葉慶成論文集

THREE-DIMENSIONAL CONSISTENT HIGHER-ORDER THEORY OF LAMINATED PLATES AND SHELLS

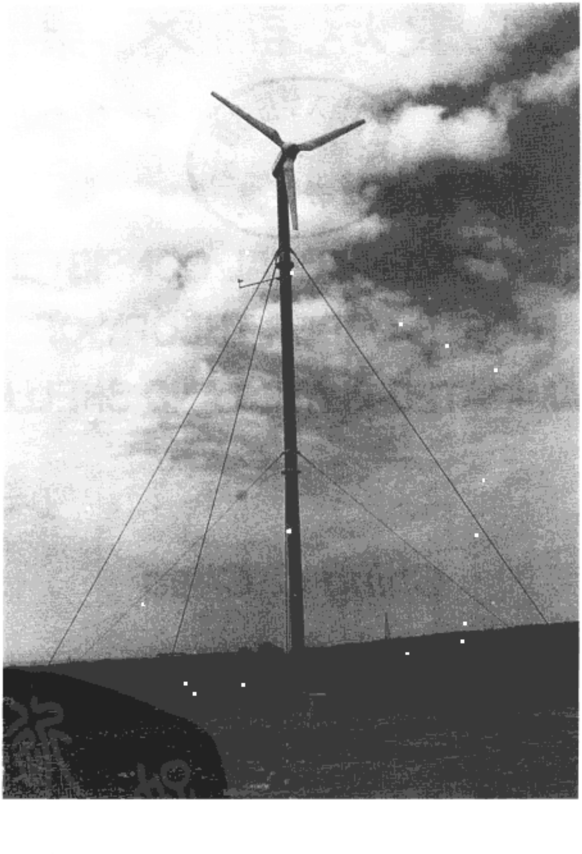
Ching C. Chao

巢慶成論文集

THREE-DIMENSIONAL CONSISTENT HIGHER-ORDER THEORY OF LAMINATED PLATES AND SHELLS

Ching C. Chao

29 March 1996



3 kilowatt glass/epoxy form sandwich wind turbine blades of 3 meters long - design, analysis and prototype manufacturing - an Energy Resources Research Project supervized by Professor C. C. Chao, Hsinchu, Taiwan, R.O.C. 1982.

To
My Parents

PREFACE

This book, entitled Three-Dimensional Consistent Higher-Order Theory of Laminated Plates and Shells, is published as a collection of technical papers which I presented at a series of international conferences followed by publication in international journals in joint efforts of my graduate students at National Tsing Hua University who did their thesis research under my supervision since 1976. The publication is made possible under the financial support of Ministry of Education on my seventieth birthday and it will be disseminated to colleagues in solid/composites mechanics all over the world as an exchange of research results and a token of friendship.

The theory was firstly developed six years ago, in which the subject problem is treated, unlike the traditional theories, in a manner of elasticity. Three-dimensional boundary conditions and interlaminar continuity are prescribed and satisfied in terms of local displacements and stresses. Three-dimensional dynamic deformation and stresses can be predicted throughout the plates and shells at all times. Successful applications have been found in stress analysis, vibration, impact, shock, water-jet cutting, solid propellant detonation, static and dynamic contact mechanics without and with friction, damage prediction, heat conduction and thermal stresses in electronic packagings. Current topics under way are the three-dimensional investigation of crack surface, penetration and failure mechanism thereof. It is hoped that the present theory will serve as a tool in solving a number of problems in solid mechanics, composites and metals as well.

The book is composed of twenty chapters in four parts. Part I is on stress and vibration analyses, Part II on impact and shock, Part III on contact mechanics, and Part IV on the first-order theory as an introductory work. Each chapter contains an article presented at an international conference and/or published in an international journal. Once again, the author grateful acknowledge the Ministry of Education in providing financial support for publication of this book. Acknowledgements are extended to The International Union of Theoretical and Applied Mechanics, International Journal of Impact Engineering, ASME Journal of vibration and Acoustics, ASME Journal of Energy Resources Technology, The International Community for Composites Engineering, Journal of Composite Materials, Composite Structures, Elsevier Science, Computational Mechanics Publications and all related international conference proceedings and journal publishers for use of the author's published articles in this book. Thanks are also due to the many graduate students for their well done research, and especially for reproducing the whole text and graphics for this book.

