x \quad (7)$$

$$y\sqrt{x^2+y^2} = \sqrt{c}\,x^3 \quad (8)$$

each side square,

$$y^2(x^2+y^2) = cx^6 \quad (9).$$

Ex6. $\left(x+y\cos\frac{y}{x}\right)dx - x\cos\frac{y}{x}\,dy = 0 \quad (1)$

It is same as the equation

$$\left(1+\frac{y}{x}\cos\frac{y}{x}\right)dx - \cos\frac{y}{x}\,dy = 0 \quad (2)$$

⑧ put $y=vx$ then $dy=x\,dv+v\,dx$

(2) reduces to

$$(1+v\cos v)dx-\cos v(x\,dv+v\,dx)=0 \quad (3)$$

$$dx-x\cos v\,dv=0 \quad (4)$$

$$\frac{dx}{x}-\cos v\,dv=0 \quad (5)$$

integrating, $\ln x-\sin v=c \quad (6)$

i.e. $\log x-\sin\frac{y}{x}=c \quad (7)$

Ex 7. $(x^2+y^2)dx-axy\,dy=0 \quad (1)$

put $y=vx$ then $dy=x\,dv+v\,dx$

(1) reduces to $(1+v^2-av^2)dx-avx\,dv=0 \quad (2)$

Separating of variables

$$\frac{dx}{x}-\frac{av\,dv}{1+(1-a)v^2}=0 \quad (4)$$

$$\frac{2dx}{x}=\frac{2av\,dv}{1+(1-a)v^2} \quad (5)$$

integrating,

$$\frac{a}{1-a}\log[1+(1-a)v^2]=\log Kx^2 \quad (6)$$

$$[1+(1-a)v^2]^{\frac{a}{1-a}}=Kx^2 \quad (7)$$

We have $[(1-a)y^2+x^2]^{\frac{a}{1-a}}=Kx^2x^{\frac{2a}{1-a}} \quad (8)$

$$[x^2+(1-a)y^2]^{\frac{a}{1-a}} = K x^{\frac{2}{1-a}} \quad (9)$$

Hence $[x^2+(1-a)y^2]^a = c x^2$ (10)

where $c = K^{1-a}$

Ex 8. $(y^2-x^2+2mxy)\,dx + (my^2-mx^2-2xy)\,dy = 0$ (1)

Put $y = vx$ (1) becomes

$$(v^2-1+2mv)\,dx + (mv^2-m-2v)(x\,dv+v\,dx) = 0 \quad (2)$$

$$(mv^3-mv-2v^2+v^2-1+2mv)\,dx$$
$$+x(mv^2-m-2v)\,dv = 0 \quad (3)$$

$$(mv^3-v^2+mv-1)\,dx + x(mv^2-2v-m)\,dv = 0 \quad (4)$$

$$\frac{mv^2-2v-m}{(mv-1)(v^2+1)}\,dv + \frac{dx}{x} = 0 \quad (5)$$

by partial fraction

$$\frac{-m}{mv-1}\,dv + \frac{2v}{v^2+1}\,dv + \frac{dx}{x} = 0 \quad (6)$$

integrating,

$$\int\frac{2v\,dv}{v^2+1} - m\int\frac{dv}{mv-1} + \int\frac{dx}{x} = \log c \quad (7)$$

$$\log\left(\frac{v^2+1}{mv-1}\right) = \log c \quad (8)$$

$$(v^2+1) = c\,(mv-1) \quad (9)$$

Hence $x^2+y^2 = c(my-x)$ (10).

Ex2. $(4x-y+2)dx+(x+y+3)dy=0$ (1)

put $x=X+\alpha$ $y=Y+\beta$

determine the values of α and β such that

$$\begin{cases} 4\alpha-\beta+2=0 & (2) \\ \alpha+\beta+3=0 & (3) \end{cases} \quad \therefore \begin{cases} \alpha=-1 \\ \beta=-2 \end{cases}$$

and (1) becomes

$$(4X-Y)dX+(X+Y)dY=0 \quad (2)$$

Put $Y=vX$ then (2) reduces to

$$(4-v)dX+(1+v)(Xdv+vdX)=0 \quad (3)$$

$$(4+v^2)dX+(1+v)Xdv=0 \quad (4)$$

$$\frac{dX}{X}+\frac{v+1}{4+v^2}dv=0 \quad (5)$$

$$\frac{dX}{X}+\frac{v\,dv}{v^2+4}+\frac{dv}{v^2+4}=0 \quad (6)$$

integrating $\int\frac{dX}{X}+\int\frac{v\,dv}{v^2+4}+\int\frac{dv}{v^2+4}=C'$ (7)

$$\log X+\frac{1}{2}\log(v^2+4)+\frac{1}{2}\tan^{-1}\frac{v}{2}=C' \quad (8)$$

$$\log(v^2+4)X^2+\tan^{-1}\frac{v}{2}=C \quad (9)$$

$$\log(Y^2+4X^2)+\tan^{-1}\left(\frac{Y}{2X}\right)=C \quad (10)$$

Subs. $Y=y+2$, $X=x+1$ back into (10)

We have

$$\tan^{-1}\left(\frac{y+2}{2x+2}\right)+\log\left[4(x+1)^2+(y+2)^2\right]=C \quad (11)$$

EX.3. $(2x-y)dx+(x-2y-3)dy=0 \quad (1)$

The equations $2\alpha-\beta=0$ and $\alpha-2\beta-3=0$ have the solution $\alpha=-1,\ \beta=-2$