Ching C. Chao

AUTOBIOGRAPHY

I was born on Mach 29, 1926 – the Chinese National Youth Day in Shuihsiu District of Kiangsi Province on Mainland China and grew up in world war II. I jointed the Chinese Air Forcė at Chungking in 1945. After graduation from the Air Technological School, I served as Lieutenant to Colonel 1948-1978. I received my master's degree from Kansas State University in 1965 and Ph.D. from Purdue University, USA in 1974, where I was elected as member of the Honor Society of Phi Kappa Phi. I was Chief of Structures with the Aircraft Industry Development Center, Taichung, Taiwan, ROC in 1974-1978. Then I started teaching and research as Professor of Power Mechanical Engineering at National Tsing Hua University, Hsinchu, Taiwan, ROC. My specialty has been in Solid Mechanics, Fluid Mechanics, Composites Mechanics, Impact and Shock Dynamics.

The Three-Dimensional Consistent Higher-Order Theory of Laminated Plates and Shells was developed in recent years with a number of successful applications in solid mechanics published in a series of international conferences and journals. I was invited to serve as International Conference Ambassador of The 13th Biennial ASME Conference on Mechanical Vibration and Noise, 1991, Miami, Florida, and Regional Co-Chairman of The First, Second and Third International Conference on Composites Engineering in 1994, 95 and 96, New Orleans, Louisiana, USA, respectively. I have been on the Editorial Board of Composites Part B: Engineering, an International Journal since 1993.

My parents Mr. and Mrs. Feng Yun Chao are in good health at the age of ninety five. My wife Yu Hua Kuang and I married in 1949 and we have six children. They all have received good education and now have good jobs in the US and Taiwan. Deep appreciation is hereby expressed to my wife for taking good care of our children and family during the past forty seven years so that I can concentrate on my research work. Many thanks to God in giving me good health, wisdom and strength. I will retire from the University August 1, 1996, but my research will go on with pleasure.

Ching C. Chao

自　傳

余生於一九二六年三月二十九日，江西修水，少年時代正值對日堅苦抗戰，一九四五年考入空軍機械學校，一九四八年畢業後，任職由空軍少尉而至上校，其間兩度爲國防部考選赴美進修，一九六六年獲堪薩斯州立大學機械工程碩士，一九七四年獲普度大學機械工程博士學位，並當選爲ΦKΦ榮譽學會會員。返國後任航空工業發展中心飛機結構課長、研究室副主任，一九七八年退伍，任教國立清華大學動力機械工程學系，兩年後升任爲教授，個人專長爲固體力學、複合材料力學、衝擊動力學與流體力學。

近年來，余手創複合材料積層板殼三維一致性高階理論，在固體力學領域一系列之基礎應用研究中，業已獲致突破性之成功．有關研究成果分別發表於各大型國際會議、國際學術期刊。一九九一年美國機械工程學會第十三屆機械振動暨噪音雙週年會，在美國佛州邁亞密舉行，本人應邀擔任國際會議大使；一九九四、九五、九六年第一、二、三屆國際複合材料工程學術會議，在美國紐奧良舉行，本人先後擔任三屆大會副主席，並自一九九三年起擔任國際複合材料工程學刊編輯委員至今。

堂上二老均健在，家父九十五歲高齡，健康良好。一九四九年余與匡玉華女士結婚，共育有子女六人，均已完成大學以上良好教育，除小女兒外均已分別在美國與台灣成家立業。感謝吾妻結縭四十七年來任勞任怨，勤儉持家，教養子女，使余無後顧之憂，乃能專注於教研工作，實爲典型賢妻良母。最後，感謝　神賜我健康、智慧、與力量，使我研究工作得心應手，勝任愉快；今年八月一日退休後，仍將繼續研究著作，興之所至，一樂也。

巢慶成　謹識
台灣新竹　清華園
1996年3月29日

CONTENTS

Chapter 1

On Free Vibration and Stability of Thick Orthotropic Plates Using a 3-D Higher-Order Theory [1]

C. C. Chao, T. P. Tung and J. S. Cherng

Department of Power Mechanical Engineering
National Tsing Hua University, Hsinchu, Taiwan, ROC

Abstract — A three-dimensional higher-order theory is developed for free vibration and stability of thick orthotropic rectangular plates. Thickness stretching and shrinking as well as transverse shear strains and rotary inertia are fully taken into account, and are shown with significant effects. The 3-D simply supported boundary requires both lateral surfaces transverse normal stress free in addition to the transverse shear stresses as in the traditional shear deformation plate theories. In keeping with the three-dimensional boundary condition, the complicated eleven term displacement field is reduced to five variables, and a simplified three-dimensional theory is formulated for vibration analysis of the orthotropic plates via the Ritz approach. Natural frequencies and buckling loads are obtained in good agreement with the existing exact solution. Also, variation of 3-D relative displacements and stresses across the thickness are found in a more reasonable manner as compared to the various existing shear deformation theories and local-global approach. This will provide an essential basis for further extension to a three-dimensional laminated plate theory for self-equilibrium of local surface traction with applied loads and interface continuity of displacements and transverse stresses.

[1] Presented at the 8th International Conference on Composite Materials held at Honolulu, Hawaii, July 15-19, 1991.

1.1 Introduction

Vibration of plates, constructed of metal or composites, has been under intensive research by many investigators. The classical plate theory, based on the Kirchhoff hypothesis of a nondeformable normal, is valid only for thin plates. In view of the smallness of shear modulii as compared to corresponding young's modulii of composites, the effect of transverse shear was incorporated in the first-order shear deformation plate theory in short as FSDPT (Whitney, 1969) on bending of laminated plates for moderate thickness with cross section rotations taken into account in addition to the mid-plane displacements. As the thickness increases, effects of thickness stretching and shrinking can not be overlooked. The very first version of higher-order theory was proposed by Whitney and Sun (1973) for extensional motion of laminated composites. Numerical results reported were restricted to cylindrical bending. In a comprehensive studies of static and vibrational problems, Reddy (1984) developed a refined higher-order shear deformation theory (HSDPT) by making the transverse shear stresses to vanish explicitly over the lateral surfaces. The transverse shear stresses was found to vary internally according to a parabolic rule without consideration of the transverse normal stresses. The transverse deflection remains unchanged through thickness.

A higher-order deformation theory of eleven parameters was used by Doong (1987) in the treatment of vibration and stability of initially stressed thick plates, in which the lateral surface traction appeared in the traditional form of stress resultants, stress couples and the higher-order moments about the mid-surface. As a matter of fact, there is no guarantee for these boundary surfaces to be stress-free in the free vibration analysis as one can see in the first-order shear deformation theory. Also analyzed were bending, buckling and vibration of the symmetric cross-ply laminated elastic plates in the work of Khdeir and Librescu (1988) using a higher-order theory, in which in addition to transverse shear deformation, the transverse normal stress was based on zero transverse normal strain. The global-local approach has been employed by many authors to obtain the in-plane and transverse shear stress components by means of the first order theory and then to have the transverse normal stress determined in accordance with a local stress equilibrium. Noor and Burton (1989) made a complete assessment of the various shear deformation theories for the laminated composite plates, over two hundred papers with extensive numerical results for comparison. However, when the plate becomes thick to a certain extent, calculation of the transverse normal stresses in this way might lead to loss of mid-surface symmetry and inconsistency with surface traction free condition in the free vibration analysis.

The present 3-D higher-order theory is developed via the Ritz approach in terms of eleven parameters for free vibration and stability of simply supported thick orthotropic rectangular plates with effects of thickness stretching-shrinking as well as transverse shear strains and rotary inertia fully taken into account. The plate is simply supported in a manner of 3-D elasticity with all edges restrained in the lateral and tangential directions and otherwise everywhere surface traction free. Emphasis is placed on vanishing of the transverse normal stresses over both lateral surfaces explicitly in addition to the transverse shear stresses as in the traditional shear deformation plate theories. Accuracy of the Ritz solution is ensured by fulfillment of all the 3-D surface and edge conditions. In the mean time, the complicated eleven term displacement field is reduced in five variables, and a simplified 3-D theory is resulted by solution form of double Fourier series in the x-y plane interlaced with cubic and quadratic polynomials of z for the displacement field in the x-, y- and z-directions respectively.

Usually, natural frequencies, critical buckling loads and the associated normal modes are of primary concern in the study of vibration and stability. However, accuracy of these results can be ensured only if the structural problem is properly solved with all the 3-D relative displacements and stresses reasonably distributed across the thickness even though they are presumably vanishingly small in magnitudes in free vibration. This will serve as an essential basis for extension to a 3-D laminated plate theory for the self-equilibrium of local surface traction and interface continuity of displacements and stresses (Chao et al., 1991). The transverse stresses are especially of importance in predicting local damage such as delamination and tensile cracking for composites under impact and other loading conditions.