$\therefore\ x=X-1, \quad y=Y-2,$

and (1) reduces to the form

$$(2X-Y)dX+(X-2Y)dY=0 \quad (2)$$

put $Y=vX$ (2) reduces to the form

$$(2-v)\,dX+(1-2v)(X\,dv+v\,dX)=0 \quad (3)$$

$$2(1-v^2)dX+(1-2v)X\,dv=0 \quad (4)$$

$$\frac{2dX}{X}+\frac{1-2v}{1-v^2}dv=0 \quad (5)$$

integrating $\displaystyle\int\frac{2dX}{X}+\int\frac{dv}{1-v^2}-\int\frac{2v\,dv}{1-v^2}=\log c' \quad (6)$

$$\log X^2+\frac{1}{2}\log\left(\frac{1+v}{1-v}\right)+\log(1-v^2)=\log c' \quad (7)$$

$$(1+v)^{\frac{3}{2}}(1-v)^{\frac{1}{2}}X^2=c' \quad (8)$$

$$(X+Y)^{\frac{3}{2}}(X-Y)^{\frac{1}{2}}=c' \quad (9)$$

$$(X+Y)^3(X-Y)=C \quad (10) \qquad \text{here } X=(x+1),\ Y=(y+2)$$

hence $(x+y+3)^3(x-y-1)=C \quad (11)$

(12) EX4. $(2x+y)dx-(4x+2y-1)dy=0$ (1)

put $2x+y=v$ then $dy=dv-2dx$

(1) reduces to

$vdx-(2v-1)(dv-2dx)=0$ (2)

$(5v-2)dx-(2v-1)dv=0$ (3)

$\left(\frac{2v-1}{5v-2}\right)dv=dx$ (4)

integrating, $\frac{2}{5}v-\frac{1}{25}\ln(5v-2)=x+c'$ (5)

$10v-\ln(5v-2)=25x+c''$ (6)

$20x+10y-\ln(10x+5y-2)=c''+25x$ (7)

Hence $5x-10y+\ln(10x+5y-2)=C$ (8).

EX5. $(2x-y+3)dx+(4x-2y+7)dy=0$ (1)

put $2x-y=v$ then $dy=2dx-dv$

the equation takes the form

$(v+3)dx+(2v+7)(2dx-dv)=0$ (2)

$(5v+17)dx-(2v+7)dv=0$ (3)

$dx-\left(\frac{2v+7}{5v+17}\right)dv=0$ (4)

$5dx-\frac{10v+35}{5v+17}dv=0$ (5)

$5dx-2dv-\frac{dv}{5v+17}=0$ (6)

integrating, $5x-2v-\frac{1}{5}\log(5v+17)=c'$ (7)

Subs. $v=2x-y$ into (7) we have

$$x+2y-\tfrac{1}{5}\log(10x-5y+17)=c' \quad (8)$$

i.e. $$5x+10y-\log(10x-5y+17)=C \quad (9)$$

Ex.6. $(15x+6y-7)dx+(5x+2y-3)dy=0$ (1)

put $5x+2y=v$ then $dy=\frac{dv-5dx}{2}$

the equation takes the form

$$(3v-7)dx+(v-3)\left(\frac{dv-5dx}{2}\right)=0 \quad (2)$$

$$(6v-14)dx+(v-3)dv-(5v-15)dx=0 \quad (3)$$

$$(v+1)dx+(v-3)dv=0 \quad (4)$$

$$dx+\frac{v-3}{v+1}dv=0 \quad (5)$$

$$dx+dv-\frac{4}{v+1}=0 \quad (6)$$

integrating, $x+v-4\log(v+1)=c'$ (7)

i.e. $$6x+2y-4\log(5x+2y+1)=c' \quad (8)$$

or $$3x+y-2\log(5x+2y+1)=c \quad (9).$$

§ II.— 4. p.21.

Ex2. $x^3y'-x^2y+y^2=0$ (1)

The wts. of each term are

$3+m-1$
$\equiv m+2,\quad 2+m,\quad 2m$

$\therefore$ (1) is isobaric when $m=2$

let $v=y/x^2$ then $y'=2xv+x^2\frac{dv}{dx}$

(14) (1) reduces to the form

$$x^3\left(2xv+x^2\frac{dv}{dx}\right)-vx^4+v^2x^4=0 \quad (2)$$

$$2v+x\frac{dv}{dx}-v+v^2=0 \quad (3)$$

$$x\frac{dv}{dx}+v(1+v)=0 \quad (4)$$

$$\frac{dv}{v(1+v)}+\frac{dx}{x}=0 \quad (5)$$

$$\frac{dv}{v}-\frac{dv}{v+1}+\frac{dx}{x}=0 \quad (6)$$

integrating, $-\ln v+\ln(v+1)=\ln x+\ln c$ (7)

$$cx=\frac{v+1}{v} \quad (8) \qquad cvx=v+1 \quad (9)$$

$$c\,\frac{y}{x^2}\,x=\left(\frac{y}{x^2}+1\right) \quad (10)$$

Hence $cxy=y+x^2$ (11)

EX3. $x^3y'+4x^2y+1=0$ (1)

the wts of each term are

$3+m-1 \equiv m+2$, $2+m$, and 0

$\therefore$ (1) is isobaric when $m=-2$

put $y=v/x^2$ $\quad y'=\frac{v'}{x^2}-\frac{2v}{x^3}$

(1) takes the form $xv'-2v+4v+1=0$ (2)

$$xv'+2v+1=0 \quad (3)$$

$$\frac{dv}{2v+1}+\frac{dx}{x}=0. \quad (4) \qquad \log(2v+1)+\log x^2=\log c \quad (5)$$