Nomenclature

a, b, h = plate dimension, sides and thickness.
$A_{ij}, D_{ij}, F_{ij}, H_{ij}$ = plate stiffness coefficients, reduced in overbar form.
C_{ij} = 3-D orthotropic elastic constants.
I_1, I_3, I_5, I_7 = mass moments of inertia, reduced in overbar form.
k, K = layer number and number of layers.
k_1, k_2 = modulus ratios C_{13}/C_{33} and C_{23}/C_{33} respectively.
$k_3, \cdots, k_8$ = functions of $\alpha = m\pi h/a$, $\beta = n\pi h/b$ respectively.
m, n = mode numbers.
t = time.
T, U = kinetic and strain energies respectively.
u_1, u_2, u_3 = displacements in x, y, z directions respectively.
$u_x, \psi_x, \xi_x, \phi_x$ = higher-order expansion of u_1, Fourier coefficients in captail letters.
$u_y, \psi_y, \xi_y, \phi_y$ = higher-order expansion of u_2, Fourier coefficients in captail letters.
w, ψ_z, ξ_z = higher-order expansion of u_3, Fourier coefficients in captail letters.
ϵ_{ij}, γ_{ij} = normal strain and shear strain components.
ρ = mass density.
σ_{ij} = stress components.

1.2 Theory

1.2.1 A Three-Dimensional Model

Considering an orthotropic rectangular plate of considerable thickness h, and sides a, b, the present higher-order theory is proposed to include the effects of thickness stretching-shrinking as well as the conventional transverse shear deformation and rotary inertia. The displacement field is assumed as

$$\begin{aligned} u_1(x,y,z,t) &= u_x(x,y,t) + z\psi_x(x,y,t) + z^2\xi_x(x,y,t) + z^3\phi_x(x,y,t) \\ u_2(x,y,z,t) &= u_y(x,y,t) + z\psi_y(x,y,t) + z^2\xi_y(x,y,t) + z^3\phi_y(x,y,t) \\ u_3(x,y,z,t) &= w(x,y,t) + z\psi_z(x,y,t) + z^2\xi_z(x,y,t) \end{aligned} \tag{1.1}$$

Observing the small strain and Generalized Hooke's law, the stress field is expressed in the form

$$\begin{aligned} \sigma_{11} &= C_{11}u_{1,x} + C_{12}u_{2,y} + C_{13}u_{3,z} \qquad & \sigma_{23} &= C_{44}(u_{2,z} + u_{3,y}) \\ \sigma_{22} &= C_{12}u_{1,x} + C_{22}u_{2,y} + C_{23}u_{3,z} & \sigma_{13} &= C_{55}(u_{1,z} + u_{3,x}) \\ \sigma_{33} &= C_{13}u_{1,x} + C_{23}u_{2,y} + C_{33}u_{3,z} & \sigma_{12} &= C_{66}(u_{1,y} + u_{2,x}) \end{aligned} \tag{1.2}$$

where C_{ij} are the stiffness coefficients of the orthotropic material, and a comma followed by x, y, z denotes the corresponding partial derivatives of the preceding variable respectively.

Combination of eqns (1.1) and (1.2) gives the stress field components as follows:

$$\begin{aligned} \sigma_{11} &= C_{11}(u_{x,x} + z\psi_{x,x} + z^2\xi_{x,x} + z^3\phi_{x,x}) \\ &\qquad + C_{12}(u_{y,y} + z\psi_{y,y} + z^2\xi_{y,y} + z^3\phi_{y,y}) + C_{13}(\psi_z + 2z\xi_z) \\ \sigma_{22} &= C_{12}(u_{x,x} + z\psi_{x,x} + z^2\xi_{x,x} + z^3\phi_{x,x}) \\ &\qquad + C_{22}(u_{y,y} + z\psi_{y,y} + z^2\xi_{y,y} + z^3\phi_{y,y}) + C_{23}(\psi_z + 2z\xi_z) \\ \sigma_{33} &= C_{13}(u_{x,x} + z\psi_{x,x} + z^2\xi_{x,x} + z^3\phi_{x,x}) \\ &\qquad + C_{23}(u_{y,y} + z\psi_{y,y} + z^2\xi_{y,y} + z^3\phi_{y,y}) + C_{33}(\psi_z + 2z\xi_z) \\ \sigma_{23} &= C_{44}(\psi_y + 2z\xi_y + 3z^2\phi_y + w_{,y} + z\psi_{z,y} + z^2\xi_{z,y}) \\ \sigma_{13} &= C_{55}(\psi_x + 2z\xi_x + 3z^2\phi_x + w_{,x} + z\psi_{z,x} + z^2\xi_{z,x}) \\ \sigma_{12} &= C_{66}(u_{x,y} + z\psi_{x,y} + z^2\xi_{x,y} + z^3\phi_{x,y} + u_{y,x} + z\psi_{y,x} + z^2\xi_{y,x} + z^3\phi_{y,x}) \end{aligned} \tag{1.3}$$

1.2.2 Simply Supported Edge Conditions

In the traditional plate theories, the simple support boundary conditions are considered such that no normal force and no normal moment exist at the edge as a whole, without reference to the local stress condition. In the present study, the boundary conditions are specified in a more realistic and rigorous manner of 3-D elasticity. The plate is roller-supported at the edges as restrained from motion in the lateral and tangential directions and otherwise everywhere surface traction free, namely for all z within the range $-h/2 \le z \le h/2$

$$\begin{aligned} &u_1(x,0,z,t) = u_1(x,b,z,t) = u_2(0,y,z,t) = u_2(a,y,z,t) = 0 \\ &u_3(0,y,z,t) = u_3(a,y,z,t) = u_3(x,0,z,t) = u_3(x,b,z,t) = 0 \\ &\sigma_{11}(0,y,z,t) = \sigma_{11}(a,y,z,t) = \sigma_{22}(x,0,z,t) = \sigma_{22}(x,b,z,t) = 0 \\ &\sigma_{12}(0,y,z,t) = \sigma_{12}(a,y,z,t) = \sigma_{21}(x,0,z,t) = \sigma_{21}(x,b,z,t) = 0 \end{aligned} \tag{1.4}$$

To satisfy the above edge conditions, the following displacement functions are assumed according to eqn (1.3).

$$
\begin{aligned}
(u_x, \psi_x, \xi_x, \phi_x) &= \sum\sum (hU_{mn}, \Psi_{xmn}, \Xi_{xmn}/h, \Phi_{xmn}/h^2) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega t \\
(u_y, \psi_y, \xi_y, \phi_y) &= \sum\sum (hV_{mn}, \Psi_{ymn}, \Xi_{ymn}/h, \Phi_{xmn}/h^2) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\omega t \qquad (1.5) \\
(w, \psi_z, \xi_z) &= \sum\sum (hW_{mn}, \Psi_{zmn}, \Xi_{zmn}/h) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega t
\end{aligned}
$$

1.2.3 Lateral Boundary Conditions

To satisfy the truly stress-free boundary conditions over both of the upper and lower lateral surfaces, i.e., $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ at $z = \pm\, h/2$, we require

$$
\begin{aligned}
& C_{13}(u_{x,x} + \frac{h}{2}\psi_{x,x} + \frac{h^2}{4}\xi_{x,x} + \frac{h^3}{8}\phi_{x,x}) + C_{23}(u_{y,y} + \frac{h}{2}\psi_{y,y} + \frac{h^2}{4}\xi_{y,y} + \frac{h^3}{8}\phi_{y,y}) \\
& \qquad + C_{33}(\psi_z + h\xi_z) = 0 \\
& C_{13}(u_{x,x} - \frac{h}{2}\psi_{x,x} + \frac{h^2}{4}\xi_{x,x} - \frac{h^3}{8}\phi_{x,x}) + C_{23}(u_{y,y} - \frac{h}{2}\psi_{y,y} + \frac{h^2}{4}\xi_{y,y} - \frac{h^3}{8}\phi_{y,y}) \\
& \qquad + C_{33}(\psi_z - h\xi_z) = 0 \qquad (1.6) \\
& C_{44}(\psi_y + h\xi_y + \frac{3h^2}{4}\phi_y + w_{,y} + \frac{h}{2}\psi_{z,x} + \frac{h^2}{4}\xi_{z,x}) = 0 \\
& C_{44}(\psi_y - h\xi_y + \frac{3h^2}{4}\phi_y + w_{,y} - \frac{h}{2}\psi_{z,x} + \frac{h^2}{4}\xi_{z,x}) = 0 \\
& C_{55}(\psi_x + h\xi_x + \frac{3h^2}{4}\phi_x + w_{,x} + \frac{h}{2}\psi_{z,x} + \frac{h^2}{4}\xi_{z,x}) = 0 \\
& C_{55}(\psi_x - h\xi_x + \frac{3h^2}{4}\phi_x + w_{,x} - \frac{h}{2}\psi_{z,x} + \frac{h^2}{4}\xi_{z,x}) = 0
\end{aligned}
$$

Combining eqns (1.6) in pairs corresponds to

$$
\begin{aligned}
& C_{13}(u_{x,x} + \frac{h^2}{4}\xi_{x,x}) + C_{23}(u_{y,y} + \frac{h^2}{4}\xi_{y,y}) + C_{33}\psi_z = 0 \\
& C_{13}(\psi_{x,x} + \frac{h^2}{4}\phi_{x,x}) + C_{23}(\psi_{y,y} + \frac{h^2}{4}\phi_{y,y}) + 2C_{33}\xi_z = 0 \\
& \psi_y + \frac{3h^2}{4}\phi_y + w_{,y} + \frac{h^2}{4}\xi_{z,y} = 0 \\
& \xi_y + \frac{1}{2}\psi_{z,y} = 0 \qquad (1.7) \\
& \psi_x + \frac{3h^2}{4}\phi_x + w_{,x} + \frac{h^2}{4}\xi_{z,x} = 0 \\
& \xi_x + \frac{1}{2}\psi_{z,x} = 0
\end{aligned}
$$

Differentiating and equating of the above, we obtain the following relations in the form of differential operators.

$$
\begin{aligned}
& \left\{ 2 - \frac{h^2}{4}(k_1 \frac{\partial^2}{\partial x^2} + k_2 \frac{\partial^2}{\partial y^2}) \right\} \xi_x = k_1 u_{x,xx} + k_2 u_{y,yx} \\
& \left\{ 2 - \frac{h^2}{4}(k_1 \frac{\partial^2}{\partial x^2} + k_2 \frac{\partial^2}{\partial y^2}) \right\} \xi_y = k_1 u_{x,xy} + k_2 u_{y,yy} \qquad (1.8) \\
& \left\{ \frac{h^2}{8}(k_1 \frac{\partial^2}{\partial x^2} + k_2 \frac{\partial^2}{\partial y^2}) - 3 \right\} \phi_x = \frac{4}{h^2} w_{,x} - \left\{ \frac{1}{2}(k_1 \frac{\partial^2}{\partial x^2} + \frac{1}{3}k_2 \frac{\partial^2}{\partial y^2}) - \frac{4}{h^2} \right\} \psi_x - \frac{1}{3}k_2 \psi_{y,xy}
\end{aligned}
$$

$$\left\{\frac{h^2}{8}(k_1\frac{\partial^2}{\partial x^2}+k_2\frac{\partial^2}{\partial y^2})-3\right\}\phi_y=\frac{4}{h^2}w_{,y}-\frac{1}{3}k_1\psi_{x,xy}-\left\{\frac{1}{2}(\frac{1}{3}k_1\frac{\partial^2}{\partial x^2}+k_2\frac{\partial^2}{\partial y^2})-\frac{4}{h^2}\right\}\psi_y$$

$$\psi_{z,z}=-2\xi_z$$

$$\xi_{z,z}=-\frac{4}{h^2}\psi_z-3\phi_z-\frac{4}{h^2}w_{,z}$$

where $k_1=C_{13}/C_{33}$, $k_2=C_{23}/C_{33}$. The last two expressions may take alternate form through the identities:

$$\xi_{y,x}=\xi_{x,y} \qquad \phi_{y,x}-\phi_{x,y}=\frac{4}{3h^2}(\psi_{x,y}-\psi_{y,x})$$

Therefore, the eleven-term 3-D modeling is reduced to one in terms of five variables, i.e., u_x, u_y, w, ψ_x, and ψ_y in observing the 3-D anisotropic Hooke's law and complete satisfaction of the over-all stress free boundary conditions over both lateral surfaces.

1.3 Ritz Procedure

In order to solve the vibration problem via the energy variational approach, we first formulate the strain energy and the kinetic energy for the 3-D model thick orthotropic plate throughout all its K layers. The strain energy is written as

$$U=\frac{1}{2}\sum_{k=1}^{K}\int_{v_k}\left\{C_{11}\epsilon_{11}^2+2C_{12}\epsilon_{11}\epsilon_{22}+C_{22}\epsilon_{22}^2+2C_{13}\epsilon_{11}\epsilon_{33}+2C_{23}\epsilon_{22}\epsilon_{33}+C_{33}\epsilon_{33}^2 +C_{44}\gamma_{23}^2+C_{55}\gamma_{13}^2+C_{66}\gamma_{12}^2\right\}dV$$

Integration across the thickness in terms of the displacement components gives

$$\begin{aligned}U=\frac{1}{2}\int_0^a\int_0^b\Big\{&A_{11}u_{x,x}^2+D_{11}(\psi_{x,x}^2+2u_{x,x}\xi_{x,x})+F_{11}(\xi_{x,x}^2+2\psi_{x,x}\phi_{x,x})+H_{11}\phi_{x,x}^2\\&+2A_{12}u_{x,x}u_{y,y}+2D_{12}(u_{y,y}\xi_{x,x}+\psi_{x,x}\psi_{y,y}+u_{x,x}\psi_{y,y})\\&+2F_{12}(\psi_{y,y}\phi_{x,x}+\xi_{y,y}\xi_{x,x}+\psi_{x,x}\phi_{y,y})+2H_{12}\phi_{x,x}\phi_{y,y}\\&+A_{22}u_{y,y}^2+D_{22}(\psi_{y,y}^2+2u_{y,y}\xi_{y,y})+F_{22}(\xi_{y,y}^2+2\psi_{y,y}\phi_{y,y})+H_{22}\phi_{y,y}^2\\&+2A_{13}u_{x,x}\psi_x+2D_{13}(\psi_x\xi_{x,x}+2\xi_x\psi_{x,x})+4F_{13}\xi_x\phi_{x,x}\\&+2A_{23}u_{y,y}\psi_x+2D_{23}(\psi_x\xi_{y,y}+2\xi_x\psi_{y,y})+4F_{23}\xi_x\phi_{y,y}+A_{33}\psi_x^2+4D_{33}\xi_x^2\\&+A_{44}(\psi_y+w_{,y})^2+2D_{44}(\psi_y+w_{,y})(3\phi_y+\xi_{z,y})+F_{44}(3\phi_y+\xi_{z,y})^2\\&+A_{55}(\psi_x+w_{,x})^2+2D_{55}(\psi_x+w_{,x})(3\phi_x+\xi_{z,x})+F_{55}(3\phi_x+\xi_{z,x})^2\\&+A_{66}(u_{x,y}+u_{y,x})^2+D_{66}\left[(\psi_{x,y}+\psi_{y,x})^2+2(u_{x,y}+u_{y,x})(\xi_{x,y}+\xi_{y,x})\right]\\&+F_{66}\left[(\xi_{x,y}+\xi_{y,x})^2+2(\psi_{x,y}+\psi_{y,x})(\phi_{x,y}+\phi_{y,x})\right]+H_{66}(\phi_{x,y}+\phi_{y,x})^2\Big\}dxdy\end{aligned} \tag{1.9}$$

In the mean time, the kinetic energy can be obtained as

$$T=\frac{1}{2}\sum_{k=1}^{K}\int_{v_k}\rho_k\left\{u_{1,t}^2+u_{2,t}^2+u_{3,t}^2\right\}dV$$

Integration across the thickness yields

$$\begin{aligned}T=\frac{1}{2}\int_0^a\int_0^b\Big\{&I_1(u_{x,t}^2+u_{y,t}^2+w_{,t}^2)\\&+I_3(\psi_{x,t}^2+2u_{x,t}\xi_{x,t}+\psi_{y,t}^2+2u_{y,t}\xi_{y,t}+\psi_{z,t}^2+2w_{,t}\xi_{z,t})\\&+I_5(\xi_{x,t}^2+2\psi_{x,t}\phi_{x,t}+\xi_{y,t}^2+2\psi_{y,t}\phi_{y,t}+\xi_{z,t}^2)+I_7(\phi_{x,t}^2,\phi_{y,t}^2)\Big\}dxdy\end{aligned} \tag{1.10}$$

where

$$(A_{ij}, D_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} C_{ij}^k (1, z^2, z^4, z^6) dz, \qquad i, j = 1,2,3,4,5,6$$

$$(I_1, I_3, I_5, I_7) = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} \rho_k (1, z^2, z^4, z^6) dz$$

According to the Ritz procedure, we take

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0 \tag{1.11}$$

in which, the configuration of the vibrating system is known at the instants t_1 and t_2. During a complete cycle of the harmonic motion, the maximum strain energy U_{max} and the maximum kinetic energy T_{max} can be obtained by substituting the double Fourier series expansion of eqn (1.5) into the energy formulation of eqns (1.9) and (1.10), and integrating over the whole plate. The energy variation in eqn (1.11) leads to

$$\left(\frac{\partial}{\partial U_{mn}}, \frac{\partial}{\partial V_{mn}}, \frac{\partial}{\partial W_{mn}}, \frac{\partial}{\partial \Psi_{xmn}}, \frac{\partial}{\partial \Psi_{ymn}} \right) (T_{max} - U_{max}) = 0 \tag{1.12}$$

Since the problem is uncoupled for each individual mode m, n, eqn (1.12), in turn, yields a system of typical eigen equations

$$[K]_{mn}\{X\}_{mn} = \Omega_{mn}^2 [M]_{mn}\{X\}_{mn} \tag{1.13}$$
$$m = 1, 2, \cdots; n = 1, 2, \cdots$$

where $[K]$, $[M]$ are the stiffness and mass matrices for the specific mode as shown in the Appendix; and Ω^2, $\{X\}$ the corresponding squared normalized natural frequency and the normal modes respectively with $\Omega_{mn} = \omega_{mn}\sqrt{\rho/C_{11}}$.

1.4 Results and Discussion

All numerical computations in this work are based on material properties corresponding to aragonite crystals as follows:

Table 1. Orthotropic properties

$C_{22}/C_{11} = 0.543103$	$C_{33}/C_{11} = 0.530172$
$C_{12}/C_{11} = 0.233190$	$C_{13}/C_{11} = 0.010776$
$C_{23}/C_{11} = 0.098276$	$C_{44}/C_{11} = 0.266810$
$C_{55}/C_{11} = 0.159914$	$C_{66}/C_{11} = 0.262931$
$C_{11} = 159.964$ GPa	$\gamma = 2.94$

First, we calculate the natural frequencies of an orthotropic square plate with thickness ratio $h/a = 0.1$. The normalized frequencies of the various modes are listed in Table 2 in good agreement with the exact solution of Scrinivas and Rao (1970) as opposed to the various approximation theories. Results of the first few modes are plotted in Fig. 1 versus the thickness ratio for comparison. Also, the normalized frequencies and buckling loads of the 3-D orthotropic rectangular plates are calculated and compared with varying aspect ratio a/b in Figs. 2 and 3 respectively. Overall, it is shown that our results are always the lowest and the closest to the exact solution.

When a structure is in free vibration, we are usually concerned only with its natural frequencies and normal modes. The actual displacement and stress fields have drawn little attention from research. However, for the prevention of mechanical resonance in structural design and for the precise prediction of dynamic response in modal analysis, importance of accurate and adequate natural frequencies and normal modes cannot be overemphasized. To understand the internal mechanism of the plate vibration, the relative magnitudes of displacements and stresses are computed across the thickness to see if the distribution is reasonable and if 3-D boundary conditions are completely satisfied.

By taking the fundamental mode as an important example, variation of the 3-D displacements, normal and shear stresses are calculated across the thickness for thick plates of thickness ratios $h/a = 0.1$ and 0.3 respectively. The Numerical results are shown in Figs. 4 to 12 in comparison with other theories, on the basis relative to the maximum central deflection $u^p_{3,max}$ and maximum longitudinal in-plane stress $\sigma^p_{11,max}$ of the 0.3 thicker plate, in which the superscript p stands for reference data of the present study. Significant increases are noted as the plate thickness is increased. The in-plane stresses reach their maximum, while the transverse normal and shear stresses all vanish at the lateral surfaces exactly. Results of the present study are much lower than any other theories. This is a reasonable indication because free vibration is the kind of motion, in which external excitation is supposed to be vanishingly small. Especially noted in Fig. 9 is the excellent antisymmetry of the normalized transverse normal stress with respect to the mid-plane as compared to the unsymmetric distribution and nonvanishingness at the lower surface for results obtained from HSDPT via the global-local approach.

1.5 Conclusion

A refined three-dimensional higher-order theory is developed for vibration and stability of thick orthotropic plates. Consideration of all the stress-free boundary conditions at both lateral surfaces and all four edges has reduced the eleven parameter theory in five variables. Numerical results obtained for natural frequencies and critical buckling loads are in good agreement with Scrinivas and Rao's exact solution. Relative 3-D displacements and stresses are found reasonable across the thickness with significant plate thickness effects, as compared to the various existing shear deformation theories and local-global approach, and thus an essential basis will be provided for further extension to a 3-D laminated plate theory for the self-equilibrium of local surface traction and interface continuity of displacements and stresses, even though their magnitudes are presumably vanishing under the circumstances of free vibration.